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§ Lecture - 4 11/5/09

Continuous functions →

Let Ω be a subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a fn. Then f is said to be:

(i) Continuous at $z_0 \in \Omega$ if

$$(z_n) \text{ seq in } \Omega, \; z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$$

[equivalently, if $\lim_{z \to z_0} f(z)$ exists and is $= f(z_0)$

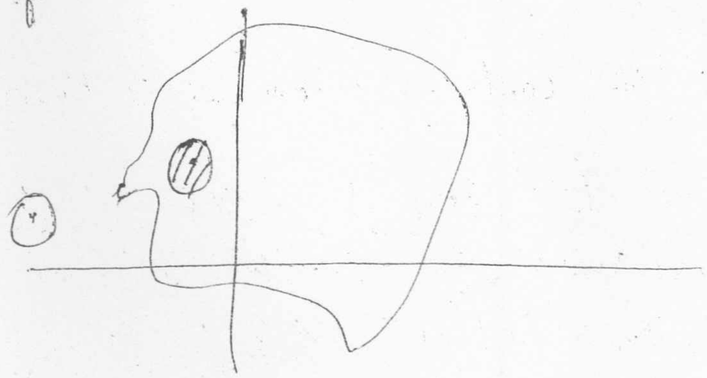
assuming z_0 is a 'limit' point of Ω.

(ii) differentiable at an interior pt. $z_0 \in \Omega$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

(iii) Analytic at an interior pt. $z_0 \in \Omega$ if f is differentiable at every pt of $B(z_0, r)$ for some $r > 0$ with $B(z_0, r) \subseteq \Omega$.

Note: $z_0 \in \mathbb{C}$ is an interior point of Ω if $B(z_0, r) \subseteq \Omega$ for some $r > 0$ whereas $z_0 \in \mathbb{C}$ is a limit point of Ω if for every $r > 0$, $B(z_0, r)$ contains some point of Ω other than z_0.



Recall:

Def^n of Limit $\rightarrow$

$\lim_{z \to z_0} f(z) = l$ means

(z_n) seq. in Ω, $z_n \neq z_0 \ \forall n$

$\Rightarrow f(z_n) \longrightarrow l$

Alternate Def^n. $\forall \epsilon > 0, \exists \delta > 0$

s.t.

$z \in \Omega$ & $0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$

When z_0 is a limit pt. of Ω, then the limit of $f(z)$ as $z \to z_0$, if it exists, is unique (Verify !)

P.T.O.

Basic Properties $\rightarrow$

① Sum & products of cont. functions are continuous.

② $f : \Omega \to \mathbb{C}$ is a fn. and $f = u + iv$ so that u, v are real valued fns. of two variables. Then f is cont. at $z_0 = x_0 + iy_0 \iff u$ is cont. at (x_0, y_0) & v is cont. at (x_0, y_0)

[But this thing is not true in case of differentiability].

③ Composites of continuous fns. are continuous. More precisely, if $f : \Omega \to \mathbb{C}$ is cont. at $z_0 \in \Omega$ and $g : \Omega' \to \mathbb{C}$, where $\Omega' \supseteq f(\Omega)$, is cont. at $f(z_0)$, then

$$g \circ f : \Omega \to \mathbb{C}$$

defined by $(g \circ f)(z) = g(f(z))$.

is continuous at z_0.

Basic Examples $\rightarrow$ poly fns, rational fns,

$f(z) = \bar{z}$, $f(z) = |z|$, etc. are cont. functions.

(Proof: Using inequality proved before)

$$\boxed{\big||z| - |w|\big| \leq |z - w|}$$

Basic Principles of Diff. ~~Equations~~ functions $\longrightarrow$

1) Sums & products of diff. fns are diff

In fact $f, g : \Omega \to \mathbb{C}$ diff at $z_0 \in \Omega$

$\Rightarrow f+g,\ fg : \Omega \to \mathbb{C}$

and

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$

$$(fg)'(z_0) = f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0)$$

~~Rk~~

0) f diff at $z_0 \Rightarrow f$ cont. at z_0

$$\lim_{z \to z_0} \left[f(z) - f(z_0) \right] = \lim_{z \to z_0} (z - z_0) \frac{f(z) - f(z_0)}{(z - z_0)}$$

$$\cancel{= 0 \times f'(z_0) = 0}$$

$$= 0 \times f'(z_0) = 0$$

Proof of ①

$$\frac{f(z) + g(z) - \left[f(z_0) + g(z_0) \right]}{z - z_0} =$$

$$\frac{f(z) - f(z_0)}{z - z_0} + \frac{g(z) - g(z_0)}{z - z_0}$$

$$\to f'(z_0) + g'(z_0)$$

and

$$\frac{(fg)(z) - (fg)(z_0)}{z - z_0}$$

$$= \frac{f(z)g(z) - f(z)g(z_0) + f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0}$$

$$= f(z)\left[\frac{g(z) - g(z_0)}{z - z_0}\right] + g(z_0)\left[\frac{f(z) - f(z_0)}{z - z_0}\right]$$

$$\to f(z_0)\,g'(z_0) + g(z_0)\,f'(z_0)$$

2) $f : \Omega \to \mathbb{C}$ diff at z_0, $f(z_0) \neq 0$

$\Rightarrow \frac{1}{f}$ is diff. at z_0 and $\left(\frac{1}{f}\right)'(z_0) =$

$$-\frac{f'(z_0)}{(f(z_0))^2}$$

(by continuity $\frac{1}{f}$ is defined on $B(z_0, r)$ for some $r > 0$)

Pf: $$\frac{\frac{1}{f(z)} - \frac{1}{f(z_0)}}{z - z_0} = \frac{-[f(z) - f(z_0)]}{[z - z_0]} \times \frac{1}{f(z)\,f(z_0)}$$

$$= -\frac{f'(z_0)}{(f(z_0))^2}$$

3) [Chain Rule]

$f : \Omega \to \mathbb{C}$, $g : \Omega' \to \mathbb{C}$

where $\Omega' \supseteq f(\Omega)$

z_0. interior pt of Ω s.t.

$f(z_0)$ interior pt. of Ω'

f diff. at z_0 & g diff. at $f(z_0)$

$\Rightarrow g\circ f$ diff. at z_0 and

$$(g\circ f)'(z_0) = g'(f(z_0))\, f'(z_0)$$

Pf: Sketch

$$\frac{(g\circ f)(z) - (g\circ f)(z_0)}{z - z_0}$$

$$\frac{g(f(z)) - g(f(z_0))}{f(z) - f(z_0)} \times \frac{f(z) - f(z_0)}{z - z_0}$$

$$\rightarrow g'(f(z_0))\, f'(z_0)$$

[To be precise, one has to ensure that $z \neq z_0 \Rightarrow f(z) \neq f(z_0)$ [Not]. This can be done by considering the cases $f'(z_0) = 0$ & $f'(z_0) \neq 0$ seperately].

Note: Above is not true if f is continuous.

(4) $f = u + iv$ diff at $z_0 = x_0 + iy_0$

$\Rightarrow u(x,y), v(x,y)$ have first order partial derivatives at (x_0, y_0) and $u_x(x_0,y_0) = v_y(x_0,y_0)$

$$u_y(x_0,y_0) = -v_x(x_0,y_0)$$

If u, v have continuous first order partial derivatives at (x_0,y_0), then the converse is true.

In fact, one has the following result:

Thm: Let $\Omega \subseteq \mathbb{C}$ and z_0 be an interior pt. of Ω. Let $f = u + iv : \Omega \to \mathbb{C}$ be a fn. Then f is analytic at z_0.

$\Leftrightarrow$ u_x, v_x, u_y, v_y exist in a ball around (x_0, y_0) and are continuous in that ball ~~at (x_0, y_0)~~ and moreover, the Cauchy-Riemann are satisfied.

$$u_x = v_y \quad \& \quad u_y = -v_x \quad \text{at } (x_0, y_0)$$

Pf: We have shown

f is diff. at $z_0 \Rightarrow u_x, u_y, v_x, v_y$ exist at (x_0, y_0) & C-R eq^{n}s hold at (x_0, y_0)

It will be shown later that

f analytic at $z_0 \Rightarrow u_x, u_y, v_x, v_y$ are cont. ~~at (x_0, y_0) in that ball~~ at (x_0, y_0)

For the converse,

assume that u_x, u_y, v_x, v_y exist in a ball around (x_0, y_0), are cont. ~~at (x_0, y_0)~~ ~~in that~~ at (x_0, y_0) & the C-R eq^{n}s hold.

Then in a ball around (x_0, y_0) we have (by Increment Lemma for fns. in two variables).

$$u(x,y) - u(x_0,y_0) = (x-x_0)\, u_x(x_0,y_0) +$$
$$u_y(x_0,y_0)(y-y_0) + r(x,y)$$

$$v(x,y) - v(x_0,y_0) = v_x(x_0,y_0)\,(x-x_0) + v_y(x_0,y_0)(y-y_0)$$
$$+ s(x,y).$$

where $r(x,y)$, $s(x,y)$ are "remainder" fns. with the property that

$$\frac{r(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \to 0 \text{ as } (x,y) \to (x_0,y_0)$$

&

$$\frac{s(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \to 0 \text{ as } (x,y) \to (x_0,y_0).$$

Now

$$\left| \frac{f(z)-f(z_0)}{z-z_0} - \left[u_x(x_0,y_0) + i\, v_x(x_0,y_0)\right] \right|$$

$$= \frac{\left| \begin{aligned} & u(x,y) + i v(x,y) - \left[u(x_0,y_0) + i v(x_0,y_0)\right] - (x-x_0)\{u_x(x_0,y_0) \\ & + (y-y_0)\, v_x(x_0,y_0) - i(x-x_0)\, v_x(x_0,y_0) \cancel{- i(y-y_0)} \\ & - i(y-y_0)\, u_x(x_0,y_0) \end{aligned} \right|}{|z-z_0|}.$$

$$= \frac{|r(x,y) + i\,s(x,y)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$ (using (-R. eq^{ns}))

$$\leq \frac{|r(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} + \frac{|s(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$\to 0 + 0 = 0$ as $z \to z_0$, i.e. as $(x,y) \to (x_0, y_0)$

This shows that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists \& is } =$$

$$u_x(x_0,y_0) + i\,v_x(x_0,y_0).$$

i.e. f is diff. at z_0.

In a similar way, we can show that f is diff. at every pt. in a ball around (x_0, y_0).

Exponential functions $\to$

We'll assume that the exponential fn. of a real variable is known (e^x for $x \in \mathbb{R}$)

Def: For $\theta \in \mathbb{R}$, we define

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

For $z \in \mathbb{C}$, we define

$$e^z = e^x(\cos y + i\sin y), \text{ if } z = x+iy$$
$$x, y \in \mathbb{R}.$$

We have,

$$e^x \cdot e^{x'} = e^{(x+x')} \quad \forall\, x, x' \in \mathbb{R}.$$

Using above,

$$e^z \cdot e^{z'} = e^x \cdot e^{x'} (\cos y + i\sin y)(\cos y' + i\sin y')$$

$$= e^{(x+x')}\Big[[\cos y \cos y' - \sin y \sin y'] + i[\cos y \sin y' + \sin y \cos y']\Big]$$

$$= e^{(x+x')}[\cos(y+y') + i\sin(y+y')].$$

$$= e^{z+z'} \quad \text{where, } z = x+iy, \; z' = x'+iy'.$$

Consider $f: \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = e^z.$$

Then $f = u + iv$

where
$$u(x,y) = e^x \cos y$$
$$v(x,y) = e^x \sin y$$

Clearly u_x, u_y, v_x, v_y exist and

$$u_x = e^x \cos y \qquad v_x = e^x \sin y$$
$$u_y = -e^x \sin y \qquad v_y = e^x \cos y$$

Thus u_x, v_x, u_y, v_y exist and are continuous on $\mathbb{R}^2$

the Cauchy-Riemann equations are satisfied:

$$u_x = v_y \quad \& \quad u_y = -v_x$$

So f is analytic on $\mathbb{C}$. Also $f'(z) = u_x + i v_x$

$$= e^x(\cos y + i \sin y$$

$$= e^z.$$