MA 205 Complex Analysis. ①

Date 14/05/09. DAILY NOTES

Lecture 7

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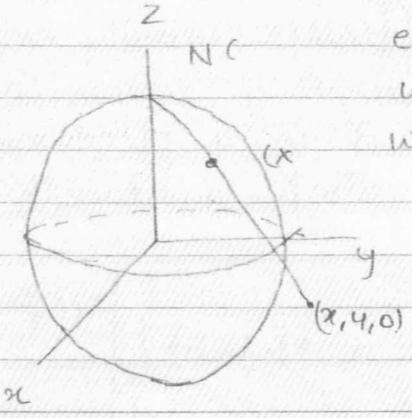
Extended complex plane or the Riemann Sphere



$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the extended complex plane, where ∞ is a symbol ~~that~~ with a property that

$$z + \infty = \infty \quad \forall\, z \in \mathbb{C}$$
$$z \cdot \infty = \infty \quad \forall\, z \in \mathbb{C} \setminus \{0\}$$
$$\frac{1}{0} = \infty$$
$$\frac{1}{\infty} = 0$$

$\hat{\mathbb{C}}$ can be regarded as a sphere in 3-space $\mathbb{R}^3$ as follows.

Consider a unit sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and $N = (0,0,1)$ be the north pole

Consider the "stereographic projection" of points on S^2 from the north pole onto the x-y plane

Geometrically, we see that every point in $\mathbb{C} = \mathbb{R}^2 = \{(x,y,0) : x, y \in \mathbb{R}\}$.

is obtained as a projection of unique point in

$$S^2 \setminus \{N\}$$

More explicitly points on the line joining

$$N = (0,0,1) \text{ and}$$

$$P = (X, Y, Z) \in S^2$$

look like

$$tP + (1-t)N = (tX, tY, tZ + (1-t))$$

We want to find points of the x-y plane on this line, consider

$$\begin{cases} x = tX \\ y = tY \\ 0 = tZ + (1-t) \end{cases}$$

$$t(Z-1) = -1 \Rightarrow t = \frac{1}{1-Z} \Rightarrow Z = \frac{t-1}{t}$$

Also

$$X^2 + Y^2 + Z^2 = 1$$

i.e.

$$\left(\frac{x}{t}\right)^2 + \left(\frac{y}{t}\right)^2 + \left(\frac{t-1}{t}\right)^2 = 1$$

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Consider $Q = (x, y, 0)$

comes to $\dot{z} = (x + iy) \in \mathbb{C}$

and the line joining N and Q.

Points on this are of the form

$$(1-t)Q + tN = ((1-t)x, (1-t)y, t)$$

We look for

$$(X, Y, Z) = ((1-t)x, (1-t)y, t)$$

such that

$$X^2 + Y^2 + Z^2 = 1$$

i.e.

$$(1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$$

$$(1-t)^2 (x^2 + y^2) = 1 - t^2$$

$$x^2 + y^2 = \frac{(1+t)}{(1-t)}$$

$$|z|^2 = \frac{1+t}{1-t}$$

$$(1-t)|z|^2 = 1 + t$$

$$t(1 + |z|^2) = |z|^2 - 1$$

$$t = \frac{|z|^2 - 1}{1 + |z|^2} = \frac{|z|^2 - 1}{|z|^2 + 1}$$

and $1 - t = \dfrac{2}{|\dot{z}|^2 + 1}$

$$P = \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

In this way we obtain a map

$$\mathbb{C} \longrightarrow S^2 \setminus \{N\}$$

given by

$$z = x+iy \longrightarrow \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

and thus identifying ∞ with $N = (0,0,1)$ we see that

$$\hat{\mathbb{C}} \longleftrightarrow S^2$$

* exercise : Show that the map

$$\mathbb{C} \longrightarrow S^2 \setminus \{N\}$$

given by

$$z = x+iy \longmapsto \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

is one-one and onto.

The unit sphere in $\mathbb{R}^3$ viewed as the external complex plane and is sometimes called the Riemann sphere.

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Fractional linear transformations.

These are maps of $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$z \longmapsto \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers with $ad-bc \neq 0$.

If we call

$$w = f(z) = \frac{az+b}{cz+d} \iff \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

then we see that f is one-one

i.e. $f(z) = f(z') \Rightarrow z = z'$

and f is onto

$$w \in \hat{\mathbb{C}} \Rightarrow \text{we can solve } \frac{az+b}{cz+d} = w$$

for z —(*)

eg. $f(z) = \frac{1}{z}$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$f(z) = az+b$

$(a, b \in \mathbb{C},\ a \neq 0)$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$

In particular the translations

$f(z) = z+b$.

and dialations

$f(z) = az \qquad a \in \mathbb{C},\ a \neq 0)$

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proof of (*)

$$w = \frac{az+b}{cz+d}$$

then $$z = \frac{dw-b}{-cw+a} \longrightarrow \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix}$$

Note.

$$\left[\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right]$$

This shows that the inverse of

$$f^{-1}(w) = \frac{dw-b}{-cw+a}.$$

is also a fractional linear transformation

Note also that if f and g are fractional linear transformation, say

$$f(z) = \frac{az+b}{bz+c}, \quad g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$

then $$(g \circ f)(z) = g(f(z))$$

$$= \frac{\alpha\left(\frac{az+b}{cz+d}\right) + \beta}{\gamma\left(\frac{az+b}{cz+d}\right) + \delta}$$

is also a fractional linear transformation exercise : If z_1, z_2, z_3 are any three distinct points in the extended complex plane $\hat{\mathbb{C}}$ then there is a unique fractional linear transf. f such that

$$f(z_1) = 0$$
$$f(z_2) = 1$$
$$\text{and } f(z_3) = \infty$$

As a consequence, we obtain

Theorem : If (z_1, z_2, z_3) and (w_1, w_2, w_3) are any triples of disctinct points in $\hat{\mathbb{C}}$, then $\exists$ a unique fractional linear transformation which maps z_i to w_i $\forall i = 1,2,3$.

proof : By exercise $\exists$ fractional linear transformation f & g such that f maps z_1, z_2, z_3 to $0, 1, \infty$ resp. and g maps w_1, w_2, w_3 to $0, 1, \infty$ resp. Now $g^{-1} f$ is a fractional linear transformation that maps z_1, z_2, z_3 to $w_1\ w_2\ w_3$ respectively.

eq. $z_1 = i$, $z_2 = -i$, $z_3 = 1+i$.
find $f(z_1) = 0$, $f(z_2) = 1$, $f(z_3) = \infty$

consider $$f(z) = \frac{az+b}{cz+d}.$$

want
$$ai + b = 0 \Rightarrow b = -ai$$
$$\frac{-ai+b}{-ci+d} = 1 \Rightarrow -ai+b = -ci+d$$
$$c(1+i)+d = 0 \Rightarrow d = -c(1+i)$$

solve the three equations, verify $ad-bc \neq 0$, get f.

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$-ai - ai = -ci - c(1+i)$

$-2ai = -2ci - c$

take $c=1$

$a = \frac{2i+1}{2i} \Rightarrow a = 1 + \frac{1}{2i} = 1 - \frac{i}{2}$

$b = -i\left(1 - \frac{i}{2}\right)$

$= -i + \frac{(-1)}{2} = -i - \frac{1}{2}$

$d = -1 - i$

we see that $ad - bc = 0$.

$f(z) = \frac{(1 - i/2)z + (-i - 1/2)}{i + (-i-1)}$

$= \frac{(i + 1/2)z + i - 1/2}{z - 1 - i}$

$= \frac{(1 - \frac{i}{2})z - i - 1/2}{z - i - 1}$