# The Borsuk-Ulam Theorem

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**Theorem**

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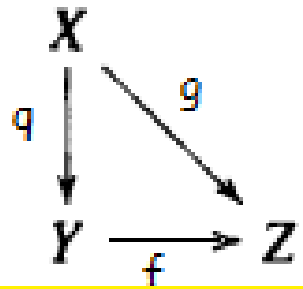
- A continuous, antipode preserving map on a circle to another circle cannot be nulhomotopic.
- There is no continuous, antipode preserving map from a sphere to a circle.
- A continuous map from a sphere to plane takes same value on at least one pair of antipodal points of the sphere.

# Quotient Topology

- Let $X, Y$ be topological spaces. A map $q : X \to Y$ is called a quotient map provided a subset $U$ of $Y$ is open in $Y$ if and only if $q^{-1}(U)$ is open in X.
- A continuous, closed, surjective map is always a quotient map.
- For example, $q : S^1 \to S^1$ such that $q(\cos\theta, \sin\theta) = (\cos 2\theta, \sin 2\theta)$ (or $q(z) = z^2$ *for* $S^1 \subset \mathbb{C}$).

# Quotient Topology

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- **Result 1**: Let $q : X \to Y$ be a quotient map and for a space $Z$, let $g : X \to Z$ be a continuous map that is constant on $q^{-1}(y)$, for $y \in Y$.
  Then $g$ induces a continuous map $f : Y \to Z$ such that $f \circ q = g$.



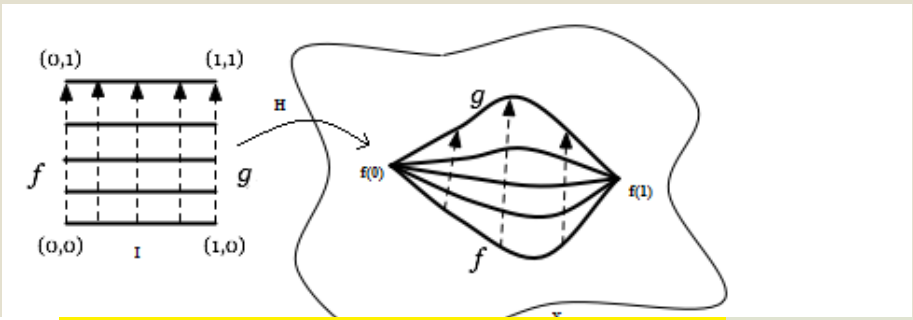
# Path homotopy

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1. A path in a topological space $X$ is a continuous map $f : [0,1] \to X$.
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2. A path homotopy between two paths $f$ and $g$, with $f(0) = g(0)$ and $f(1) = g(1)$, is a continuous function $H : [0,1] \times [0,1] \to X$ such that
   - $H(t,0) = f(t)$
   - $H(t,1) = g(t)$; $\forall t \in [0,1]$.
   - $H(1,s) = f(1)$ ; $H(0,s) = f(0)$ ; $\forall s \in [0,1]$.

   The paths $f$ and $g$ are called path homotopic to each other. It is denoted by $f \sim g$.



# The Fundamental Group

Let $X$ be a topological space and $x_0 \in X$. Consider set of all loops at $x_o$ i.e. $f(0) = f(1) = x_0$. We define an operation $\star$ on this set as

$$(f \star g)(t) = \begin{cases} f(2t) & : t \in [0, \frac{1}{2}] \\ g(2t-1) & : t \in [\frac{1}{2}, 1]. \end{cases}$$

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• The set of **path homotopy classes of loops at** $x_0$, with the operation $\star$, is called The Fundamental Group of $X$ at base point $x_0$. It is denoted by $\pi_1(\mathbf{X}, \mathbf{x_0})$.

- The class of nulhomotopic loops, at $x_0$, is identity for the group.
- For the class $[f]$, the class $[\bar{f}]$ of path $\bar{f}(t) = f(1-t)$, is the inverse.

• The fundamental group of the circle $\pi_1(S^1, b_0) \cong \mathbb{Z}$.

## Definition

Let $h : (X, x_0) \longrightarrow (Y, y_0)$ be continuous map between two spaces then **homomorphism induced by** $h$ is given $h_* : \pi_1(X, x_0) \longrightarrow \pi_2(Y, y_0)$ such that $h_*([f]) = [h \circ f]$.

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## Lemma (1)

*Let $h : S^2 \longrightarrow X$ be continuous map then following are equivalent.*
*(1) h is nulhomotopic.*
*(2) h can be extended to a continuous map from $B^2 \longrightarrow X$.*
*(3) $h_*$ is trivial homomorphism between fundamental groups.*

# Covering Spaces and maps

## Definition

Let $p : E \to B$ be surjective and continuous map. Then it is called **a covering map** if $\forall b \in B, \quad \exists U \subset B$ open, containing $b$, such that $p^{-1}(U)$ is disjoint union of open sets $V_i \subset E$ with each $V_i$ homeomorphic to $U$ under $p$.

## Definition

Let $p : E \to B$ be a covering map and $f : [0,1] \to B$ be a path starts at $b_0 \in B$. Then $\tilde{f} : [0,1] \to E$ is called **unique path lifting** if $p \circ \tilde{f} = f$ with path lifting starting at $p(b_0) = e \in E$.

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## Lemma

*If $f$ and $g$ are two paths in $B$ with $f(0) = g(0), \ f(1) = g(1)$ and $f \sim g \Rightarrow$ then path liftings $\tilde{f}(0) = \tilde{g}(0), \ \tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f} \sim \tilde{g}$*

# The Borsuk-Ulam Theorem

**Definition**

For a point $x \in S^n$, it's **antipodal** point is given by $-x$.

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A map $h : S^n \to \mathbb{R}^n$ is called **antipodal preserving** if $h(-x) = -h(x)$ for $\forall x \in S^n$.

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**Theorem (1)**

*If $h : S^1 \to S^1$ is continuous, antipodal preserving map then $h$ is not nulhomotopic.*

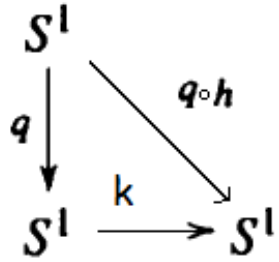
## Proof: Step 1

Let $b_0 = (1, 0) \in S^1$.
We first prove the theorem for the case $h(b_0) = b_0$. Hence $h(-b_0) = -b_0$ .
Let $q : S^1 \to S^1$ be a quotient map given by $q(z) = z^2$.
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Now for $z \in S^1$, $q^{-1}(z) = \{z, -z\}$.

Also $h(-z) = -h(z)$, we have $q(h(-z)) = q(h(z))$.
By result 1, $q \circ h$ induces **a continuous map k** : $\mathbf{S^1} \to \mathbf{S^1}$ such that $k \circ q = q \circ h$ with $k(b_0) = b_0$.



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Let $\tilde{f}$ be a path from $b_0$ to $-b_0$, then $f := q \circ \tilde{f} \in \pi_1(S^1, b_0)$ nontrivial element. Hence $\tilde{f}$ is path lifting of $f$ with same initial point $b_0$ but different end point.

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Also $h \circ \tilde{f}$ is also path from $b_0$ to $-b_0$.
Now $k_*[f] = [k \circ (q \circ \tilde{f})] = [q \circ (h \circ \tilde{f})]$. [ $\because$ $k \circ q = q \circ h$ ]
As $h \circ \tilde{f}$ is non trivial, $k_*$ is nontrivial on $\mathbb{Z}$ $\Rightarrow$ $k_*$ is injective.

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We have $q_* : \mathbb{Z} \to \mathbb{Z}$, as a homomorphism from $x$ to $2x$, is an injective map. Also $k_*$ is injective.
$\Rightarrow k_* \circ q_*$ is injective. Now $k_* \circ q_* = q_* \circ h_*$, thus $h_*$ is also injective.

Hence, by lemma 1, $h$ is not nulhomotopic.

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Hence, by lemma 1, $h$ is not nulhomotopic.

For $h(b_0) \neq b_0$, consider a rotation map $\rho : S^1 \to S^1$ is antipode preserving with $\rho(h(b_0)) = b_0$.
So if $h$ is nulhomotopic with homotopy $H$, then $\rho \circ h$ is nulhomotopic with homotopy $\rho \circ H$. Hence we are done.

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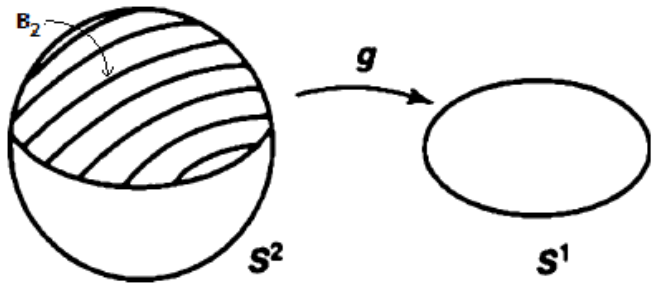
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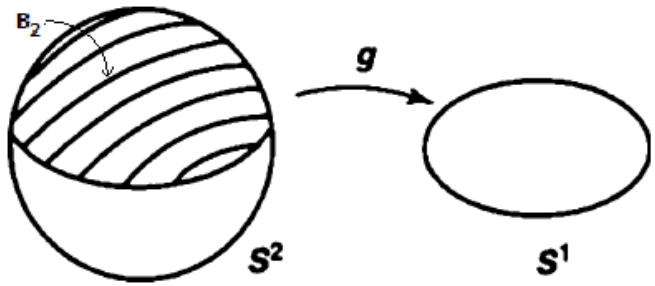
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**Proof.**

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Let $S^1$ be the equator of $S^2$.
Then the restriction of $g$ to $S^1$ say $h : S^1 \to S^1$ is also continuous, antipode preserving map to itself.
Hence it is not nullhomotopic. But $h$ has a continuous extension to upper hemisphere of $S^2$, homeomorphic to the the ball $B^2$.
A contradiction by lemma (1). □

# Borsuk-Ulam theorem for $S^2$

**Theorem**

*Given a continuous map $f : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.*

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**Proof.**

Suppose $f(x) \neq f(-x)$, $\forall x \in S^2$. Then

$$g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$$

will be a continuous map from $S^2$ to $S^1$ with $g(-x) = -g(x)$.
A contradiction to the theorem 2. □

# References

**Topology**
**James Munkres,**
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# Thank You