Indian Institute of Technology Bombay

MA 414: Algebra I

Spring 2007 SRG

Mid Semester Examination

Date & Time: 23 Feb 2008, 10.30 AM – 12.30 PM Max Marks : 25

Note: *Throughout n denotes a positive integer, S_n the group of permutations of $\{1, 2, \ldots, n\}$, and $\mathrm{GL}_n(\mathbb{R})$ the group of $n \times n$ nonsingular matrices with entries in the set $\mathbb{R}$ of all real numbers.*

Q. 1 Define the cycle-type of a permutation in S_n. Given any $\sigma, \tau \in S_n$, show that σ and τ are conjugate to each other if and only if σ and τ have the same cycle-type. [2 marks]

Q. 2 Define the center of a group. Prove that for $n \geq 3$, the center of S_n is trivial and for $n \geq 1$, the center of $\mathrm{GL}_n(\mathbb{R})$ consists precisely of the scalar matrices, that is, matrices of the form cI_n where $c \in \mathbb{R}$. [3 marks]

Q. 3 Define when a group is said to be (i) simple, and (ii) solvable. Show that a simple group is solvable if and only if it is abelian. Further show that an abelian group is simple if and only if it is cyclic of prime order. [3 marks]

Q. 4 Let G be a group. Define the upper central series $\{Z_i(G)\}_{i \geq 0}$ and the lower central series $\{G^i\}_{i \geq 0}$ of G. Show that if $Z_n(G) = G$ for some $n \geq 1$, then $G^n = \{1\}$. Prove any auxiliary results that you may require. [2 marks]

Q. 5 Let G be a transitive subgroup of S_n.

(a) If $N \trianglelefteq G$ and if $O_1, \ldots, O_k$ denote the disjoint orbits with respect to the natural action of N on $\{1, 2, \ldots, n\}$, then show that for any $j = 2, \ldots, k$, there exists $\sigma_j \in G$ such that $\sigma_j(O_1) = O_j$. Deduce that $k \mid n$ and $|O_j| = \frac{n}{k}$ for $j = 1, \ldots, k$.

(b) If $\{1\} \neq N \trianglelefteq G$ and if N is p-group for some prime p, then show that $p \mid n$.

(c) If G is nilpotent, then for every prime p dividing $|G|$, show that $p \mid n$. [5 marks]

Q. 6 (a) Define the composition series of a group.

(b) State the Jordan-Hölder Theorem.

(c) Determine a composition series for the dihedral group D_{12} of order 24 and the symmetric group S_6 of order 720. [3 marks]

Q. 7 Give an example of a group G such that every finite group is isomorphic to a subgroup of G. Justify your answer. [2 marks]

Q. 8 Show that the group Q_8 of quaternions is not the semidirect product of any two of its subgroups. [2 marks]

Q. 9 Let G be a finitely generated abelian group, with group operation written additively. Show that $T := \{x \in G : nx = 0 \text{ for some } n \geq 1\}$ is a finite subgroup of G. [3 marks]

Bonus Problems: Write the complete statement and a solution of any of the bonus problems from any of the Exercise sets given thus far.