Scherier Theorem

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In this report, we describe a proof of the Scherier theorem for subnormal series of group. This is a fundamental result in the theory of series. The proof involves another interesting lemma called the butterfly lemma (also known as the Zassenhaus lemma). As a consequence of the Schreier theorem, we shall also outline a proof of Jordan-Höder theorem for composition series.

Basic Definitions and Examples

Notation: Let G be a group. If H is a subgroup (resp., proper subgroup) of G, then we denote it by $H \leq G$ (resp., $H < G$). Similarly, if H is a normal subgroup (resp., proper normal subgroup) of G, then we denote it by $H \trianglelefteq G$ (resp., $H \triangleleft G$).

Definition 1 *A subnormal (or subinvariant) series of a group G is a finite sequence $H_0, H_1, \ldots, H_n$ of subgroups of G such that H_i is a normal subgroup of H_{i+1} with $H_0 = \{e\}$ and $H_n = G$. We denote it by*

$$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G.$$

Definition 2 *A normal (or invariant) series of group G is a subnormal series $\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$ of G such that each H_i is a normal subgroup of G.*

Definition 3 *A length of a subnormal (resp., normal) series $\{e\} = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G$ (that is, each H_i is proper normal subgroup of H_{i+1}) of a group G is n.*

Example 4 (a) *Consider the group $\mathbb{Z}$ under addition. Then the series*

$$\{e\} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}$$

is subnormal as well as normal.

(b) *Similarly, consider the group $D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = e, \sigma\tau = \tau\sigma^{-1} \rangle$, then the series*

$$\{e\} < \{e, \tau\} < \{e, \tau, \sigma^2, \sigma^2\tau\} < D_8$$

is subnormal, but not normal.

Definition 5 *A subnormal (resp., normal) series $\{K_j\}$ is a refinement of a subnormal (resp., normal) series $\{H_i\}$ of a group G if $\{H_i\} \subseteq \{K_j\}$, i.e., each H_i is one of the K_j.*

Example 6 *For example, the series*

$$\{0\} \lhd 32\mathbb{Z} \lhd 16\mathbb{Z} \lhd 8\mathbb{Z} \lhd 4\mathbb{Z} \lhd \mathbb{Z}$$

is refinement of the series

$$\{0\} \lhd 16\mathbb{Z} \lhd 4\mathbb{Z} \lhd \mathbb{Z} \ .$$

Definition 7 *Two subnormal (resp., normal) series $\{H_i\}$ and $\{K_j\}$ of the same group G are isomorphic if there is a one-to-one correspondence between the collections of factor groups $\{H_{i+1}/H_i\}$ and $\{K_{j+1}/K_j\}$ such that the corresponding factor groups are isomorphic.*

Example 8 *For example, the following two series of $\mathbb{Z}_{15}$,*

$$\{0\} \lhd \langle 5 \rangle \lhd \mathbb{Z}_{15}$$

and

$$\{0\} \lhd \langle 3 \rangle \lhd \mathbb{Z}_{15},$$

are isomorphic.

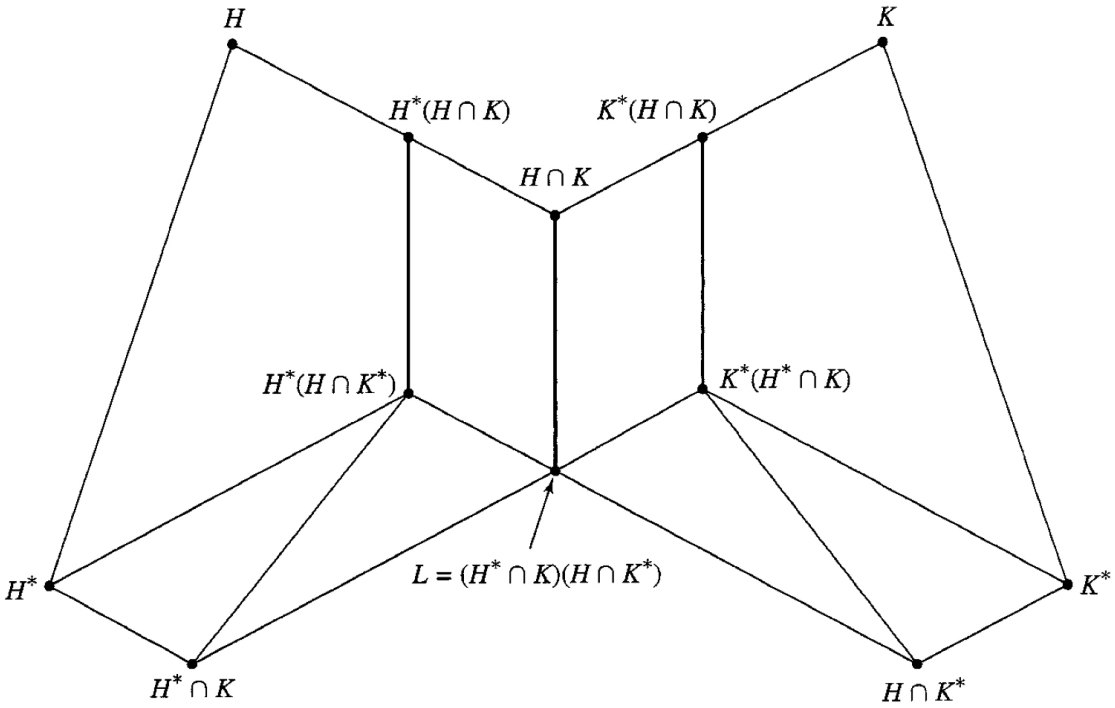
Observe that any two isomorphic subnormal (resp., normal) series must have the same length.

Butterfly Lemma

Lemma 9 [Butterfly (or Zassenhaus) Lemma] *Let H and K be subgroups of a group G and let H^* and K^* be normal subgroups of H and K, respectively. Then*

1. *$H^*(H \cap K^*)$ is normal subgroup of $H^*(H \cap K)$.*
2. *$K^*(H^* \cap K)$ is normal subgroup of $K^*(H \cap K)$.*
3. *$H^*(H \cap K)/H^*(H \cap K^*) \cong K^*(H \cap K)/K^*(H^* \cap K) \cong (H \cap K)/[(H^* \cap K)(H \cap K^*)]$.*

Proof:



(1) Since H^* is a normal subgroup of H, it follows that $H^*(H \cap K^*)$ and $H^*(H \cap K)$ both are subgroup of H. Clearly, $H^*(H \cap K^*)$ is a subgroup of $H^*(H \cap K)$. For any $g \in H^*(H \cap K)$, we have

$$gH^*(H \cap K^*)g^{-1} = (gH^*g^{-1})(g(H \cap K^*)g^{-1}) \subseteq H^*(H \cap K).$$

Hence $H^*(H \cap K^*)$ is a normal subgroup of $H^*(H \cap K)$.

(2) Similar to part (1) (interchange the roll of H and K).

(3) Since both $H \cap K^*$ and $H^* \cap K$ are normal subgroups of $H \cap K$, then $L = (H \cap K^*)(H^* \cap K)$ is also a normal subgroup of $H \cap K$.

Let $\phi : H^*(H \cap K) \longrightarrow (H \cap K)/L$ be defined as follows. For $h \in H^*$ and $x \in H \cap K$, let $\phi(hx) = xL$.

Claim: ϕ is a well-defined and a group homomorphism.

For if, let $h_1, h_2 \in H^*$ and $x_1, x_2 \in H \cap K$. If $h_1x_1 = h_2x_2$, then $h_2^{-1}h_1 = x_2x_1^{-1} \in H^* \cap (H \cap K) = H^* \cap K \subseteq L$, so $x_1L = x_2L$.
Thus ϕ is a well defined.

Since H^* is normal in H, there is $h_3 \in H^*$ such that $x_1h_2 = h_3x_1$. Then

$$\begin{aligned}\phi((h_1x_1)(h_2x_2)) &= \phi((h_1h_3)(x_1x_2)) \\ &= (x_1x_2)L \\ &= (x_1L)(x_2L) \\ &= \phi(h_1x_1)\phi(h_2x_2).\end{aligned}$$

Thus ϕ is a homomorphism.

Obviously ϕ is onto $(H \cap K)/L$. Finally if $h \in H^*$ and $x \in H \cap K$, then

$$\begin{aligned}\phi(hx) = L &\iff xL = L \\ &\iff x \in L \\ &\iff hx \in H^*(H^* \cap K)(H \cap K^*) = H^*(H \cap K^*).\end{aligned}$$

Thus $\mathrm{Ker}(\phi) = H^*(H \cap K^*)$.

Hence

$$H^*(H \cap K)/H^*(H \cap K^*) \cong (H \cap K)/[(H^* \cap K)(H \cap K^*)].$$

Similarly one can prove

$$K^*(H \cap K)/K^*(H^* \cap K) \cong (H \cap K)/[(H^* \cap K)(H \cap K^*)].$$

Schreier Theorem

Theorem 10 [**Schreier Theorem**] *Two subnormal (resp., normal) series of a group G have isomorphic refinements.*

Proof: Let

$$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$$

and

$$\{e\} = K_0 \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_m = G$$

be two subnormal series for G.

For each $i \in \{0, 1, \ldots, n-1\}$, consider the chain of groups

$$H_i = H_i(H_{i+1} \cap K_0) \leq H_i(H_{i+1} \cap K_1) \leq \cdots \leq H_i(H_{i+1} \cap K_m) = H_{i+1}.$$

This inserts $m-1$ not necessarily distinct groups between H_i and H_{i+1}.

Let $H_{i,j} = H_i(H_{i+1} \cap K_j)$, where $0 \leq i \leq n-1$ and $0 \leq j \leq m$. Then we obtain the chain of groups

$$\begin{aligned}
\{e\} = H_{0,0} \leq H_{0,1} &\leq H_{0,2} \leq \ldots &&\leq H_{0,m-1} \leq H_{1,0} \\
&\leq H_{1,1} &&\leq H_{1,2} \leq \ldots &&\leq H_{1,m-1} \leq H_{2,0} \\
&\leq H_{2,1} &&\leq H_{2,2} \leq \ldots &&\leq H_{2,m-1} \leq H_{3,0} \\
&\leq \ldots \\
&\leq H_{n-1,1} &&\leq H_{n-1,2} \leq \ldots &&\leq H_{n-1,m-1} \leq H_{n,0} = G
\end{aligned}$$

Above chain contains $nm+1$ not necessarily distinct groups, and $H_{i,0} = H_i$ for each i.

Similarly let $K_{j,i} = K_j(K_{j+1} \cap H_i)$, where $0 \leq j \leq m-1$ and $0 \leq i \leq n$. Then we obtain the chain of groups

$$\begin{aligned}
\{e\} = K_{0,0} \leq K_{0,1} &\leq K_{0,2} \leq \ldots &&\leq K_{0,n-1} \leq K_{1,0} \\
&\leq K_{1,1} &&\leq K_{1,2} \leq \ldots &&\leq K_{1,n-1} \leq K_{2,0} \\
&\leq K_{2,1} &&\leq K_{2,2} \leq \ldots &&\leq K_{2,n-1} \leq K_{3,0} \\
&\leq \ldots \\
&\leq K_{m-1,1} &&\leq K_{m-1,2} \leq \ldots &&\leq K_{m-1,n-1} \leq K_{m,0} = G
\end{aligned}$$

Above chain contains $mn + 1$ not necessarily distinct groups, and $K_{j,0} = K_j$ for each j.

By butterfly lemma, we have

$$H_i(H_{i+1} \cap K_{j+1})/H_i(H_{i+1} \cap K_j) \cong K_j(K_{j+1} \cap H_{i+1})/K_j(K_{j+1} \cap H_i),$$

that is,

$$H_{i,j+1}/H_{i,j} \cong K_{j,i+1}/K_{j,i}$$

for $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$.

The above isomorphic of relation give a one-to-one correspondence of isomorphic factor groups between above chains of subgroups of G. Hence above chains are require refinements of given subnormal series.

For normal series, where all H_i and K_j are normal in G, we merely observe that all the group $H_{i,j}$ and $K_{j,i}$ formed are also normal in G, so the same proof applies.

Jordan-Hölder Theorem

Definition 11 *A subnormal series* $\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = G$ *of group* G *is a composition series if all the factor groups* H_{i+1}/H_i *are simple.*

Example 12 (a) *For example, The series*

$$\{e\} < A_n < S_n$$

for $n \geq 5$ *is a composition series (and also a principal series) of* S_n*.*

(b) *We claim that* $\mathbb{Z}$ *has no composition series. To see this, let*

$$\{0\} = H_0 < H_1 < \cdots < H_{n-1} < H_n = \mathbb{Z}$$

be a subnormal series. Then H_1 *must be of the form* $r\mathbb{Z}$ *for some* $r \in \mathbb{Z}^+$*, which is not simple for all* $r \in \mathbb{Z}^+$*.*

Remark 13 *It is not necessary that every group has composition series.*

Theorem 14 [Jordan-Hölder Theorem] *Any two composition series of a group* G *are isomorphic.*

Proof: Let

$$\{e\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_n = G$$

and

$$\{e\} = K_0 \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \cdots \trianglelefteq K_m = G$$

be two composition series for G. By Schreier Theorem, they have isomorphic refinements. But since all factor groups are already simple, neither series has any further refinement. Thus $\{H_i\}$ and $\{K_j\}$ must already be isomorphic.

References

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