MA 205. Complex Analysis.

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lecture 6 Dated - 13/05/09.

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Consider $\Omega = \mathbb{C} \setminus \{0\}$ and

$$u(x,y) = \log \sqrt{x^2+y^2}$$

show that $u(x,y)$ is harmonic in Ω

but $u(x,y)$ has no harmonic conjugate on Ω

(However, $u(x,y)$ has a harmonic conjugate on $\mathbb{C} \setminus [-\infty, 0]$

Indeed $u = \log|z| = \text{Re}(\text{Log } z)$

and $\text{Log } z$ is analytic on $\mathbb{C} \setminus [-\infty, 0]$

Theorem :

If Ω is an open ball or an open rectangle with sides parallel to the axes and $u : \Omega \to \mathbb{R}$ is harmonic, then $u(x,y)$ has a harmonic conjugate v in Ω and moreover, v is unique upto addition by a constant

proof :

To find v such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

let $(x_0, y_0) \in \Omega$

given any $(x,y) \in \Omega$

Consider

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}\, dy + h(x)$$

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Now $\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[\int_{y_0}^{y} \frac{\partial u}{\partial x} dy + h(x)\right]$

$$= \int_{y_0}^{y} \frac{\partial^2 u}{\partial x^2} dy + h'(x).$$

$$= -\int_{y_0}^{y} \frac{\partial^2 u}{\partial y^2} dy + h'(x)$$

$$= -\frac{\partial u}{\partial y}\Big|_{(x,y)} + \frac{\partial u}{\partial y}\Big|_{(x,y_0)} + h'(x).$$

Thus $v_x = -u_y$ provided

$$h'(x) = -\frac{\partial u}{\partial y}(x, y_0).$$

$$\Rightarrow \int h'(x)dx = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(x, y_0)\, dx.$$

$$h(x) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(x, y_0)\, dx.$$

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,y)dy - \int_{x_0}^{x} \frac{\partial u}{\partial y}(x, y_0)\, dx.$$

then v has continuous first order partial and

$$v_x = -u_y \; ; \; u_x = v_y.$$

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Hence $f = u + iv$ has the property that u_x, u_y, v_x, v_y exists and are continuous and C-R equations hold and so f is analytic.

For uniqueness, if v_1, v_2 are two harmonic conjugates of u, then

$$\frac{\partial}{\partial x}(v_1 - v_2) = -\left[\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right] = 0.$$

and $$\frac{\partial}{\partial y}(v_1 - v_2) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = 0.$$

both the partials are zero hence v is unique upto the addition by a constant provided Ω is "path connected". Hence $v_1 - v_2$ is constant since we considered Ω to be path connected.

Path :

A path in $\mathbb{C}$ is a function $\gamma : [a,b] \to \mathbb{C}$ for some interval $[a,b]$ in $\mathbb{R}$

A path γ is said to be

i) continuous if γ is continuous, i.e. the functions $x, y : [a,b] \to \mathbb{R}$ are continuous where

$$x = Re(\gamma) \text{ and } y = Im(\gamma)$$

i.e. $\gamma(t) = x(t) + iy(t)$ for $t \in [a,b]$

(ii) smooth if x, y are differentiable on (a,b) and derivatives are continuous.

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(iii) piecewise smooth if we can find a partition $\{a = t_0, t_1, t_2 \ldots\ldots, t_n = b\}$ of $[a,b]$ such that x, y are diferentiable on (t_{i-1}, t_i) $\forall\ i \in 1, 2, \ldots\ldots n$ and also continuous.

(iv) simple if $\gamma(s) \neq \gamma(t)$ for $s \neq t$ in $[a,b]$

(v) closed if $\gamma(a) = \gamma(b)$.

(vi) simple closed if $\gamma(a) = \gamma(b)$ and $\gamma(s) \neq \gamma(t)$ for $s \neq t$ in (a,b)

Def

A subset Ω of $\mathbb{C}$ is said to be path connected if any two points of Ω can be joined by a continuous path in Ω, i.e. for any $z, w \in \Omega$ $\exists$ a constant function $\gamma : [a,b] \to \mathbb{C}$ such that

(i) $\gamma(a) = z$ and $\gamma(b) = w$ and

(ii) $\gamma(t) \in \Omega$ $\forall\ t \in [a,b]$

eg.

1) Any convex subset of $\mathbb{C}$ is connected

A subset Ω of $\mathbb{C}$ is convex if $z, w \in \Omega$ $\Rightarrow tz + (1-t)w \in \Omega$ for $0 \leq t \leq 1$

In particular ball, a rectangle are connected

2. A star shaped region is connected

($\Omega \in \mathbb{C}$ is star shaped If $\exists$ atleast one point $z_0 \in \Omega$ such that $z \in \Omega \Rightarrow$ $tz_0 + (1-t)z \in \Omega$ $\forall\ t \in [0,1]$

Path Integrals :

let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path. in $\mathbb{C}$

let $\Omega \subseteq \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a continuous path in Ω i.e.

$$\gamma(t) = x(t) + iy(t) \in \Omega \quad \forall\ t \in [a,b]$$

define $$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt}$$

Example

$$f(z) = \frac{1}{z} \quad \text{for } \mathbb{C} \setminus \{0\}$$

$$\gamma = \cos t + i\sin t \qquad 0 \le t \le 2\pi$$

$$\gamma' = (-\sin t + i\cos t)$$

$$\int f(z)\,dz = \int_0^{2\pi} \frac{-\sin t + i\cos t}{\cos t + i\sin t}\,dt$$

$$= \int_0^{2\pi} \frac{(-\sin t + i\cos t)(\cos t - i\sin t)}{1}\,dt$$

$$= \int_0^{2\pi} -\cos t\sin t + \cos t\sin t + i\cos^2 t + i\sin^2 t$$

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$$= i\int_0^{2\pi} dt$$

$$= 2\pi i.$$

Note : The definition of $\int_\gamma f(z)\,dz$ easily extends to the case when γ is a piecewise smooth path rather than a smooth path.