Indian Institute of Technology Bombay

MA 403: Real Analysis

End-Semester Examination

Date & Time: 22.11.2008, 9.00 AM – 12.00 Noon Max Marks : 50

Note: *Begin your answer to each question on a new page of the answerbook. Answers to all the subparts of a question should appear together. You are not required to solve the Bonus Problems. But you are welcome to attempt them if time permits. You must justify all your answers. Unless otherwise specified, use the usual metric on* $\mathbb{R}^k$ *given by* $d(\underline{x}, \underline{y}) = |\underline{x} - \underline{y}|$ *for* $\underline{x}, \underline{y} \in \mathbb{R}^k$.

Q. 1 (a) State precisely both parts of the Fundamental Theorem of Calculus.

(b) Let $f : [a, b] :\to \mathbb{R}$ be a differentiable function such that f' is Riemann integrable on $[a, b]$. Show that $\displaystyle\int_a^b [2xf(x) + x^2 f'(x)]dx = b^2 f(b) - a^2 f(a)$.

(c) For $x \in [0, 1]$, define $G(x) := \displaystyle\int_x^1 \frac{dt}{\sqrt{1 + t + t^2}}$. Show that G is differentiable on $[0, 1]$ and find G'.

(d) Find $\displaystyle\lim_{n\to\infty} \left(\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \cdots + \frac{n}{n^2 + n^2}\right)$. [10 marks]

Q. 2 Let X be a set. Define the discrete metric on X. With respect to this metric, find all bounded subsets, all closed subsets, all compact subsets and all connected subsets of X. [5 marks]

Q. 3 (a) State precisely the Heine-Borel Theorem. State with reasons whether the following subsets of $\mathbb{R}^2$ are compact: (i) $\{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 2\}$, (ii) $\{(1/m, 1/n) : m, n \in \mathbb{N}\}$.

(b) Let X denote the set of all rational numbers with the metric defined by $d(p, q) := |p - q|$ for $p, q \in X$, and let $E := \{p \in X : p^2 < 2\}$. Is E closed in X? Is E compact? [5 marks]

Q. 4 Let X be a metric space and $p, q \in X$. What is meant by a path from p to q in X? If E is a path-connected subset of X, show that E is connected. Is the subset $\mathbb{R}^2 \setminus \{(0, 0)\}$ of $\mathbb{R}^2$ connected? [5 marks]

Q. 5 Let X and Y be metric spaces and $f, g : X \to Y$ be continuous. Prove the following: (i) If $D \subset X$ is dense and $f_{|D} = g_{|D}$, then $f = g$. (ii) If $K \subset X$ is compact, then $f(K)$ is a compact subset of Y. [5 marks]

Q. 6 (a) For $n \in \mathbb{N}$ and $x \in [0, 1]$, let $f_n(x) := x/(nx + 1)$. Show that the sequence (f_n) converges uniformly on $[0, 1]$. Find $\lim_{n\to\infty} \int_0^1 f_n(x)dx$. Does the sequence (f_n') converge uniformly on $[0, 1]$?

(b) Let E be a set and $f_k : E \to \mathbb{R}$ for $k \in \mathbb{N}$. If the series $\sum_{k=1}^{\infty} f_k$ converges uniformly on E, then show that the sequence (f_k) converges uniformly to 0 on E. If $E := \mathbb{R}$, and $f_k(x) := \frac{x^k}{k!}$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$, then show that the sequence (f_k) does not converge uniformly on $\mathbb{R}$. Do the series $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ and $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$ converge uniformly on $\mathbb{R}$? [10 marks]

Q. 7 (a) State precisely the Polynomial Approximation Theorem of Weierstrass for functions on $[0, 1]$, and the Trigonometric Polynomial Approximation Theorem of Fejér for functions on $[-\pi, \pi]$, explaining the notations used.

(b) For $t \in [-1, 1]$, let $g(t) := |t|$. Is there a sequence of polynomial functions defined on $[-1, 1]$ which converges to g uniformly on $[-1, 1]$?

(c) Let $f : [-\pi, \pi] \to \mathbb{R}$ be defined by $f(x) := \pi^2 - x^2$. Show that there is a sequence of trigonometric polynomial functions defined on $[-\pi, \pi]$ which converges to f uniformly on $[-\pi, \pi]$ and which involves none of the functions given by $\sin kx$, where $k \in \mathbb{N}$, and $x \in [-\pi, \pi]$.

(d) Suppose f is a continuous function on $[-\pi, \pi]$ such that $f(\pi) = f(-\pi)$ and all Fourier coefficients of f are equal to 0. Show that $f(x) = 0$ for all $x \in [-\pi, \pi]$. [10 marks]

Bonus Problems

B 1 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. If there is $n_0 \in \mathbb{N}$ such that $\int_0^1 x^n f(x)dx = 0$ for all $n \geq n_0$, then show that $f(x) = 0$ for all $x \in [0, 1]$.

B 2 Let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function. Show that there is a sequence (P_n) of polynomial functions defined on $[0, 1]$ such that (P_n) converges uniformly to f on $[0, 1]$ and (P_n') converges uniformly to f' on $[0, 1]$.

— *Paper Ends* —