Indian Institute of Technology Bombay

MA 524 Algebraic Number Theory

Spring 2010 SRG

Quiz 1

Date: February 5, 2010 Weightage: 10 %
Duration: 60 minutes Max. Marks: 20

---

1. Let L/K be a field extension of degree n.
   (a) Define the discriminant $D_{L/K}(\alpha_1, \ldots, \alpha_n)$ of n elements $\alpha_1, \ldots, \alpha_n$ of L.
   (b) Compute the following discriminants: (i) $D_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}}(1, \sqrt{3})$, (ii) $D_{\mathbb{Q}(\zeta_4)/\mathbb{Q}}(1, \zeta_4)$, and (iii) $D_{\mathbb{Q}(\zeta_{27})/\mathbb{Q}}(1, \zeta_{27}, \zeta_{27}^2, \ldots, \zeta_{27}^{18})$. Here ζ_n denotes a primitive nth root of unity (in $\mathbb{C}$). [4 marks]

2. Let v be a discrete valuation of a field K. If $R_v := \{x \in K : v(x) \geq 0\}$ and $M_v := \{x \in K : v(x) > 0\}$, then show that R_v is a subring of K and M_v is the unique maximal ideal of R_v. Further show that if K contains $\mathbb{Q}$ as a subfield, then R_v contains $\mathbb{Z}$ as a subring, and $M_v \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. (Note that as per usual conventions, $v(0) = \infty$ and $\infty > 0$.) [3 marks]

3. Determine all the discrete valuations of $\mathbb{Q}$. Justify your answer. [3 marks]

4. Determine explicitly the ring of (algebraic) integers of the quadratic field K when (i) $K = \mathbb{Q}(\sqrt{7})$, and (ii) $K = \mathbb{Q}(\sqrt{17})$. [3 marks]

5. Let p be an odd prime number and ζ be a primitive pth root of unity (in $\mathbb{C}$). Show that the ideal generated by p in the ring $\mathbb{Z}[\zeta]$ is not prime. Further, give an example of a prime ideal of $\mathbb{Z}[\zeta]$ lying above $p\mathbb{Z}$. [3 marks]

6. Let α be a root of the irreducible cubic polynomial $X^3 + 2X + 1$ in $\mathbb{Q}[X]$ and let $K = \mathbb{Q}(\alpha)$. Compute $D_{K/\mathbb{Q}}(1, \alpha, \alpha^2)$. Is it possible that the integral closure of $\mathbb{Z}$ in K is generated as a $\mathbb{Z}$-module by elements $\beta_1, \beta_2, \beta_3$ such that $\mathbb{Z}[\alpha] \subsetneq \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 + \mathbb{Z}\beta_3$? Justify your answer. [4 marks]

Bonus Problems

1. Prove that every quadratic field is a subfield of a cyclotomic field.

2. Determine all the discrete valuations of $\mathbb{C}(X)$. Justify your answer.