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lecture 8 Date - 15/05/09

Notes by PUSHPENDRA

Complex Integration.

path in $\mathbb{C}$: $\gamma : [a, b] \to \mathbb{C}$

path is closed $\to$ $\gamma(a) = \gamma(b)$

path is smooth $\to$ $\gamma(\cdot)$ is differentiable in $[a, b]$

path is piecewise smooth $\longrightarrow$ $\exists$ a partition $\{a = t_0, t_1 \dots t_n = b\}$ of $[a, b]$ such that γ is differentiable on (t_{i-1}, t_i) $\forall\ i = 0, \dots n$. and γ' is continuous on $(t_{i-1}, t_i)\ \forall\ i = 0, \dots n$

path Integral :

$f(z)$ is continuous in $\Omega \subseteq \mathbb{C}$, γ is a path in Ω

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

where $\gamma : [a, b] \to \Omega$ is a piecewise smooth path. in Ω

also

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(\gamma(t))\,\gamma'(t)\,dt.$$

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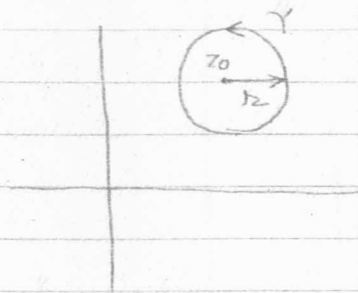
eq. representation of a circle $|z-z_0| = r$

can be $z = z_0 + re^{i\theta}$ $\quad \theta \in [0, 2\pi]$



$$\int_\gamma \frac{1}{z-z_0}\, dz$$

$$= \int_0^{2\pi} \frac{1}{re^{it}} \cdot ire^{it}\, dt$$

$$= 2\pi i$$

- If instead the $\gamma(t)$ is derected in opp. direction then

$$\gamma(t) = z = z_0 + re^{-it} \qquad t \in [0, 2\pi]$$

$$\int \frac{1}{z-z_0}\, dt = -2\pi i$$

- If circle is traced twiced

$$\gamma(t) = z = z_0 + e^{it} \qquad t \in [0, 4\pi]$$

$$\int_\gamma \frac{1}{z-z_0}\, dt = 4\pi i$$

NOTE : This

$\frac{1}{2\pi i} \int_\gamma \frac{1}{z-z_0}\, dz$ gives the "winding number" of γ around z_0 which is poisitive if γ is traced in anticlockwise direction & negetive of it is traced in clock wise direction.

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For $\gamma(t) = z_0 + re^{it}$ $\quad t \in [0, 2\pi]$

$$\int_\gamma (z-z_0)^m \, dz = \int_0^{2\pi} r^m e^{imt} (rie^{it} \, dt)$$

$$= \cancel{\tfrac{i}{r}m} \; ir^{m+1} \int_0^{2\pi} e^{i(m+1)t} \, dt$$

$$= 0.$$

CAUCHY'S THEOREM :

Let $f : \Omega \to \mathbb{C}$ be analytic on an open connected set Ω. If γ is a ~~piecewise~~ smooth and closed curve in Ω such that γ and its interior is in Ω i.e. $\gamma = \partial D$ for $D \subset \Omega$

$$\int_\gamma f(z) \, dz = 0$$

Sketch of proof :

let $f = u + iv$

$$\int f(z) \, dz = \int (u+iv)(dx+idy)$$

...

$$= \int (u\,dx - v\,dy) + i \int (u\,dy + v\,dx)$$

Using Green's theorem.

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

then $\int f(z)\,dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$$+ i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$\because$ f is analytic in D and hence satisfies Cauchy Riemann equation

$$= \iint_D (0)\,dx\,dy + i \iint (0)\,dx\,dy$$

$$= 0$$

Applications of Cauchy's theorem :

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1) Cauchy's Integral Formula :

Suppose $f : \Omega \to \mathbb{C}$ is analytic and $z \in \Omega$. If γ is a piecewise smooth curve such that γ and its interior is in Ω and z is ints interior, i.e. $\gamma = \partial D$ for some $D \subseteq \Omega$ with z an interior point of D.

P.T.O.

NOTE : Cauchy's theorem also holds if ∂D is a union of finitely many piecewise smooth closed curves.

Applications of Cauchy's theorem

Cauchy's integral formula:

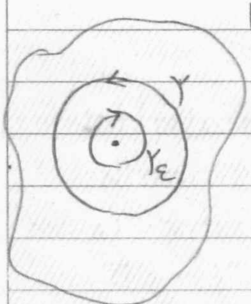
(let us prove the following first).

Suppose $f : \Omega \to \mathbb{C}$ is an analytic and $z \in \Omega$. Let γ be a circle centered at z of some radius r, where we assume that $\{w \in \mathbb{C} : |w - z| \leq r\} \subseteq \Omega$.

Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} \, dw.$$

proof: Consider a small circle γ_ε of radius $\varepsilon < r$ centered at z. let $D = \{w \in \mathbb{C} : \varepsilon \ |w - z| < r\}$



Now the function

$$g(w) = \frac{f(w)}{w - z}$$

is analytic on D as well as its boundary ∂D which consists of γ and γ_ε traversed in opposite directions. By cauchy's theorem

$$\int_{\partial D} g(w) \, dw = 0.$$

$$\Rightarrow \int_\gamma g(w) \, dw = \int_{\gamma_\varepsilon} g(w) \, dw.$$

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$$\frac{1}{2\pi i}\int_{\gamma_\varepsilon} g(\omega)\,d\omega - f(z)$$

$$= \frac{1}{2\pi i}\int \frac{f(\omega)}{\omega - z}\,d\omega - \frac{f(z)}{2\pi i}\int_{\gamma_\varepsilon} \frac{1}{\omega - z}\,dz$$

$$= \frac{1}{2\pi i}\int \frac{f(\omega) - f(z)}{\omega - z}\,d\omega.$$

$$= \frac{1}{2\pi \cancel{i}}\int \frac{f(z+\varepsilon e^{it}) - f(z)}{\cancel{\varepsilon} e^{it}} \cdot i\cancel{\varepsilon} e^{it}\,dt.$$

$$= \frac{1}{2\pi}\int_{\gamma_\varepsilon} f(z+\varepsilon e^{it}) - f(z)$$

as $\varepsilon \to 0$ the integral tends to 0.

$= 0.$

i.e $\int_{\gamma} g(\omega)\,d\omega - f(z) = 0.$

or $f(z) = \int_{\gamma} g(\omega)\,d\omega$

$$= \int \frac{f(\omega)}{\omega - z}\,d\omega.$$

Remark : A similar proof shows that the Cauchy's Integral formula is valid if γ is a finite union of piecewise smooth closed curves, provided f is analytic on

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an open subset containing γ and its "inside"

2) Formula for Derivatives :-

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$= \lim_{h\to 0} \frac{1}{2\pi i}\int \frac{f(w)}{h}\left[\frac{1}{(w-z-h)} - \frac{1}{(w-z)}\right] dw$$

$$= \lim_{h\to 0} \frac{1}{2\pi i}\int \frac{f(w)}{h}\left[\frac{\cancel{w}-\cancel{z}-\cancel{w}+\cancel{z}+h}{(w-z)(w-z-h)}\right] dw$$

$$= \frac{1}{2\pi i}\int f(w) \lim_{h\to 0}\left[\frac{1}{(w-z)(w-z-h)}\right] dw.$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(w)\, dw}{(w-z)^2}$$

Similarly

$$f'' = \frac{2}{2\pi i}\int_\gamma \frac{f(w)\, dw}{(w-z)^3}$$

By induction we can conclude that

$f^{(m)}(z)$ exists and is equal to.

$$f^{m}(z) = \frac{m!}{2\pi i} \int \frac{f(w)\, dw}{(w-z)^{m+1}}$$