MA 523

HW 2

Problems stated in the class + (i) – (vii) below

10 Exercises

(i) Evaluate $\sum_{d|n} \mu(d)\sigma(d)$ in terms of the distinct prime factors of n.

(ii) Let $\Lambda(n) = \log p$ if n is a power of a prime p and let $\Lambda(n) = 0$ otherwise (Λ is called von Mangoldt's function). Evaluate $\sum_{d|n} \Lambda(d)$. Express $\sum \Lambda(n)/n^s$ in terms of $\zeta(s)$.

(iii) Let a run through all the integers with $1 \le a \le n$ and $(a, n) = 1$. Show that $f(n) = (1/n)\sum a$ satisfies $\sum_{d|n} f(d) = \frac{1}{2}(n+1)$. Hence prove that $f(n) = \frac{1}{2}\phi(n)$ for $n > 1$.

(iv) Let a run through the integers as in (iii). Prove that $(1/n^3)\sum a^3 = \frac{1}{4}\phi(n)(1 + (-1)^k p_1 \cdots p_k/n^2)$, where $p_1, \ldots, p_k$ are the distinct prime factors of n (>1).

(v) Show that the product of all the integers a in (iii) is given by $n^{\phi(n)} \prod_{d|n} (d!/d^d)^{\mu(n/d)}$.

(vi) Show that $\sum_{n \le x} \mu(n)[x/n] = 1$. Hence prove that $|\sum_{n \le x} \mu(n)/n| \le 1$.

(vii) Let m, n be positive integers and let d run through all divisors of (m, n). Prove that $\sum d\mu(n/d) = \mu(n/(m, n))\phi(n)/\phi(n/(m, n))$. (The sum here is called Ramanujan's sum.)

(viii) Prove that $\sum_{n=1}^{\infty} \phi(n)x^n/(1-x^n) = x/(1-x)^2$. (Series of this kind are called Lambert series.)

(ix) Prove that $\sum_{n \le x} \phi(n)/n = (6/\pi^2)x + O(\log x)$.