MA 403 Real Analysis I

Exercise Set 4

(1) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all x, $y \in \mathbb{R}$. If f is continuous at 0, then show that (i) f is continuous at every $c \in \mathbb{R}$ and (ii) $f(sx) = sf(x)$ for all s, $x \in \mathbb{R}$. Deduce that there exists $r \in \mathbb{R}$ such that $f(x) = rx$ for all $x \in \mathbb{R}$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x)f(y)$ for all x, $y \in \mathbb{R}$. If f is continuous at 0, then show that f is continuous at every $c \in \mathbb{R}$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(xy) = f(x)f(y)$ for all x, $y \in \mathbb{R}$. If f is continuous at 1, then show that f is continuous at every $c \in \mathbb{R}$, except possibly at $c = 0$. Give an example of such a function which is continuous at 1 as well as at 0. Also, give an example of such a function which is continuous at 1, but not at 0.

(4) Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x)+f(y)$ for all x, $y \in (0, \infty)$. If f is continuous at 1, then show that f is continuous at every $c \in (0, \infty)$.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 1 - x & \text{if } x \text{ is irrational.} \end{cases}$

Determine the points $c \in \mathbb{R}$ at which f is continuous. Justify your answer.

(6) Let $f : [0, \infty) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/q & \text{if } x = p/q, \text{where } p,\ q \in \mathbb{N} \text{ and have no common factor,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f is discontinuous at each rational in $[0, \infty)$ and it is continuous at each irrational in $[0, \infty)$. [Note: This function is known as Thomae's function.]

(7) Let $D \subseteq \mathbb{R}$, $c \in D$ and $f : D \to \mathbb{R}$ be such that f is continuous at c. Show that $|f| : D \to \mathbb{R}$ is continuous at c. Is the converse true?

(8) Let $D \subseteq \mathbb{R}$, $c \in D$ and f, $g : D \to \mathbb{R}$ be such that f and g are continuous at c. Show that the functions $\max(f, g)$, $\min(f, g) : [a, b] \to \mathbb{R}$ given by

$$\max(f, g)(x) = \max\{f(x), g(x)\} \quad \text{and} \quad \min(f, g)(x) = \min\{f(x), g(x)\}$$

for $x \in [a, b]$, are continuous at c.

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ 3 - x & \text{if } 1 \le x \le 2. \end{cases}$

Show that f assumes every value between 0 and 2 exactly once on [0,2], but f is not continuous on [0,2].

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 3x/2 & \text{if } 0 \le x < 1/2, \\ (3x - 1)/2 & \text{if } 1/2 \le x \le 1. \end{cases}$

Show that $f([0, 1]) = [0, 1]$. Is f continuous on [0,1]? Does f have the IVP on [0,1]?

(11) Show that the cubic $x^3 - 6x + 3$ has exactly three real roots.

(12) Let $f : [a, b] \to \mathbb{R}$ be continuous. Given $c_1, \ldots, c_n$ in $[a, b]$, show that there is $c \in [a, b]$ such that $f(c) = \dfrac{f(c_1) + \cdots + f(c_n)}{n}$. (Hint: IVP)

(13) If a function f satisfies one of the following conditions, then can it be continuous? Why?

(i) $f : [1, 10] \to \mathbb{R},\ f(1) = 0,\ f(10) = 11,$ range of $f \subseteq [-1, 0] \cup [1, 11]$.

(ii) $f : [0, 1] \to \mathbb{R}$ and range of $f = (-1, 1)$.

(iii) $f : [-1, 1] \to \mathbb{R}$ and range of $f = [0, \infty)$.

(14) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x/(1+x) & \text{if } x \geq 0, \\ x/(1-x) & \text{if } x < 0. \end{cases}$
Show that f is continuous and bounded on $\mathbb{R}$, and moreover,

$$\inf\{f(x) : x \in \mathbb{R}\} = -1 \quad \text{and} \quad \sup\{f(x) : x \in \mathbb{R}\} = 1,$$

but there are no r, s in $\mathbb{R}$ such that $f(r) = -1$ and $f(s) = 1$.

(15) Let D and E be subsets of $\mathbb{R}$ such that D is closed and bounded. If $f : D \to E$ is bijective and continuous, then show that $f^{-1} : E \to D$ is continuous.

(16) Analyze the following functions for uniform continuity.
(i) $f(x) = x,\ x \in \mathbb{R}$, (ii) $f(x) = 1/x,\ x \in (0, 1]$,
(iii) $f(x) = x^2,\ x \in (0, 1)$, (iv) $f(x) = \sqrt{1 - x^2},\ x \in [-1, 1]$.

(17) Let D and E be subsets of $\mathbb{R}$. Also, let $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions such that the range of f is contained in E. If f is uniformly continuous on D and g is uniformly continuous on E, then show that $g \circ f$ is uniformly continuous on D.

(18) Let $f : [a, b] \to \mathbb{R}$ be a continuous function satisfying $f(a) = f(b)$. Show that there are c, $d \in [a, b]$ such that $d - c = (b - a)/2$ and $f(c) = f(d)$. Deduce that for every $\epsilon > 0$, there are x, $y \in [a, b]$ such that $0 < y - x < \epsilon$ and $f(x) = f(y)$.

(19) Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Prove the following.
(i) If D is bounded and f is uniformly continuous on D, then f is bounded on D. Is this true if f is merely continuous on D?
(ii) Let (x_n) be a Cauchy sequence in D. If f is uniformly continuous on D, then $(f(x_n))$ is also a Cauchy sequence. Is this true if f is merely continuous on D?

(20) Let $r \in \mathbb{Q}$ and $r \geq 0$. If $f : [0, \infty) \to \mathbb{R}$ is defined by $f(x) = x^r$, show that f is uniformly continuous if and only if $r \leq 1$.

(21) Let f, $g : D \to \mathbb{R}$ be uniformly continuous on D. Are the functions $f + g$, fg, $1/f$ (provided $f(x) \neq 0$ for all $x \in D$) uniformly continuous on D? What if D is a bounded subset of $\mathbb{R}$? What if D is a closed subset of $\mathbb{R}$? What if $D = [a, b]$? Justify your answers.

(22) Show that $\lim_{x \to c} f(x)$ does not exist if $f : \mathbb{R} \to \mathbb{R}$ is given by
(i) $f(x) = [x] - x,\ c = 1$, (ii) $f(x) = \dfrac{|x + 1|}{x + 1},\ c = -1$.

(23) Consider f, $g : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Under which of the following conditions does $\lim_{x \to c} f(x)g(x)$ exist? Justify.
(i) $\lim_{x \to c} f(x)$ exists.
(ii) $\lim_{x \to c} f(x)$ exists and g is bounded on $\{x \in \mathbb{R} : 0 < |x - c| < \delta\}$ for some $\delta > 0$.
(iii) $\lim_{x \to c} f(x) = 0$ and g is bounded on $\{x \in \mathbb{R} : 0 < |x - c| < \delta\}$ for some $\delta > 0$.
(iv) $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

(24) Let I be an interval and $f : I \to \mathbb{R}$ be a function that is convex on I, or concave on I. Show that f is continuous at every point of I except possibly the endpoints of I. [Hint: Let c be an interior point of I and $c_1, c_2 \in I$ be such that $c_1 < c < c_2$. If f is convex on I, then

$$f(c_1) + \frac{f(c) - f(c_1)}{c - c_1}(x - c_1) \leq f(x) \leq f(c) + \frac{f(c_2) - f(c)}{c_2 - c}(x - c) \quad \forall x \in (c, c_2), \text{ and}$$

$$f(c) + \frac{f(c) - f(c_2)}{c_2 - c}(c - x) \leq f(x) \leq f(c_1) + \frac{f(c) - f(c_1)}{c - c_1}(x - c_1) \qquad \forall x \in (c, c_2).]$$