MA 205 Complex Analysis

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lecture 9. Dated - 20/05/09.

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We prooved Cauchy's theorem using Green's theorem. Strictly speaking, Green's theorem requires that u_x, u_y, v_y, v_x exist and are continuous in Ω and also that ∂D is positively oriented (i.e. ∂D is so oriented so that D is to left as we traverse along ∂D. Thus proof is valid if $f = u+iv$ is analytic and f' is continuous. However, Goursat prooved that Cauchy's theorem is valid even when the continuity of f' is not assumed.

In fact we have the following:

Def:

let Ω be a connected subset of $\mathbb{C}$. Then Ω is said to be simply connected if every closed path in Ω can be "shrunk" to a point. more precisely if $\gamma : [a,b] \longrightarrow \Omega$ is a constant path such that $\gamma(a) = \gamma(b)$; then there is a $z_0 \in \Omega$ and a continuous function

$$H : [a,b] \times [0,1] \longrightarrow \Omega$$

such that

i) $H(s,0) = \gamma(s) \quad \forall s \in [a,b]$

ii) $H(s,1) = z_0 \quad \forall s \in [a,b]$

iii) For each $t \in [0,1]$, the path given by $s \mapsto H(s,t)$ is closed $[a,b] \to \Omega$

i.e. $H(a,t) = H(b,t)$

such a map is called a "Homotopy" between γ and the path γ_0 given by $\gamma_0(s) = z_0$ $\forall\ s \in [a,b]$

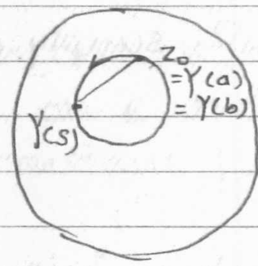
eg.

A disc or a rectangle or more generally any convex subset of $\mathbb{C}$ is simply connected

<u>proof</u> : let $\Omega \subseteq \mathbb{C}$ be convex, then Ω is path connected and hence connected

Further if

$$\gamma : [a,b] \longrightarrow \Omega$$



is a closed path in Ω

Consider

$$H(s,t) = (1-t)\,\gamma(s) + t z_0$$

for $t \in [0,1]$
& $s \in [a,b]$

where $z_0 = \gamma(a) = \gamma(b)$. Clearly H is a homotopy. and let γ and H are const path from z_0

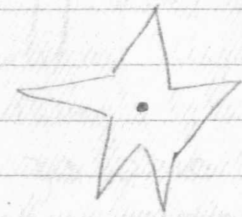
<u>General ~~forma~~ form of Cauchy's Theorem.</u>

If $\Omega \subseteq \mathbb{C}$ is open connected and simply connected then

$$\int_\gamma f(z)\,dz = 0$$

for every piecewise ~~simp~~ smooth simple closed ~~co~~ curve γ in Ω and analytic function $f : \Omega \to \mathbb{C}$

exercise : show that every star ~~connected~~ shaped region in $\mathbb{C}$ is simply connected.



Remark : For $\Omega \subseteq \mathbb{C}$

Ω path connected $\Rightarrow$ Ω connected

In case Ω is open,

Ω connected $\Rightarrow$ Ω path connected

Further if Ω is open connected then

Ω simply connected $\Leftrightarrow$ $\mathbb{C} \setminus \Omega = \{z \in \mathbb{C} : z \notin \Omega\}$ is connected.

Cauchy Integral formula

let Ω be open subset of $\mathbb{C}$ and $z \in \Omega$ and If γ denotes a circle of radius $r > 0$ centered at z such that γ and its inside are subset of Ω

i.e. $\{\omega \in \mathbb{C} : |\omega - z| \leq r\} \subseteq \Omega$

Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\omega)}{\omega - z} \, d\omega$$

Further

$$f'(z) ; f''(z), \ldots$$

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\gamma} \frac{f(\omega)}{(\omega - z)^{m+1}} \, d\omega .$$

where γ is oriented in anticlockwise direct^n

Note : The result is valid more generally for a piecewise smooth curve or simple closed curve.

Applications of Cauchy Theorem.

Liouville's theorem :

<u>Def</u> : A function $f : \mathbb{C} \to \mathbb{C}$ which is analytic on $\mathbb{C}$ is called entire function.

An entire bounded function is constant i.e. if $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ and bounded on $\mathbb{C}$ then f is a constant function

<u>proof</u> : let $z_0 \in \mathbb{C}$ or $z_0 = 0$

let γ be circle of radius R centered at z_0 say

$$\gamma(t) = z_0 + Re^{it} \qquad \text{for } t \in [0, 2\pi]$$

Since f is bounded on $\mathbb{C}$, $\exists\ M > 0$ such that

$$|f(z)| \leq M \qquad \forall\ z \in \mathbb{C}$$

By cauchy Integral formula

$$f'(z_0) = \frac{1}{2\pi i} \int \frac{f(w)}{(w - z_0)^2}\, dw.$$

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int \frac{f(w)}{(w - z_0)^2}\, dw \right|$$

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$$= \frac{1}{2\pi} \int \frac{M}{|(\omega - z_0)^2|} \, d\omega$$

$$= \frac{M}{2\pi R^2} \int_{\gamma} d\omega$$

$$= \frac{\cancel{2\pi} R . M}{\cancel{2\pi} R^2}$$

$$= \frac{M}{R}$$

$$\therefore \quad |f'(z_0)| \le \frac{M}{R}$$

for $R > 0$. But $\frac{M}{R} \to 0$ as $R \to \infty$

Hence we must have

$$f'(z_0) = 0$$

since z_0 is arbitrary $\forall \; z \in \mathbb{C}$
hence f is constant.

<u>Corollary</u> : (Fundamental theorem of Algebra

Every non constant polynomial with coefficients in $\mathbb{C}$ has atleast one root in $\mathbb{C}$

<u>proof</u>: let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a non-constant polynomial with

co-efficient in $\mathbb{C}$ i.e. $n \geq 1$ and $a_n \neq 0$.

Now p defines function from $\mathbb{C}$ to $\mathbb{C}$ which is clearly analytic on $\mathbb{C}$

Suppose

$p(z)$ has no roots in $\mathbb{C}$.

Then $\frac{1}{p(z)}$ is analytic on $\mathbb{C}$ for $p(z) \neq 0$

X Now

$$|p(z)| \leq |a_n||z^n| + |a_{n-1}||z^{n-1}| + \cdots + |a_1||z|$$

$$\leq |z^n| \left(|a^n| + \frac{|a_{n-1}|}{|z|} + \cdots \frac{|a_1|}{|z^{n-1}|} + \frac{|a_0|}{|z^n|} \right)$$

Note that $p(z)$ is a non-constant polynomial

$$|p(z)| \to \infty \quad \text{as} \quad |z| \to \infty$$

$$\frac{1}{|p(z)|} \to 0 \quad \text{as} \quad |z| \to \infty$$

In particular $\exists R > 0$ such that

$$\frac{1}{|p(z)|} < 1 \qquad \forall z \in \mathbb{C} \text{ with } |z| > R$$

on the other hand, the disk

$D = \{ z \in \mathbb{C} : |z| \leq R \}$ is a closed and a bounded ~~and~~ set and $\frac{1}{p(z)}$ is continuous

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Hence it is bounded on D

Thus $\frac{1}{p(z)}$ is bounded on $\mathbb{C}$

So by Liouville's theorem

$$\frac{1}{p(z)} \text{ is a constant function.}$$

Hence $p(z)$ is a constant function which is a contradiction.

So $p(z)$ must have a root in $\mathbb{C}$

Corollary of Corollary: If $p(z)$ is a polynomial of degree n with co-efficients in $\mathbb{C}$ then $\exists$ $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_r \in \mathbb{C}$ and positive Integers $e_1, e_2\, e_r$ such that

$$p(z) = a_n (z-\alpha_1)^{e_1}(z-\alpha_2)^{e_2} \ldots (z-\alpha_r)^{e_r}$$

for $a_n \in \mathbb{C}$ and $a_n \neq 0$.

with $e_1 + e_2 + e_3 \ldots e_r = n$.

In other words $p(z)$ has exactly n roots when counted with multiplities

proof Induct on n and use FTA.