Complex Analysis.

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Lecture 5. 12/05/09.

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let $\Omega \subseteq \mathbb{C}$ and z_0 be interior of Ω i.e.
$B(z_0, r) := \{ z \in \mathbb{C} : |z - z_0| < r \} \subseteq \Omega$ for some $r > 0$

$f : \Omega \to \mathbb{C}$ is analytic at z_0 if $\exists\, r > 0$ such that $B(z_0, r) \subseteq \Omega$ and f is differentiable at every point of $B(z_0, r)$

Simple eg. $f(z) = z$ is analytic on $\mathbb{C}$

Since we have shown that sums, products, reciprocals (with non-zero dinominator) of analytic functions are analytic.

we see that

every polynomial function is analytic on $\mathbb{C}$ and every rational function is analytic wherever defined i.e. $\frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials is analytic for all $z_0 \in \mathbb{C}$ for which $q(z_0) \neq 0$.

Exponential function :-

Unlike the real exponential function e^x, the complex function e^z is not one-one since

$$e^{iy} = e^{i(y + 2n\pi)} \qquad \forall\, n \in \mathbb{Z}$$

In fact

$$e^z = e^{z'} \implies x = x' \; ; \; y = y' + 2n\pi \qquad \forall\, n \in \mathbb{Z}$$

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Multivalued functions —

A "function" such as arg z which for one valve of z takes several values is sometimes called "multivalued function". One way to make it precise is to think of it as a set-valued function, e.g. if θ is an argument of $0 \neq z \in \mathbb{C}$ then

$$\arg(z) = \{ \theta + 2n\pi \, ; \, n \in \mathbb{Z} \}$$

On the other hand, the principal argument function

$$\operatorname{Arg}(z) : \mathbb{C} \setminus \{0\} \longrightarrow (-\pi, \pi]$$

is a "single valued" function.

In general if

$$f : \Omega \to \mathbb{C}$$

is not one-one and $\Omega' = f(\Omega)$ then the "inverse function" f^{-1} is a multivalued function on Ω' with

$$f^{-1}(w) = \{ z \in \Omega : f(z) = w \}.$$

NOTE :- $f(\mathbb{C}) = \mathbb{C} / \{0\}$ In fact for any $w \in \mathbb{C}$ with $w \neq 0$ we can find $x \in \mathbb{R}$ such that $e^{x} = |w|$ and if y is the argument of w then $w = e^{x+iy}$.

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In particular, the inverse of the exponential fn is a multivalued function on $\mathbb{C} \setminus \{0\}$, called the logarithmic function and denoted by $\log_e$ or $\ln$. Thus.

$$\ln w = z \iff e^z = w$$

$$\iff |w| = |e^z| = e^x \text{ ~~(real part~~}$$

~~and $\arg(w) =$~~ .

and $\arg(w) = \arg(e^z) = y + 2n\pi$ where $z = x+iy$.

$$\iff x = \log |w|$$

$$\text{and } y = \arg(w) + 2n\pi$$

for all $n \in \mathbb{Z}$

Thus $\boxed{\ln w = \ln|w| + i\,(\arg(w) + 2n\pi)}$

replacing argument by principal argument we oblain a "single valued" function denoted by

$$\boxed{\ln w = \ln|w| + i \operatorname{Arg}(w)}$$

called the principal branch of the logarithmic function

$$\ln : \mathbb{C} \setminus \{0\} \longrightarrow \{x+iy \in \mathbb{C} : -\pi < y \leq \pi\}$$

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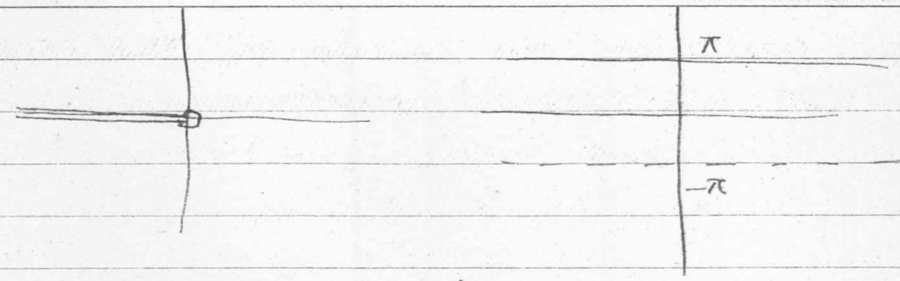

observe that the principal argument fn $Arg : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is continuous on $\mathbb{C} \setminus \{z \in \mathbb{C} : Re(z) \leq 0 \ \& \ Im(z) = 0\}$.

In fact
log is analytic on $\mathbb{C} \setminus \{$negetive x-axis$\}$
This can be shown by using

Inverse function theorem of multivariable calculus.

If $u(x,y)$, $v(x,y)$ have continuous first order partial derivatives on an open ~~set~~ subset of Ω of $\mathbb{R}^2$ and if

$$J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

at an interior point ~~of~~ (x_0, y_0) of Ω

Then a small ball ~~of~~ U around (x_0, y_0)

the function
$$(x,y) \longrightarrow (u(x,y), v(x,y))$$
is one-one and the image of U is an open set

As a consequence, we have.

Theorem : let $\Omega \subseteq \mathbb{C}$ be open and $z_0 \in \Omega$.
If $f : \Omega \to \mathbb{C}$ is analytic at z_0, and if $f'(z) \neq 0$, then there is a small ball V centered at z_0 such that $V \subseteq \Omega$ and
f is one one on V,
$f(V)$ is open
and, moreover,

$$f^{-1} : f(V) \longrightarrow V$$

is analytic at $w_0 = f(z_0)$ and

$$(f^{-1})'(w) = \frac{1}{f'(z)} \quad \text{for } w \in f(V)$$

where $z \in V$ is such that $f(z) = w$

Roughly speaking

$$w = w(z), \quad \left.\frac{dw}{dz}\right|_{z=z_0} \neq 0$$

$$\Rightarrow \quad z = z(w) \quad \text{near} \quad w_0 = w(z_0)$$

and $$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}}$$

proof :- The fact that there is

$V \subset \Omega$ such that f is one-one on V and $f(V)$ is open follows from the

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Inverse function theorem since
$|f'(z)|^2 = |J(u,v)(x_0,y_0)|$.

For analyticity observe that

$$\frac{f^{-1}(\omega) - f^{-1}(\omega_0)}{\omega - \omega_0}$$

$$= \frac{z - z_0}{f(z) - f(z_0)} \quad \text{if } \omega = f(z)$$

$$= \frac{1}{\dfrac{f(z) - f(z_0)}{z - z_0}}$$

$$= \frac{1}{f'(z_0)} \quad \text{as } \omega \to \omega_0.$$

we can apply this to.

$$f(z) = e^z$$

to deduce that

Log is analytic on $\mathbb{C} \setminus \{z \in \mathbb{C} : Re(z) \leq 0, Im(z) = 0\}$

open set and moreover

$$Log'(\omega_0) = \frac{1}{e^{z_0}} \quad \text{where } z_0 \text{ is such that } e^{z_0} = \omega_0.$$

$$= \frac{1}{\omega_0}.$$

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Thus $\frac{d}{dw}(\mathrm{Log}\, w) = \frac{1}{w}$

Trigonometric and Hyperbolic function :

for $z \in \mathbb{C}$ we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}$$

$$\sin hz = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

From the analyticity of complex exponential function and the chain rule, we see that all above functions are analytic on $\mathbb{C}$ except for $\tan z$, $\tanh z$ where $\cos z$ and $\cosh z$ are zero.

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Harmonic Functions :

Def : A function $u(x,y)$ with continuous second order partial derivatives, which satisfies the Laplace equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

is called a harmonic function.

If $\Omega \subset \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is analytic and $f = u + iv$, then u, v have continuous partial derivatives of every order (will be shown later) and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

by continuity from mixed partials.

Thus.

$$\nabla^2 u = 0.$$

Similarly

$$\nabla^2 v = 0.$$

Thus the real and imaginary parts

of an analytic functions are harmonic.

Question : given a harmonic function $u(x,y)$ on a domain $\Omega \subseteq \mathbb{R}^2$ does there exist a harmonic function $v(x,y)$ such that $u+iv$ is analytic on Ω?

[Such a function v is called a harmonic conjugate of u]

eg $u(x,y) = xy$

$$u_x = y \,,\; u_y = x$$

$$u_{xx} = u_{yy} = 0.$$

So u is harmonic. To find v such that $u+iv$ is analytic we solve.

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = y \qquad v = \frac{y^2}{2} + h(x).$$

$$\frac{\partial v}{\partial x} = h'(x) = -x.$$

$$h(x) = -\frac{x^2}{2} + c.$$

$$v = \frac{y^2}{2} - \frac{x^2}{2}$$

$$f(z) = xy + i\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + ic$$

$$f(z) = \frac{iz^2}{2} + ic.$$