$(h = h_1 + ih_2)$ then u_x, u_y, v_x, v_y exist at (x_0, y_0) and

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}}$$

i.e., This equations are called Cauchy-Rieman equations.

$\forall (x_0, y_0)]$

Lecture 3

08/05/09: Notes by Banothu. Raghu.

Review:

Sequences:

A seq. (z_n) in $\mathbb{C}$ is,

i) bounded if $\exists\ M \in \mathbb{R}$ s.t. $|z_n| \le M \ \forall n \in \mathbb{N}$

ii) convergent if $\exists\ z \in \mathbb{C}$ s.t. for every $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $|z_n - z| < \varepsilon \ \forall n \ge n_0$.

(i.e., $z_n \in B(z, \varepsilon)$)

In this case z is unique and denoted by $\lim_{n\to\infty} z_n$.

iii) Cauchy if for every $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t. $|z_n - z_m| < \varepsilon \ \forall n, m \ge n_0$.

Basic Properties of Sequences:

Let (z_n) be a seq in $\mathbb{C}$

$\rightarrow (z_n)$ convergent $\Rightarrow (z_n)$ bounded

[converse not true in general].

iii) (z_n) convergent $\Longleftrightarrow$ (z_n) Cauchy.

$\longrightarrow$ (z_n) convergent $\Longleftrightarrow$ $(Re(z_n))$ and $(Im(z_n))$ are both convergent.

{resp: cauchy $\Rightarrow$ (resp: Cauchy}

Further if (z_n) and (w_n) are any sequences in $\mathbb{C}$, then

(a) $\hookrightarrow$ $z_n \longrightarrow z$ & $w_n \longrightarrow w$ $\Rightarrow$ $z_n + w_n \longrightarrow z+w$ and $z_n w_n \longrightarrow zw$.

$\hookrightarrow$ $w_n \longrightarrow w$ and $w \neq 0$ $\Rightarrow$ $\exists\ n_0 \in \mathbb{N}$ s.t. $w_n \neq 0$ $\forall\ n \geq n_0$ and moreover, $\frac{1}{w_n} \to \frac{1}{w}$

Proof: (a) we have,

$$|z_n + w_n - (z+w)| \leq |z_n - z| + |w_n - w|$$

so given $\varepsilon > 0$, $\exists\ n_1, n_2 \in \mathbb{N}$ s.t.

$$|z_n - z| < \frac{\varepsilon}{2} \quad \forall\ n \geq n_1$$

and $\quad |w_n - w| < \varepsilon/2 \quad \forall\ n \geq n_2$

Let $\quad n_0 = \max\{n_1, n_2\}$. Then

$$|(z_n + w_n) - (z+w)| < \varepsilon/2 + \varepsilon/2 \quad \forall\ n \geq n_0$$

so $\quad z_n + w_n \longrightarrow z + w$.

$$|z_n w_n - zw| = |z_n w_n - z w_n + z w_n - zw|$$

$$\leq |z_n - z||w_n| + |z||w_n - w|$$

$$\leq M|z_n - z| + |z||w_n - w|.$$

$\hookrightarrow$ Next (w_n) cgt $\Rightarrow$ (w_n) bded. So $\exists\ M > 0$ s.t. $|w_n| \leq M\ \forall n$.

Let $\varepsilon > 0$ be given. Choose n_1, n_2 s.t.

$$|z_n - z| < \frac{\varepsilon}{2M} \quad \forall\, n > n_1$$

and $|w_n - w| < \frac{\varepsilon}{2(|z|+1)} \quad \forall\, n \geq n_1$.

Then
$$\begin{aligned}
|z_n w_n - zw| &= |z_n w_n - z w_n + z w_n - zw| \\
&\leq |z_n - z|\,|w_n| + |z|\,|w_n - w| \\
&\leq M\,|z_n - z| + |z|\,|w_n - w| \\
&< M \cdot \frac{\varepsilon}{2M} + |z| \cdot \frac{\varepsilon}{2(|z|+1)} \\
&< \varepsilon \quad \forall\, n \geq n_0 = \max(n_1, n_2).
\end{aligned}$$

* Proof of (b) is exercise.

Continuity:

Def:

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$ be a function.

Then f is continuous at $z \in \Omega$ if,

(z_n) seq. in Ω, $z_n \longrightarrow z \Rightarrow f(z_n) \longrightarrow f(z)$.

We say f is continuous (on Ω) if it is continuous at every $z \in \Omega$.

From basic properties of sequences, we can deduce the following:

Basic Properties of cont. functions:

$\hookrightarrow$ $f, g : \Omega \longrightarrow \mathbb{C}$ cont. at $z \in \Omega$

$\Rightarrow$ $f+g$ is cont. at $z \in \Omega$

$\Rightarrow$ fg is not cont. at $z \in \Omega$.

Further if $g(z) \neq 0$ and g is cont. at z,

then $\boxed{\exists\ r > 0 \quad s.t.\ g(\omega) \neq 0 \quad \forall\ \omega \in B(z,r) \cap \Omega} \rightarrow (*)$

Moreover,
$$\frac{1}{g} : B(z,r) \cap \Omega \rightarrow \mathbb{C} \text{ is cont. at } z.$$

Proof:

Continuity of $f+g$ and fg is immediate from defn. and properties of sequences.

Further suppose 'g' is cont. at z and

$g(z) \neq 0$.

Suppose, if possible $(*)$ is not true.

Then, $\forall\ r > 0$, $\exists\ \omega \in B(z,r) \cap \Omega$ s.t $g(\omega) = 0$.

Thus

for $r = 1$, $\exists\ \omega_1 \in B(z,r) \cap \Omega$ s.t $g(\omega_1) = 0$.

for $r = 1/2$, $\exists\ \omega_2 \in B(z,r) \cap \Omega$ s.t $g(\omega_2) = 0$.

and ~~so~~ so on, i.e., for,

$r = 1/n$, $\exists\ \omega_n \in B(z, 1/n)$ s.t. $g(\omega_n) = 0$.

Now $\omega_n \rightarrow z$, But $g(\omega_n) = 0\ \forall\ n$

and since $g(z) \neq 0$, $g(\omega_n) \not\rightarrow g(z)$.

This contradicts the continuity of g at

Now it follows from properties of sequences

that $\frac{1}{g}$ is cont. at z.

Consequence:

Every polynomial function is cont. on $\mathbb{C}$ and every rational function is cont. wherever it is defined.

Differentiability:

Def^n:

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a function. Let z_0 be an interior point of Ω. Then 'f' is (complex) differentiable at 'z_0' if

$$\lim_{\substack{h \to 0 \\ (h \in \mathbb{C})}} \frac{f(z_0+h) - f(z_0)}{h} \quad \text{exists.}$$

(equivalently if $\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

In this case, the limit is denoted by $f'(z_0)$ and called the derivative of 'f' at z_0.

Def^n:

If Ω is an open set, then $f : \Omega \to \mathbb{C}$ is differentiable (on Ω) if f is differentiable at every $z_0 \in \Omega$.

If we write $f = u + iv$ or $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ then 'f' is diff at $z_0 = x_0 + iy_0$ $\Rightarrow$ the first order partial derivative of u, v

at (x_0, y_0) exist and

$$\boxed{\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= v_x(x_0, y_0) \end{aligned}}$$

Cauchy - Rieman equations.

Further,

$$\begin{aligned} f'(z_0) &= u_x(x_0, y_0) + i\, v_x(x_0, y_0) \\ &= \frac{1}{i} \left[u_y(x_0, y_0) + i\, v_y(x_0, y_0) \right] \\ &= v_y(x_0, y_0) - i\, u_y(x_0, y_0). \end{aligned}$$

Observe that,

Jacobian of (u, v) at $(x_0, y_0) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$\left.\begin{aligned} &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \; \cancel{= \text{Laplacian}} \\ &= \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= |f'(z_0)|^2. \end{aligned}\right\} \text{ at } (x_0, y_0)$$

Fact:

Conversely if u, v have continuous partial derivatives ~~and~~ at (x_0, y_0) and if they Cauchy-Rieman equations are satisfied then $f = u + iv$ is diff at $z_0 = x_0 + iy_0$.

Pf: Exercise. {Hint: Use Taylor's Theorem}

Eg: $\rightarrow f(z) = z$ is diff. at every $z_0 \in \mathbb{C}$

and $f'(z) = 1$.

$\rightarrow f(z) = \bar{z}$ is not diff. at any $z_0 \in \mathbb{C}$

$\rightarrow f(z) = |z|^2$ is diff. only at $z = 0$.

Differentiability:

Analytic:

Def^n:

Let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a function. Let z_0 be an interior point of Ω. Then f is said to be analytic or holomorphic at z_0 if $\exists$ $r > 0$ s.t. $B(z_0, r) \subseteq \Omega$ and f is differentiable at every point of $B(z_0, r)$.

11/05/09

Review:

Continuous Functions:

Let Ω be a subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a function. Then, 'f' is said to be:

$\hookrightarrow$ continuous at $z_0 \in \Omega$ if (z_n) seq. in Ω, $z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$ [equivalently if $\lim_{z \to z_0} f(z)$ exists and is $= f(z_0)$.]

assuming z_0 is a "limit point" of Ω.

$\hookrightarrow$ differentiable at an interior point $z_0 \in \Omega$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.