The Borsuk Ulam Theorem

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1 Quotient Topology

1.1 Quotient Map

1.1.1 Definition

Let X,Y be topological spaces. A map $q : X \to Y$ is called **a quotient map** if a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

- A continuous, closed, surjective map is a quotient map.

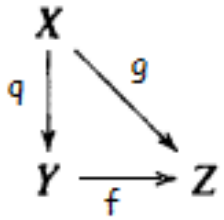
1.1.2 Example

The map $q : S^1 \to S^1$ such that $q(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$ (or $q(z) = z^2 \ for \ S^1 \subset \mathbb{C}$).

1.1.3 Proposition

Let $q : X \to Y$ be a quotient map and $g : X \to Z$ be a continuous map such that $q^{-1}(y)$ is constant for all $y \in Y$. Then g induces a continuous map $f : Y \to Z$ such that $f \circ q = g$.

Proof: Given that $g(q^{-1}(y))$ is singleton set in Z. Let $f : Y \to Z$ such that $f(y) = g(q^{-1}(y))$. Here f is a well defined map with $f(q(x)) = g(x) \ for \ x \in X$.(see figure 1)
Now g is continuous. Let $V \subset Z$ be an open set, hence $g^{-1}(V) = q^{-1}(f^{-1}(V))$ is open in X. Since q is a quotient map, $(f^{-1}(V))$ is open.
Hence f is continuous.



■

Figure 1:

2 Covering Spaces

2.1 Covering Map

2.1.1 Definitions

1. Let $p : E \to B$ be surjective and continuous map. Then it is called **a covering map** if $\forall\ b \in B,\ \ \exists\ U \subset B$ open, containing b, such that $p^{-1}(U)$ is disjoint union of open sets $V_i \subset E$ with each V_i homeomorphic to U under p.

2. Let $p : E \to B$ be a covering map and $f : [0,1] \to B$ be a path starts at $b_0 \in B$. Then $\tilde{f} : [0,1] \to E$ is called **unique path lifting** if $p \circ \tilde{f} = f$ with path lifting starting at $p(b_0) = e \in E$.

2.1.2 Lemma

If f and g are two paths in B with $f(0) = g(0),\ f(1) = g(1)$ and $f \sim g \Rightarrow$ then path liftings $\tilde{f}$ and $\tilde{g}$ satisfy $\tilde{f}(0) = \tilde{g}(0),\ \tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f} \sim \tilde{g}$.

3 The Fundamental Group

3.1 Path Homotopy

3.1.1 Definitions

- A continuous map $f : [0.1] \to X$ is called **path** in X.
- A path with $f(0) = f(1) = x_0$ is called **loop** based at $x_0 \in X$.
- For given two paths f and g in X, with $f(0) = g(0)$ and $f(1) = g(1)$, **a path homotopy** is a continuous function $H : [0,1] \times [0,1] \to X$ such that,
  - $H(t,0) = f(t)$
  - $H(t,1) = g(t)$; $\forall t \in [0,1]$.
  - $H(1,s) = f(1)\ ;\ H(0,s) = f(0)\ ;\ \forall s \in [0,1]$.

  The paths f, g are called **path homotopic** to each other and denoted by $f \cong g$.

3.1.2 Lemma

$\cong$ is an equivalence relation

Proof: 1. If $H(t,s) = f(t)$ then $f \cong f$, hence **reflexive**.

2. For $f \cong g$ with homotopy H, the map $H'(t,s) = H(t,1-s)$ gives $g \cong f$. Hence **symmetric**

3. Let $f \cong g$ and $g \cong h$ with homotopies H and G respectively, then $F : [0,1] \times [0,1] \to X$ given by

   $$F(t,s) = \begin{cases} H(t,2s) & : s \in [0, \frac{1}{2}] \\ G(t,2s-1) & : s \in [\frac{1}{2}, 1]. \end{cases}$$

   Then F is a path homotopy between f and h. Hence $\cong$ is **transitive**.
   Therefore $\cong$ is **an equivalence relation**. ■

3.2 The Fundamental Group

Let X be a topological space and $x_0 \in X$. Consider set of all loops at x_o. We define an operation $\star$ on this set as

$$(f \star g)(t) = \begin{cases} f(2t) & : t \in [0, \frac{1}{2}] \\ g(2t-1) & : t \in [\frac{1}{2}, 1]. \end{cases}$$

Now consider path homotopy classes and define a product operation $\star$ as $[f] \star [g] = [f \star g]$.
This is a well defined operation that satisfies group properties.

• The set of **path homotopy classes of loops at** x_0, with the operation $\star$, is called **the fundamental group** of X at base point x_0. It is denoted by $\pi_1(X, x_0)$.
- The class of nulhomotopic loops, at x_0, is identity for the group.
- For the class $[f]$, the class $[\bar{f}]$ of path $\bar{f}(t) = f(1-t)$, is the inverse.

• The fundamental group of the circle $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$.

3.2.1 Definition

Let $h : (X, x_0) \to (Y, y_0)$ be continuous map between two spaces then **homomorphism induced by** h is given $h_* : \pi_1(X, x_0) \to \pi_2(Y, y_0)$ such that $h_*([f]) = [h \circ f]$.

3.2.2 Proposition

Let $h : S^2 \to X$ be continuous map then following are equivalent.
(1) h is homotopic to a constant map, called as nulhomotopic.
(2) h can be extended to a continuous map from $B^2 \to X$.
(3) h_* is trivial homomorphism between fundamental groups.

Proof: (1) $\Rightarrow$ (2). Let $H : S^1 \times [0, 1] \to$ be the homotopy between h and a constant map. Let $\pi : S^1 \times [0, 1] \to B^2$ defined as $\pi(x, t) = (1-t)x$. It is continuous,closed, surjective map. Hence it is a quotient map.
Since H is constant on $S^1 \times 1$, by proposition 1.1.3 , it induces a continuous map $k : B^2 \to X$ as a extension of h.

(2) $\Rightarrow$ (3). Consider the inclusion $j : S^1 \to B^2$ with $h = k \circ j$ and hence $h_* = k_* \circ j_*$.
The fundamental group of B^2 is trivial $\Rightarrow$ homomorphisms j_* and k_* are trivial. $\Rightarrow$ h_* is trivial.

(3) $\Rightarrow$ (1). Let $p_0 : [0, 1] \to S^1$ such that $p_0(t) = (\sin 2\pi t, \cos 2\pi t)$ is a covering map. Also it is a loop at b_0 which generates $\pi_1(S^1, b_0)$.
Now h_* is trivial, so the loop $f = h \circ p_0$ is identity in $\pi_1(S^1, b_0)$. So let F be path homotopy between f and constant path at x_0.
Consider $p_0 \times id : [0, 1] \times [0, 1] \to S^1 \times [0, 1]$ is a quotient map.
It maps $(0, t)$ and $(1, t)$ to (b_0, t) and injective elsewhere. Since F maps $(0, t)$ and $(1, t)$ to x_0, it induces a homotopy between h and constant map at x_0. ■

4 The Borsuk Ulam Theorem

4.1 Definitions

1. For a point $x \in S^n$, it's **antipodal** point is given by $-x$.
2. A map $h : S^n \to \mathbb{R}^n$ is called **antipodal preserving** if $h(-x) = -h(x)$ for $\forall x \in S^n$.

4.2 Theorem 1

If $h : S^1 \to S^1$ is continuous, antipodal preserving map then h is not nulhomotopic.

Proof: Let $b_0 = (1, 0) \in S^1$.
For $h(b_0) \neq b_0$, consider a rotation map $\rho : S^1 \to S^1$ is antipode preserving with $\rho(h(b_0)) = b_0$. So if h is nulhomotopic with homotopy H, then $\rho \circ h$ is nulhomotopic with homotopy $\rho \circ H$. Hence it suffices to prove the theorem for the case $h(b_0) = b_0$. Hence $h(-b_0) = -b_0$.

Let $q : S^1 \to S^1$ be a quotient map given by $q(z) = z^2$. Now for $z \in S^1$, $q^{-1}(z) = \{z, -z\}$.
Also $h(-z) = -h(z)$, we get $q(h(-z)) = q(h(z))$.
By proposition 1.1.3, $q \circ h$ induces **a continuous map $\mathbf{k} : \mathbf{S^1} \to \mathbf{S^1}$** such that $k \circ q = q \circ h$ with $k(b_0) = b_0$.

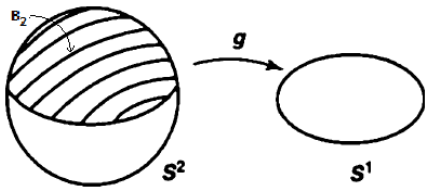
Here q is a covering map. Let $\tilde{f}$ be a path from b_0 to $-b_0$, then $f := q \circ \tilde{f} \in \pi_1(S^1, b_0)$ nontrivial element. Hence $\tilde{f}$ is path lifting of f with same initial point b_0 but different end point.
Also $h \circ \tilde{f}$ is also path from b_0 to $-b_0$.
Now $k_*[f] = [k \circ (q \circ \tilde{f})] = [q \circ (h \circ \tilde{f})]$. [$\because$ $k \circ q = q \circ h$]
As $h \circ \tilde{f}$ is non trivial, k_* is nontrivial on $\mathbb{Z}$ $\Rightarrow$ k_* is injective.

We have $q_* : \mathbb{Z} \to \mathbb{Z}$, as a homomorphism from x to $2x$, is an injective map. Also k_* is injective.
$\Rightarrow k_* \circ q_*$ is injective.
Now $k_* \circ q_* = q_* \circ h_*$, thus h_* is also injective.
By proposition 3.2.2, h is not nulhomotopic. ■

4.3 Theorem 2

There is no continuous antipode preserving map $g : S^2 \to S^1$.

Proof: Suppose $g : S^2 \to S^1$ is continuous and antipode preserving.
Let S^1 be the equator of S^2.



Then the restriction of g to S^1 say $h : S^1 \to S^1$ is also continuous, antipode preserving map to itself. By Theorem 1, it is not nullhomotopic.
But h has a continuous extension to upper hemisphere of S^2, homeomorphic to the the ball B^2.
A contradiction to proposition 3.2.2. ■

4.4 The Borsuk Ulam Theorem

Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.

Proof: Suppose $f(x) \neq f(-x),\ \forall x \in S^2$. Then

$$g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$$

will be a continuous map from S^2 to S^1 with $g(-x) = -g(x)$.
A contradiction to the theorem 2. ■

References

[1] Topology, James Munkres, Prentice Hall Publication.