Indian Institute of Technology Bombay

MA 414 Algebra I

Spring 2008 SRG

Quiz 1

Date: February 8, 2007 Weightage: 10 %
Duration: 60 minutes Max. Marks: 20

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1. Let G be a group and let $Z = Z(G)$ denote the center of G. Prove or disprove the following.
   (a) If G/Z is cyclic, then G is cyclic.
   (b) If G/Z is cyclic, then G is abelian.
   (c) If G/Z is abelian, then G is abelian.
   (d) If G/Z is abelian, then G is nilpotent.
   (e) If G/Z is abelian, then G is solvable.
   (f) If G/Z is nilpotent, then G is nilpotent.
   (g) If G/Z is solvable, then G is solvable.
   (h) If G/Z is solvable, then G is nilpotent. [8 marks]
2. Let p and q be primes. Show that every group of order p^2q is solvable. Is it true that every group of order p^2q is nilpotent? Justify your answer. [4 marks]
3. Let G be a finite group with $|G|$ divisible by a prime p. If P is a Sylow p-subgroup of G and N is a normal subgroup of G containing P, then show that $NN_G(P) = G$. [3 marks]
4. Let G be a finite abelian group. If $g_1, \ldots, g_h$ are elements of G of orders $e_1, \ldots, e_h$ respectively, then is it true that the order of their product is $\mathrm{LCM}(e_1, \ldots, e_h)$? Justify your answer. [2 marks]
5. Let G be a nilpotent group of class n. If $\{1\} = N_0 \le N_1 \le \cdots \le N_n = G$ is any central series of G of length n, then show that $G^{n-i} \le N_i \le Z_i(G)$ for $i = 0, 1, \ldots, n$. [3 marks]

Bonus Problems

1. Prove that S_n is not isomorphic to a subgroup of A_{n+1} for any $n \ge 2$.
2. Let $B_n(\mathbb{R})$ denote the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of all $n \times n$ nonsingular upper triangular matrices with real entries, and let $U_n(\mathbb{R})$ be the set of those matrices in $B_n(\mathbb{R})$ which have all diagonal entries equal to 1. Show that
   (a) $U_n(\mathbb{R})$ is a normal subgroup of $B_n(\mathbb{R})$.
   (b) $U_n(\mathbb{R})$ is nilpotent for all $n \ge 1$.
   (c) $B_n(\mathbb{R})$ is solvable for all $n \ge 1$.
   (d) $B_n(\mathbb{R})$ is not nilpotent for all $n \ge 2$.