Indian Institute of Technology Bombay
MA 414 Algebra I

Spring 2008 SRG

Exercise Set 2

1. Let p and q be primes. Show that every group of order pq is solvable. Is it true that every group of order pq is nilpotent? Justify your answer.

2. If G_1 and G_2 are solvable groups, then show that $G_1 \times G_2$ is also a solvable group.

3. If H and K are solvable subgroups of a group G such that $H \trianglelefteq G$, then show that HK is a solvable subgroup of G.

4. Prove the Fundamental Theorem of Arithmetic using the Jordan-Hölder Theorem.

5. Determine a composition series for the following groups: D_4, A_4, $S_3 \times \mathbb{Z}_2$, D_6.

6. Show that the infinite cyclic group $\mathbb{Z}$ has no composition series. More generally, prove that an abelian group has a composition series if and only if it is finite.

7. If G is a nilpotent group and $H \leq G$, then show that H is nilpotent. Further, if $H \trianglelefteq G$, then show that G/H is nilpotent.

8. If H is a subgroup of a group G such that $H \leq Z(G)$ and G/H is nilpotent, then show that G is nilpotent. Is it true that if H is a normal subgroup of G such that H and G/H is nilpotent, then G is necessarily nilpotent? Justify your answer.

9. Let H and K be subgroups of a group G. Let $[H, K]$ denote the subgroup of G generated by all the elements of the form $[h, k] := hkh^{-1}k^{-1}$ where h varies over H and k varies over K. Show that if $H \trianglelefteq G$, then $[H, K]$ is a subgroup of H, and if also $K \trianglelefteq G$, then $[H, K]$ is a normal subgroup of K as well as of H.

10. Let G be a finite group. Show that G is nilpotent if and only if any two elements of G of coprime orders commute with each other.

11. Let D_n denote the dihedral group of order $2n$. Show that D_n is nilpotent if and only if n is a power of 2.

12. Let G be a nilpotent group of class n. If $\{1\} = N_0 \leq N_1 \leq \cdots \leq N_n = G$ is any central series of G of length n, then show that $G^{n-i} \leq N_i \leq Z_i(G)$ for $i = 0, 1, \ldots, n$.

13. Let G be a group and $H \leq G$. Then H is said to be a characteristic subgroup of G if $\sigma(H) \subseteq H$ for every automorphism σ of G. Show that if H is a characteristic subgroup of G, then H is a normal subgroup of G and every characteristic subgroup of H is a characteristic subgroup of G. Give an example to show that a normal subgroup of G need not be a characteristic subgroup of G.

14. Let G be a group. Show that $Z_i(G)$ is a characteristic subgroup of G for each $i \geq 0$, and also that G^j as well as $G^{(j)}$ are characteristic subgroups of G for each $j \geq 0$.

15. Let G be a group. The Frattini subgroup of G, denoted by $\Phi(G)$, is defined to be the intersection of all maximal subgroups of G. [Note that the empty intersection of subgroups of G is G itself.] Prove the following.

    (a) $\Phi(G)$ is a characteristic subgroup of G.

    (b) If $H \trianglelefteq G$, then $\Phi(H) \trianglelefteq \Phi(G)$.

    (c) If $|G| > 1$, then $\Phi(G)$ is the set of all nongenerators of G. Here an element $g \in G$ is said to be a nongenerator of G if any subset of G which together with g generates G is itself a generating set of G.

    (d) If G is finite and H, K are normal subgroups of G such that $H \subseteq K \cap \Phi(G)$ and K/H is nilpotent, then K is nilpotent.

    (e) If G is finite and $H \trianglelefteq G$, then H is nilpotent if and only if $[H, H] \subseteq \Phi(G)$.

    (f) If G is finite, then $\Phi(G)$ is nilpotent.

    (g) If G is finite, then G is nilpotent if and only if $[G, G] \subseteq \Phi(G)$.

Bonus Problems

1. Let $B_n(\mathbb{Q})$ denote the subgroup of $\mathrm{GL}_n(\mathbb{Q})$ consisting of all $n \times n$ nonsingular upper triangular matrices with rational entries, and let $U_n(\mathbb{Q})$ be the set of those matrices in $B_n(\mathbb{Q})$ which have all diagonal entries equal to 1. Show that

    (a) $U_n(\mathbb{Q})$ is a normal subgroup of $B_n(\mathbb{Q})$.

    (b) $U_n(\mathbb{Q})$ is nilpotent for all $n \geq 1$.

    (c) $B_n(\mathbb{Q})$ is solvable for all $n \geq 1$.

    (d) $B_n(\mathbb{Q})$ is not nilpotent for all $n \geq 2$.

2. Let G be a finite group.

    (a) If G has a normal subgroup H whose order is the smallest prime p dividing $|G|$, then show that H is a subgroup of $Z(G)$.

    (b) If for each $H \trianglelefteq G$, the quotient G/H has normal subgroups of all orders dividing $|G/H|$, then show that G must be nilpotent. Is it true that if for each $H \trianglelefteq G$, the quotient G/H has subgroups of all orders dividing $|G/H|$, then G is nilpotent? Justify your answer.