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1 C*-Algebras

1.1 Basic Definitions

Definition 1.1.1 A C*-algebra A is a non-empty set with the following operations:

1. Addition (which is commutative and associative)
2. Multiplication (which is associative)c
3. Multiplication by complex scalars
4. Involution $a \mapsto a^*$ (with $a^{**} = a$)

- Both types of Multiplication distribute over addition.
- $(ab)^* = b^*\ a^*$
- $(\lambda a + \beta b)^* = \overline{\lambda} a^* + \overline{\beta} b^*$
- $\lambda(ab) = (\lambda a)b_2 = a(\lambda b)$ and $(\lambda\beta a) = \lambda(\beta a) \quad \forall\ a, b \in A$ and $\forall\ \alpha, \beta \in \mathbb{C}$

A has a norm $||.||$ such that

1. $||\lambda a + \beta b|| \leq |\lambda|\ ||a|| + |\beta|\ ||b||$ (Sub-Linear)
2. $||ab|| \leq ||a||\ ||b||$ (Sub-Multiplicative)
3. $||a^* a|| = ||a||^2$ (C*-norm condition) $\quad \forall\ a, b \in A$ and $\forall\ \alpha, \beta \in \mathbb{C}$

A is complete in the metric $d(a, b) = ||a - b||$ (Banach Algebra Condition)

1. If A has *the* multiplicative identity then A is said to be a *unital* C*-algebra.
2. If $ab = ba \ \ \forall\ a, b \in$ A then A is said to be Commutative C*-algebra.

Remarks 1.1.2

1. Topology on A is given by the metric topology induced by the C*-norm such that $||.||$ is a continuous function.

2. The algebraic operations are continuous w.r.t. the topology on A.

3. Norm Topology on C*-algebra A is the coarsest/smallest topology such that $||.||$ is a continuous function.

Proposition 1.1.3 For a in C*-algebra $A, ||a^*|| = ||a||$

Proof. $||a||^2 = ||a * a|| \leq ||a^*||\ ||a|| \implies ||a|| \leq ||a^*||$
Replacing a by a^* in the above inequality gives the conclusion. □

Examples 1.1.4

1. Given a compact topological Hausdorff space X, Consider the space

$$C(X) = \{f : X \to \mathbb{C} : f \text{ is continous on } X\} \text{ and}$$
$$||f|| := \sup\{|f(x)| : x \in X\}$$

with pointwise operations and involution $f^*(x) = \overline{f(x)}\ , x \in X$.
$(C(X), ||.||)$ is a commutative unital C*-algebra.
In general, we will see that every commutative unital C*-algebra arises this way.

2. $M_n(\mathbb{C})$ is a non-commutative unital C*-algebra with the norm given by

$$||a|| = \{||ax||_2 : ||x||_2 \leq 1, x \in \mathbb{C}^n\}, \forall a \in M_n(\mathbb{C})$$

and involution as conjugate transpose.

3. Let, $\mathcal{H}$ be a Hilbet space. Then the space $\mathrm{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ with the operator norm and adjoint operation is a non-commutative C*-algebra.
If $\mathcal{H}$ is n-dimensional then $\mathrm{B}(\mathcal{H}) = M_n(\mathbb{C})$ which is a C*-algebra.

4. A and B are C*-algebras. The Direct Sum $A \oplus B = \{(a, b) : a \in A, b \in B\}$ is a C*-algebra with the norm given by

$$||(a, b)|| = \max\{||a||, ||b||\}$$

and pointwise operations.
Similarly, Direct sum of any n C*-algebras is again a C*-algebra.

1.2 Algebraic and Topological Structure of C*-Algebra

Definition 1.2.1 Let, A be an algebra with unity e and $a \in A$.

- a is said to be invertible in A if $\exists\ b \in A : ab = ba = e$.
  The set of all invertible elements in A is denoted by $\mathrm{G}(A)$.
- Spectrum of $a : \sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda e - a \text{ is not invertible in } A\}$
- Spectral Radius : $r(A) = \mathrm{Sup}\ \{|\lambda| : \lambda \in \sigma_A(a)\}$
  (allowing $r(A)$ to be infinity)

Proposition 1.2.2 Let, A be a unital Banach Algebra and $a \in A$.

1. $||a - e|| < 1 \implies a \in \mathrm{G}(A)$
2. $\mathrm{G}(A)$ is an open subset of A.

Proof.

1. Consider the series,

   $$\sum_{n=0}^{\infty} ||(e-a)^n|| \leq \sum_{n=0}^{\infty} ||e-a||^n$$

   which converges in $\mathbb{C}$ as $||a - e|| < 1$.

   $\implies \sum_{n=0}^{\infty}(e-a)^n$ is absolutely convergent in A.
   A is Banach $\implies \exists\ b \in A : b = \sum_{n=0}^{\infty}(e-a)^n$.

   $$\begin{aligned} ab &= a \lim_{m\to\infty} \sum_{n=0}^{m} (e-a)^n \\ &= (e-(e-a)) \lim_{m\to\infty} \sum_{n=0}^{m} (e-a)^n \\ &= \lim_{m\to\infty} \sum_{n=0}^{m} (e-a)^n - \lim_{m\to\infty} \sum_{n=0}^{m} (e-a)^{n+1} \\ &= \lim_{m\to\infty} (e-(e-a)^{m+1}) \\ &= 1 \end{aligned}$$

   Similarly, $ba = 1 \implies a \in \mathrm{G}(A)$.

2. To show: Given $a \in A$ $\exists\ \epsilon > 0 : ||a - b|| < \epsilon \implies b \in \mathrm{G}(A)$.
   Choose $\epsilon = ||a||^{-1} > 0$ ($\because a \in \mathrm{G}(A)$). Then,

$$||ba^{-1} - 1|| = ||a^{-1}(b - a)|| \leq ||a^{-1}||\ ||b - a|| \leq ||a^{-1}||\ ||a^{-1}||^{-1} = 1$$

   By above part, $ba^{-1} \in \mathrm{G}(A) \implies b \in \mathrm{G}(A)$.

□

Proposition 1.2.3 (Gelfand) A be a unital Banach algebra and $a \in A$.

1. $\sigma_A(a) \neq \phi$
2. $\sigma_A(a)$ is compact subset of $\{z \in \mathbb{C} : |z| \leq ||a||\}$

Proof. (Compactness)

1. **(Boundedness)**
   $|\lambda| > ||a|| \implies ||\lambda^{-1}a|| < 1 \implies e - \lambda^{-1}a \in G(A) \implies \lambda e - a \in G(A)$
   Hence, $\lambda \notin \sigma_A(a) \implies \sigma_A(a) \subseteq \{z \in \mathbb{C} : |z| \leq ||a||\}$
2. **(Closed)**
   Assume, $(\lambda_n)_n \subseteq \sigma_A(a) : (\lambda_n)_n \to \lambda$ in $\mathbb{C}$
   Continuity of Algebraic operations $\implies (a - \lambda_n e)_n \to (a - \lambda e)$ in A
   $(a - \lambda_n)_n \in A \backslash G(A)$ (which is closed as $G(A)$ is open in A)
   $\implies a - \lambda e \in G(A)$. Hence, $\lambda \in \sigma_A(a)$.

□

Proposition 1.2.4 (Beurling) A be a unital Banach Algebra and $a \in A$.

1. $(||a^n||^{1/n})$ is a decreasing sequence bounded by $||a||$.
2. $\lim\limits_{n \to \infty} ||a^n||^{1/n} = r(a) = \inf ||a^n||^{1/n}$
3. $r(a) \leq ||a||$

Proof.

1. Since $||a^n|| \leq ||a||^n$ holds in any normed algebra, the sequence is bounded by $||a||$.

2. By definition of $r(a), \exists\ m \in \mathbb{N} : ||a^m||^{1/m} < r(a) + \epsilon$.

   Now, write any $n = p(n)m + q(n)$, for some $0 \leq q(n) \leq m - 1$ which implies that

   $$\tfrac{p(n)}{n} = \tfrac{1}{m}(1 - \tfrac{q(n)}{m}) \to \tfrac{1}{m}$$

   as $n \to \infty$. Hence

   $$||a^n||^{1/n} \leq ||a^m||^{p(n)/n}||a||^{q(n)/n} \to ||x^m||^{1/m} < r(a) + \epsilon$$

   for all sufficiently large n.

3. $$\begin{aligned} \lambda \in \sigma_A(a) &\implies \lambda^n \in \sigma_A(a^n) \ \because (\lambda^n e - a^n) = (\lambda e - a)\ p(\lambda) \\ &\implies |\lambda^n| \leq ||a^n|| \\ &\implies |\lambda| \leq ||a^n||^{1/n} \end{aligned}$$

   $r(a) \leq \inf ||a^n||^{1/n} \leq \liminf_n ||a^n||^{1/n} \leq ||a||$

□

Proposition 1.2.8 If a is a self-adjoint element of unital C*-algebra A then $r(a) = ||a||$

Proof. $||a^2|| = ||a^*a|| = ||a||^2$. Inductively , we can show that

$$||a^{2^n}|| = ||a||^{2^n} \quad \forall n \in \mathbb{N}$$

$r(a) = \lim_{n\to\infty} ||a^{2^n}||^{\frac{1}{2^n}} = \lim_{n\to\infty} ||a|| = ||a||$ □

Corollary 1.2.9 An algebra *-homomorphism between two C*-algebras is always contractive.

Proof. Let, A and B be C*-algebras and $f : A \to B$ be an algebra *-homomorphism.

- f preserves invertible elements $\implies \sigma_B(f(a)) \subseteq \sigma_A(a)\ \forall a \in A$
- Let, $a \in A$ be self-adjoint. Then, $||f(a)|| = r(f(a)) \leq r(a) \leq ||a||$.
- For general element a in A. We have,

$$||f(a)||^2 = ||f(a)^* f(a)|| = ||f(a^* a)|| \leq ||a * a|| = ||a||^2.$$

  Hence, $||f(a)|| \leq ||a||\ \forall a \in A$.

□

Corollary 1.2.10 Norm on any C*-algebra A is unique.

Proof. Let, $a \in A$.We have,

$$||a|| = ||a^* a||^{1/2} = r(a^* a)^{1/2}$$

which is independent of the normed structure of A.

□

2 Projective Limit Algebra

2.1 Projective Topology

Let,$\{A_i| \ i \in I\}$ is a family of TVS. Let, A be a VS and $\{f_i : A \to A_i \mid i \in I\}$ is a family of linear maps.The *projective topology* on A is the coarsest topology on A w.r.t. to which f_i is continuous for every $i \in I$.
Local basis of A is given by

$$\{ \underset{i\in F}{\cap} f_i^{-1}(U_i) \mid U_i \text{ run over local basis of } A_i \ \forall i \in F, F \subseteq I \text{ Finite } \}$$

Examples

1. The Product topology on cartesian product of TVS is the *Projective Topology* w.r.t the projection maps.

2. The weak topology on Banach space A is the *Projective Topology* w.r.t. all linear functionals on A.

Proposition 2.1.1 (Notations as Above)
Projective topology on A w.r.t (A_i, f_i) makes it a TVS.

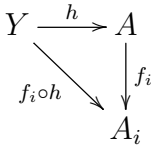
Proof.

1. The addition map $(x, y) \mapsto x + y$ is continuous.
   Assume, U be open in A and $x + y \in U$. Then $\underset{i\in F}{\cap} f_i^{-1}(U_i) \subseteq U$ where, U_i run over local basis of $A_i \ \forall i \in F, F \subseteq I$ finite.
   A_i is TVS $\implies \exists \ V_i$ open in $A_i : f_i(x) + f_i(y) \in V_i + V_i \subseteq U_i \ \forall i \in F$
   $\implies x + y \in \underset{i\in F}{\cap} f_i^{-1}(V_i + V_i) \subseteq U_i$

2. Similarly, the map $(\alpha, x) \mapsto \alpha x$ is continuous.

□

Lemma 2.1.2 A be a VS topologized with *projective topology* w.r.t $\{(A_i, f_i) \mid i \in I\}$ as above. Given a TVS Y and a linear map $h : Y \to A$, h is continuous $\iff$ $f_i \circ h$ is continuous $\forall i \in I$.



Proof.

1. Assume that $h : Y \to A$ is continuous. Given the *projective topology* on A, each f_i is continuous. Hence, $f_i \circ h$ is continuous $\forall i \in I$

2. Assume that $f_i \circ h$ is continuous $\forall i \in I$. Take U be any open neighbourhood of 0 in A. Then $\exists\ F \subseteq I$ Finite and U_i is local basis of $A_i\ \forall\ i \in F$ such that $\underset{i\in F}{\cap} f_i^{-1}(U_i) \subseteq U$. Therefore,

$$h^{-1}(U) \supseteq h^{-1}(\underset{i\in F}{\cap} f_i^{-1}(U_i)) = \underset{i\in F}{\cap} h^{-1}(f_i^{-1}(U_i)) = \underset{i\in F}{\cap} (f_i \circ h)^{-1}(U_i)$$

Continuity of $f_i \circ h(\forall i \in I)$ implies that $\underset{i\in F}{\cap} (f_i \circ h)^{-1}(U_i)$ is open neighbourhood of 0 in Y. Hence, $h^{-1}(U)$ is neighbourhood of 0 in Y.

□

Lemma 2.1.3 A be a VS with *projective topology* w.r.t $\{(A_i, f_i) \mid i \in I\}$ as above. A is Hausdorff space $\iff$ $\forall x \neq 0$ in $A\ \exists\ i \in I$ and U_i in local basis of A_i such that $f_i(x) \notin U_i$

Proof.

1. Suppose that A is Hausdorff space and $0 \neq x \in A$. Then $\exists$ open neighbourhood U of 0 such that $x \notin U$.
$\implies$ $\exists\ F \subseteq I$ finite and U_i in local basis of A_i $(\forall i \in F)$ such that $\underset{i\in F}{\cap} f_i^{-1}(U_i) \subseteq U \implies x \notin f_i^{-1}(U_i), \forall i \in F$
$\implies$ $f_i(x) \notin U_i\ \forall i \in F$.

2. Conversely, assume that $\forall x \neq 0$ in $A\ \exists\ i \in I$ and U_i in local basis of A_i such that $f_i(x) \notin U_i$. This implies, $x \notin f_i^{-1}(U_i)$ which is an open neighbourhood of 0 in A. Hence, A is Hausdorff space.

□

Proposition 2.1.4 Given an algebra A, a family of topological algebras $\{A_i \mid i \in I\}$ and algebra morphisms $\{f_i : A \to A_i \mid i \in I\}$, *projective topology* on A makes it a topological algebra.

Proof.

1. Proposition 2.1.1 $\implies$ *Projective topology* on A makes it a TVS and each f_i a continuous function.

2. Right and Left multiplication on A is continuous.

   (a) For $x \in A$, Right multiplication $r_x : A \to A$, $y \mapsto yx$

   $$(f_i \circ r_x)(y) = f_i(y)f_i(x) = r_{f_i(x)}(f_i(y)) = (r_{f_i(x)} \circ f_i)(y) \quad (1)$$

   $\forall\ i \in I, y \in A$

   (b) As $f_i(x) \in A_i$ and A_i is a TA,we get that $r_{f_i(x)} : A_i \to A_i$ is continuous and so $r_{f_i(x)} \circ f_i$ is continuous.
   (1) $\implies$ $f_i \circ r_x$ is continuous $\forall i \in I$ and so Lemma 2.1.2 $\implies$ r_x is continuous.

   Similarly, we can obtain that left multiplication is continuous on A .

□

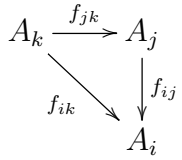
Lemma 2.1.5 Given a family of topological algebras $\{A_i \mid i \in I\}$, the Cartesian product set $A = \prod_{i \in I} A_i$ can be made a topological algebra.

Proof. Consider, the canonical projection map $\pi_i : A \to A_i$.
Proposition 2.1.4 $\implies$ *Projective topology* on A w.r.t. $\{(A_i, \pi_i) \mid i \in I\}$ makes it a topological algebra and each π_i a continuous function.
The *Projective topology* on A is actually the product topology. □

2.2 Projective System of TAs and Projective Limit

Definition 2.2.1 Assume that we have a family $\{A_i \mid i \in I\}$ of algebras , where $(I, \leq)$ is a directed set. Also , we have a family of algebra morphsims $\{f_{ij} : A_j \to A_i \mid i \leq j \text{ in } I\}$ such that

- $f_{ij} \circ f_{jk} = f_{ik} \quad \forall\ i \leq j \leq k \text{ in } I$ .....(2.1)
  i.e. the diagram below commutes.



- $f_{ii} = id_{A_i}$(identity map on A_i) $\quad \forall\ i \in I$ .....(2.2)

The family $\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$, A_i and f_{ij} as above satisfying (2.1) and (2.2), is called **Projective System of Algebras**.

Definition 2.2.2 Given a projective system of algebras

$$\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$$

Consider the Cartesian Product algerba $X = \prod_{i \in I} A_i$ and its subset

$$A = \{(x_i) \in X \mid f_{ij}(x_j) = x_i \ \ \forall i \leq j \text{ in } I\ \}.$$

A is called *the* projective limit of algebras, denoted by, $\varprojlim(A_i, f_{ij}, I)$.

Remark 2.2.3 Under the algebra operations of X , A is subalgebra of X. For any $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in A$, scalars α , β and $i \leq j$ in I

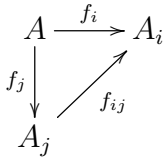
- $\alpha x + \beta y = (\alpha x_i + \beta y_i)_{i \in I}$ which belongs to X and
  $f_{ij}(\alpha x_j + \beta y_j) = \alpha f_{ij}(x_j) + \beta f_{ij}(y_j) = \alpha x_i + \beta y_i \implies \alpha x + \beta y \in A$
- $xy = (x_i y_i)_{i \in I} \in X$ and
  $f_{ij}(x_j y_j) = f_{ij}(x_j) f_{ij}(y_j) = x_i y_i \implies xy \in A$

Given a projective system of algebras $\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$,we define a family $\{f_i \mid i \in I\}$ of algebra morphisms as

$$f_i = \pi_i|_A \; A :\to A_i, \forall i \in I \; ,$$

where π_i the canonical projection map of $X = \prod_{i \in I} A_i$ onto A_i. Note that

1. f_i need not be an onto map.
2. $f_{ij} \circ f_j = f_i \;\; \forall i \leq j$ in I. i.e. The following diagram commutes.



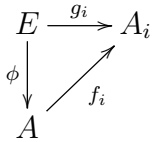
Proposition 2.2.4 Projective system of algebras $\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$ with projective limit A has the following universal property:
Given an algebra E and a family of algebra morphisms

$$\{g_i : E \to A_i \mid i \in I \; \} \text{ with } f_{ij} \circ g_j = g_i \; \forall i \leq j \text{ in } I$$

There exists a unique homomorphism

$$\phi : E \to A \text{ such that } g_i = f_i \circ \phi \; \forall i \in I$$

,i.e. the following diagram commutes.



Proof. Define $\phi : E \to A$ as $\phi(x) = (g_i(x))_{i \in I}$

1. Existence of ϕ :
   (a) $(g_i(x))_{i \in I} \in A \; \forall \; x \in E$ because
   $f_{ij}(g_j(x)) = (f_{ij} \circ g_j)(x) = g_i(x) \forall i \leq j$ in I
   (b) ϕ is well defined as each g_i is well defined.
   (c) ϕ is homomorphism as each g_i is homomorphism and the operations on A are pointwise.

(d) $(f_i \circ \phi)(x) = f_i(\phi(x)) = f_i(g_i(x)_{i\in I}) = g_i(x)\ \forall i \in I, x \in E$

2. Uniqueness of ϕ :
   Given homomorphism $\rho : E \to A$ such that $g_i = f_i \circ \rho\ \forall i \in I$, we get,

$$\phi(x) = (g_i(x))_{i\in I} = (f_i(\rho(x)))_{i\in I} = ((\rho(x))_i)_{i\in I} = \rho(x)$$

$\implies \phi \equiv \rho$

□

Lemma 2.2.5 Projective Limit A of a projective system of algebras $\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$ is unique upto unique isomorphism.

Proof. Let, A_1 and A_2 be projective limits of given projective system.
By the universal property, $\exists$ homomorphisms

$$\phi : A_1 \to A_2 \text{ and } \rho : A_2 \to A_1$$

$\implies \phi \circ \rho : A_2 \to A_2$ is a homomorphism but $id_{A_2} : A_2 \to A_2$ is one such homomorphism satisfying the required conditions.
Uniqueness of homomorphism implies that

$$\phi \circ \rho = id_{A_2}$$

Similarly, $\rho \circ \phi = id_{A_1}$.
Hence, A_1 and A_2 are isomorphic and the isomorphism is unique. □

Definition 2.2.6 (1)

1. Projective System of Topological Algebras is a projective system of algebras $\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$, where each A_i is topological algebra and each f_{ij} $(i \leq j)$ is a continuous algebra morphism.

   Similarly, we can define a Projective system of Topological *-algebras.

2. *The* Projective Limit, say A, of projective system of TAs

$$\{(A_i, f_{ij}) \mid i \leq j \text{ in } I\}$$

is *the* Projective Limit E of associated projective system of algebras and topology on A is *projective topology* w.r.t.$\{(A_i, f_i) \mid f_i = \pi_i|_A \ i \in I\}$.

Example 2.2.6 (2)

1. $\mathbb{C}^n$ is a topological*-algebra with

$$||(x_1, x_2, ..., x_n)|| = \max \{|x_1|, |x_2|, ..., |x_n|\}$$

2. $(\mathbb{N}, \leq)$ is a directed set with natural ordering.

3. $\forall n \leq m$ in $\mathbb{N}$, define the map $f_{nm} : \mathbb{C}^m \to \mathbb{C}^n$ as

$$f_{nm}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_m) = (x_1, x_2, ..., x_n)$$

   which is continuous algebra *-homomorphism.

4. $\forall n \leq m \leq p$ in $\mathbb{N}$, $f_{np} = f_{nm} \circ f_{mp}$ and $f_{nn} = id_{\mathbb{C}^n}$

Hence, $(\mathbb{C}^n, f_{nm}, \mathbb{N})$ forms a projective system of C*-algebras.

Remark 2.2.7 (Notations as above)

1. Note that as each f_i is an algebra morphism, Proposition 2.1.4 implies that A is a topological algebra endowed with *Projective topology.*

2. Projective system of TAs satisfies the following universal property: Given an algebra A and a family of *continuous* algebra morphisms

$$\{g_i : A \to A_i \mid i \in I \} \text{ such that } f_{ij} \circ g_j = g_i \ \ \forall \ i \leq j \text{ in } I$$

   There exists a unique *continuous* homomorphism

$$\phi : E \to A \text{ such that } g_i = f_i \circ \phi \ \forall i \in I$$

   In Proposition 2.2.4, we have shown that

   $\phi : E \to A$ defined by $\phi(x) = (g_i(x))_{i \in I}$ is the unique homomorphism.

   Consider, $U = \bigcap\limits_{i \in F} f_i^{-1}(U_i), F \subseteq I$ Finite and U_i in local basis of A_i

$$\phi^{-1}(U) = \phi^{-1}(\bigcap_{i\in F} f_i^{-1}(U_i)) = \bigcap_{i\in F} \phi^{-1}(f_i^{-1}(U_i)) = \bigcap_{i\in F} g_i^{-1}(U_i)$$

But $\bigcap_{i\in F} g_i^{-1}(U_i)$ is open in E as each g_i is continuous.
Hence, $\phi^{-1}(U)$ is open in E implying the continuity of ϕ .

3. Projective Limit A is unique upto unique isomorphism which follows from Lemma 2.2.5 .

4. If each A_i is Hausdorff then A is Hausdorff.

   For any $a = (a_i)_{i\in I} \neq 0$ in A, there exists some $i \in I$ such that $a_i \neq 0$. $a_i = f_i(a)$ implies $\exists\ i \in I$ such that $a_i = f_i(a) \neq 0$. As A_i is Hausdorff implies $\exists\ U_i$ in local basis of A_i such that $f_i(a) \notin U_i$.
   Lemma 2.1.3 $\implies$ A is Hausdorff.

5. If each A_i is Hausdorff TA then A is closed subalgebra of $X = \prod_{i\in I} A_i$.

   Proof.

   (a) Remark 2.2.3 $\implies$ A is subalgebra of X.

   (b) Let, $a = (a_i) \in \overline{A}$ $\implies$ $\exists$ a net(a_α) in $A : (a_\alpha)_\alpha \to a$, where each a_α is of the form $(a_{\alpha i})_{i\in I}$ in A.

      i. f_j is continuous
      $\implies$ $(f_j(a_\alpha))_\alpha \to f_j(a)$ $\implies$ $(a_{\alpha j})_\alpha \to a_j$ (1)

      ii. $\forall i \leq j$ in I, f_{ij} is continuous. Therefore,
      $(f_{ij}(a_{\alpha j}))_\alpha \to f_{ij}(a_j)$
      $\implies$ $(a_{\alpha i})_\alpha \to f_{ij}(a_j)$ (2)

   As A is Hausdorff, (1) and (2) implies that $\forall i \leq j$ in I, $f_{ij}(a_j) = a_i$.
   Hence, $a \in A$. Therefore, A is closed.

   □

6. If each A_i is complete Hausdorff TA then so is A.

Proof. Each A_i is complete, therefore, X is complete. (Cartesian Product of complete spaces is complete under product topology).
A, being closed subalgebra of X, is complete. □

3 Pro C*-algebra

3.1 Basic Properties of Pro C*-algebra

Definition 3.1.1 (Pro C*-algebra)
A Pro C*-algebra is a complete Hausdorff Topological $*$-algebra endowed with the Projective topology with respect to a family of C*-seminorms.

Examples 3.1.2

1. $\mathbb{C}^{\mathbb{N}}$ under pointwise operations and * as conjugation. With the family of C*-seminorms $(p_n)_{n\in\mathbb{N}}$ given by

$$p_n(x_1, x_2, ...) = \max\{|x_1|, |x_2|, ..., |x_n|\}$$

   $\mathbb{C}^{\mathbb{N}}$ becomes a Pro C*-algebra.

2. C($\mathbb{N}$) under pointwise operations and * as conjugation. With the family of C*-seminorms $(q_n)_{n\in\mathbb{N}}$ given by

$$q_n(f) = \max\{|f(1)|, |f(2)|, ..., |f(n)|\}\ \forall f \in C(\mathbb{N})$$

   C($\mathbb{N}$) becomes a Pro C*-algebra.

3. C($\mathbb{R}$) under pointwise operations and * as conjugation. With the family of C*-seminorms $(p_K)_K$ (K is compact in $\mathbb{R}$) given by

$$p_K(f) = \sup\{|f(x)| : x \in K\}$$

   C($\mathbb{R}$) becomes a Pro C*-algebra.

We will now see one important characterization of Pro C*-algebra using Projective limit and Projective system of Topological *-algebra.

Proposition 3.1.3 A topological $*$-algebra E is pro C*-algebra if and only if it is the projective limit of projective system of C*-algebras and continuous $*$-homomorphisms.

Proof.
Let E be a pro C*-algebra and $(p_i)_i$ be family of C*-seminorms.
(Topology on E is the projective topology w.r.t $(p_i)_i$). Take

$$E_i := E/p_i^{-1}\{0\} = \{x_i \mid x_i = x + p_i^{-1}\{0\} \ \forall \ x \in E\}$$

1. E_i is C*-algebra $\forall i \in I$. Define

$$||.|| : E_i \to \mathbb{R} \text{ by } ||x_i|| = ||x + p_i^{-1}\{0\}|| = p_i(x).$$

   (a) $||.||$ is an algebra norm on E_i.

   (b) **Claim:** E_i is complete.
   Assume,$(x_n + p_i^{-1}\{0\})_n$ $(x_n \in E)$ be Cauchy sequence in E_i.
   Given $\epsilon > 0$ $\exists$ N such that $\forall n, m \geq N$

$$\begin{aligned} ||(x_n + p_i^{-1}\{0\}) - (x_m + p_i^{-1}\{0\})|| \leq \epsilon &\implies ||x_n - x_m + p_i^{-1}\{0\}|| < \epsilon \\ &\implies p_i(x_n - x_m) < \epsilon \\ &\implies x_n - x_m \in p_i^{-1}[0, \epsilon) \\ &\implies (x_n) \text{ is Cauchy in } E \\ &\implies \exists\ x \in E \text{ such that } (x_n)_n \to x \\ &\implies \exists\ M : x_n - x \in p_i^{-1}[0, \epsilon)\ \forall n \geq M \\ &\implies ||(x_n + p_i^{-1}\{0\}) - (x + p_i^{-1}\{0\})|| < \epsilon \\ &\implies (x_n + p_i^{-1}\{0\})_n \to x + p_i^{-1}\{0\} \text{ in } E_i \end{aligned}$$

   (c) Define *: $E_i \to E_i$ by $x_i^* = x^* + p_i^{-1}\{0\}$
   * is continuous because, $||x_i^* - y_i^*|| = p_i(x-y)$ and p_i is continuous.

   (d) $||.||$ is a C*-norm.

      i. $||x_i^*|| = ||x^* + p_i^{-1}\{0\}|| = p_i(x^*) = p_i(x) = ||x_i||$

      ii. $||x_i x_i^*|| = p_i(xx^*) = (p_i(x))^2 = ||x_i||^2$

2. Define partial order $\leq$ on I as $i \leq j \iff p_i(x) \leq p_j(x)\ \forall x \in E$. Then, $(I, \leq)$ is a directed set.

3. Let $i \leq j$. $f_{ij} : E_j \to E_i$ as $f_{ij}(x_j) = x_i$ is a well defiend continuous algebra homomorphism.

   (a) $\forall i \leq j \leq k$ in I, $f_{ij} \circ f_{jk} = f_{ik}$ and $f_{ii} = id_{E_i}$

(b) $\forall i \leq j, f_{ij}$ is a $*$-homomorphism.

$$f_{ij}(x_j^*) = f_{ij}(x^* + p_j^{-1}\{0\}) = x^* + p_i^{-1}\{0\} = (f_{ij}(x_j))^*$$

(c) $f_i : \varprojlim E_i \to E_i$ given by $f_i((x_i)_i) = x_i$ (Projection map)

$f_{ij} \circ f_j((x_i)_i) = f_{ij}(f_j(x_i)_i) = f_{ij}(x_j) = x_i = f_i((x_i)_i)\ \forall i \leq j$ in I

$\implies f_{ij}f_j = f_i\ \forall\ i \leq j$ in I

4. (E_i, f_{ij}, I) forms a projective system of C*-algebras and $*$-homomorphisms. Denote $\varprojlim(E_i, f_{ij}, I)$ by $\varprojlim E_i$. Then

$$\varprojlim E_i = \{(x_i)_i \in \textstyle\prod_i E_i \mid f_{ij}(x_j) = x_i \forall\ i \leq j \text{ in } I\}$$

(a) Since each E_i is complete Hausdorff TA, Remark 2.2.7 (6) $\implies$ $\varprojlim E_i$ is complete Hausdorff TA

(b) Define involution on $\varprojlim E_i$ as pointwise invoution of E_i's, i.e. $(x_i)_i^* = (x_i^*)_i$ which is a continuous function because each involution on E_i is continuous.

$\implies \varprojlim E_i$ is complete Hausdorff topological *-algebra.

5. Consider $\rho_i : E \to E_i$ given by $\rho_i(x) = x_i$ which is a well defined surjective continuous open *-homomorphism.

6. Define $\phi : E \to \varprojlim E_i$ as $\phi(x) = (\rho_i(x))_i$

(a) $(\rho_i(x))_i \in \varprojlim E_i$ because $f_{ij}(\rho_j(x)) = f_{ij}(x_j) = x_i = \rho_i(x)$

(b) ϕ is well defined because each ρ_i is.

(c) ϕ is $*$-homomorphism because each ρ_i is $*$-homomorphism and algebraic operations are pointwise.

(d) ϕ is injective, continuous and open map.

(e) $\phi : E \to \phi(E)$ is an isomorphism of topological $*$-algebra. Hence, $\phi(E)$ is a closed vector subspace of $\varprojlim(E_i, f_{ij}, I)$.

$$\overline{\phi(E)} = \varprojlim(\overline{f_i(\phi(E))}, f_{ij}|_{\overline{f_i(\phi(E))}}, I)$$

But , we have

$$f_i(\phi(E)) = \rho_i(A) = E_i \implies \overline{f_i(\phi(E))} = \overline{E_i} = E_i$$
$$\implies \overline{\phi(E)} = \phi(E) = \varprojlim(E_i, f_{ij}, I)$$
$$\implies A \cong \varprojlim(E_i, f_{ij}, I)$$

Conversely, Let E be a projective limit of projective system (E_i, f_{ij}, I) of C*-algebras and $*$-homomorphisms.

1. Remark 2.2.7 (6) $\implies$ E endowed with *projective topology* (which is subspace topology induced by product topology on $\prod_i E_i$) is a complete Hausdorff TA algebra.

2. Define involution on E as pointwise invoution of E_i's, i.e. $(x_i)_i^* = (x_i^*)_i$ which is continuous because each involution on E_i is continuous.

3. Define $p_i : E \to E_i$ as

$$p_i((x_i)_i) = ||x_i||_i$$

where $||.||_i$ is C*-norm on $E_i \ \forall \ i \in I$

(a) p_i is a C* semi-norm.

i. $p_i((x_i)_i) = ||x_i||_i \geq 0$

ii. Sub-additivity

$$\begin{aligned} p_i((x_i)_i + (y_i)_i) &= p_i((x_i + y_i)_i) \\ &= ||x_i + y_i||_i \\ &\leq ||x_i||_i + ||y_i||_i \\ &= p_i((x_i)_i) + p_i((y_i)_i) \end{aligned}$$

iii. $p_i(\lambda(x_i)_i) = p_i((\lambda x_i)_i) = ||\lambda x_i||_i = |\lambda| \ ||x_i||_i = |\lambda| p_i((x_i)_i)$

iv. $(x_i)_i = 0 \implies x_i = 0 \ \forall \ i \in I \implies ||x_i||_i = 0$
$\implies p_i((x_i)_i) = 0$

v. Sub-multiplicaitve

$$\begin{aligned} p_i((x_i)_i(y_i)_i) &= p_i((x_i y_i)_i) \\ &= ||x_i y_i||_i \\ &\leq ||x_i||_i ||y_i||_i \\ &= p_i((x_i)_i) p_i((y_i)_i) \end{aligned}$$

vi. *-homomorpshim

$$p_i((x_i)_i^*) = p_i((x_i^*)_i) = ||x_i^*|| = ||x_i||_i^* = p_i((x_i)_i)^*$$

vii. C*-condition

$$p_i((x_i)_i^*(x_i)_i) = p_i((x_i^*x_i)_i) = ||x_i^*x_i|| = ||x_i||_i^2 = p_i((x_i)_i)^2$$

(b) $p_i(x) = ||f_i(x)||_i$ where $f_i(x) = x_i$(projection into i-th coordinate)
As $||.||_i$ and f_i are continuous, therefore, p_i is continuous.

(c) *Projective topology* on E w.r.t $(f_i)_i$ and $(p_i)_i$ coincide.
Because , $p_i^{-1}[0,\epsilon) = f_i^{-1}(||.||^{-1}[0,\epsilon))\ \forall \epsilon > 0$
which means that local bases of E under both the projective topologies coincide.

Hence, E is complete hausdorff topological *-algebra whose topology is determined by a family of continuous C*semi-norms $(p_i)_{i\in I}$. □

Remark 3.1.4 (Notations as above in the proof)

1. E is unital $\iff$ each E_i is unital.

   Proof. Assume that u be the unity in E.
   For all $i \in I$ take $u_i = f_i(u)$
   $\implies u_ix_i = f_i(u)f_i(x) = f_i(ux) = f_i(x) = x_i$
   Similarly, we get $x_iu_i = x_i \implies$ each E_i is unital.

   Conversely, Suppose for each $i \in I, u_i$ is the unity in E_i.
   $u = (u_i)_i \in E = \varprojlim(E_i, f_{ij}, I)$ because $\forall i \le j$ in I and x_i in E_i,

$$\begin{aligned} x_if_{ij}(u_j) &= \rho_i(x)f_{ij}(u_j) \\ &= f_{ij}(\rho_j(x))f_{ij}(u_j) \\ &= f_{ij}(\rho_j(x)u_j) \\ &= f_{ij}(x_ju_j) \\ &= f_{ij}(x_j) = x_i \end{aligned}$$

$\implies f_{ij}(u_j) = u_i$.
Hence, u is the unity in E since the multiplication in E is pointwise and u_i is unity of each E_i. □

2. Suppose E is unital.Then :
$x \in E$ is invertible $\iff$ each $\rho_i(x) = x_i$ is invertible in E_i.

Proof. Let u be the unity in E and $x \in E$ is invertible.
$\implies$ $\exists\ y \in E$ such that $xy = yx = u$.
As $\rho_i(u) = u_i$ is unity in E_i, we have

$$x_i y_i = \rho_i(xy) = \rho_i(u) = u_i.$$

Similarly, $y_i x_i = u_i \implies x_i$ is invertible in $E_i\ \forall\ i \in I$

Conversely, Let $x \in E$ such that each x_i is invertible in E_i.

$$\forall\ i \in I\ \exists\ y_i \in E_i \text{ such that } x_i y_i = y_i x_i = u_i.$$

Now $y := (y_i)_i \in E = \varprojlim(E_i, f_{ij}, I)$ because $\forall\ i \leq j$ in I,

$$\rho_i(x) f_{ij}(y_j) = f_i j(\rho_j(x)) f_{ij}(y_j) = f_{ij}(x_j y_j) = f_{ij}(u_j) = u_i$$

Uniqueness of inverse $\implies f_{ij}(y_j) = y_i$
Since multiplication is pointwise, we have that $xy = yx = u$
Hence, $x \in E$ is invertible □

3.2 Unitization and Spectrum

Definition 3.2.1 E be a pro C*-algebra .Then its unitization $E^+ = E \oplus \mathbb{C}$ is the unitization of E considered as *-algebra.

Proposition 3.2.2 If E is a pro C*-algebra then E^+ is a pro C*-algebra.

Proof. Let E be a pro C*-algebra. By Proposition 3.1.3, there exists an topological*-isomorphism

$$\phi : E \to \varprojlim(E_i, f_{ij}, I) \text{ given by } \phi(x) = (\rho_i(x))_i$$

1. $\forall\ i \in I, E_i$ is a C*-algebra, 1.4.2 $\implies$ $E_i^+ = E_j \oplus \mathbb{C}$ is a C*-algebra.

2. Define $f_{ij}^+ : E_j^+ \to E_i^+$ as

$$f_{ij}^+(x, \lambda) = (f_{ij}(x), \lambda)\ \forall x \in E_i, \lambda \in \mathbb{C}, i \leq j \text{ in } I$$

   (a) f_{ij}^+ is a well defined map because f_{ij} is a well defined map.

   (b) f_{ij}^+ is *-homomorphism.

      i. Vector addition and scalar multiplication are pointwise in E_i^+. So, f_{ij}^+ is a linear map as it is linear in each component.

      ii. f_{ij}^+ is multiplicative.

$$\begin{aligned} f_{ij}^+((x, \lambda)(y, \beta)) &= f_{ij}^+(xy + \beta x + \lambda y, \lambda\beta) \\ &= (f_{ij}(xy + \beta x + \lambda y, \lambda\beta), \lambda\beta) \\ &= (f_{ij}(x)f_{ij}(y) + \beta f_{ij}(x) + \lambda f_{ij}(y), \lambda\beta) \\ &= (f_{ij}(x), \lambda)(f_{ij}(y), \beta) \\ &= f_{ij}^+(x, \lambda) f_{ij}^+(y, \beta) \end{aligned}$$

      iii. f_{ij}^+ is *-homomorphism.

$$\begin{aligned} f_{ij}^+((x, \lambda)^*) &= f_{ij}^+(x^*, \overline{\lambda}) \\ &= (f_{ij}(x^*), \overline{\lambda}) \\ &= (f_{ij}(x)^*, \overline{\lambda}) \\ &= (f_{ij}(x), \lambda)^* \\ &= (f_{ij}^+(x, \lambda))^* \end{aligned}$$

   (c) f_{ij}^+ is a continuous on E_i^+ because E_i^+ is endowed with product topology and each component of f_{ij}^+ is a continuous function.

   (d) $\forall\ i \leq j \leq k$ in I,

      i. $f_{ij}^+(f_{jk}^+(x, \lambda)) = (f_{ij}(f_{jk}(x)), \lambda) = (f_{ik}(x), \lambda) = f_{ik}^+(x, \lambda)$

      ii. $f_{ii}^+(x, \lambda) = (f_{ii}(x), \lambda) = (x, \lambda)$

Hence, (E_i^+, f_{ij}^+, I) forms a projective system of C*-algebras and *-homomorphisms.

3.

$$\varprojlim E_i^+ = \{(x,\lambda) = (x_i,\lambda_i)_i \in \prod_{i\in I} E_i^+ \mid f_{ij}^+(x_j,\lambda_j) = (x_i,\lambda_i)\ \forall\ i \leq j \in I\}$$
$$= \{(x,\lambda) = (x_i,\lambda_i)_i \in \prod_{i\in I} E_i^+ \mid (f_{ij}(x_j),\lambda_j) = (x_i,\lambda_i)\ \forall\ i \leq j \in I\}$$
$$= \{(x,\lambda) = (x_i,\lambda_i)_i \in \prod_{i\in I} E_i^+ \mid f_{ij}(x_j) = x_i, \lambda_j = \lambda_i\ \forall\ i \leq j \in I\}$$

Since for any $(x_i,\lambda_i)_i \in \varprojlim E_i^+$ and for any $i,j \in I\ \exists k \in I$ such that $i \leq k$ and $j \leq k \implies \lambda_i = \lambda_k = \lambda_j$

$$\implies \varprojlim E_i^+ = \{(x_i,\lambda)_i \in \textstyle\prod_{i\in I} E_i^+ \mid (x_i)_i \in \varprojlim E_i,\ \lambda \in \mathbb{C}\}$$

(a) Topology on $\varprojlim E_i^+$ is *projective topology* w.r.t.$(f_i^+)_{i\in I}$ where $f_i^+(x_i,\lambda)_i = (x_i,\lambda)$

(b) $f_{ij}^+(f_j^+(x_i,\lambda)_i) = f_{ij}^+(x_j,\lambda) = (x_i,\lambda) = f_i^+(x_i,\lambda)_i$

4. The map

$$\psi : \varprojlim E_i^+ \to \varprojlim E_i \oplus \mathbb{C} \text{ as } \psi(x_i,\lambda)_i = ((x_i)_i,\lambda)$$

defines a topogical $*$-isomorphism. Hence,

$$\varprojlim E_i^+ \cong \varprojlim E_i \oplus \mathbb{C} \cong E \oplus \mathbb{C} = E^+$$

□

Definition 3.2.3 Given a unital pro C*-algebra E and $x \in E$, spectrum of E is defined as $\sigma(x)$=$\{\lambda \in \mathbb{C} \mid \lambda - x$ is not invertible$\}$.

If E is not unital, then the spectrum is taken w.r.t E^+.

Example 3.2.4 $\mathbb{C}^{\mathbb{N}} = \varprojlim \mathbb{C}^n$ and $x = (1,2,3,4,.....)$ Then

$$\sigma(x) = \{1,2,3,4,...\} = \mathbb{N} = \bigcup_{n\in\mathbb{N}} \sigma_{\mathbb{C}^n}(1,2,3...,n)$$

which shows that $\mathbb{C}^{\mathbb{N}}$ is not a C*-algebra but a Pro C*-algebra.

The natural question arises that under what conditions a Pro C*-algebra will be a C*-algebra.

Lemma 3.2.5 Suppose that $E = \varprojlim(E_i, f_{ij}, I)$. Then for every $x \in E$, we have

$$\sigma(x) = \underset{i\in I}{\cup}\sigma(\rho_i(x))$$

where $\rho_i : E \to E_i$ is the canonical map.

Proof.

1. $\lambda \in \sigma(x) \implies \lambda - x$ is not invertible in E.
   3.1.5 (2) $\implies$ $\exists\ i \in I : \rho_i(\lambda - x) = \lambda - \rho_i(x)$ is not invertible in E_i.
   $\implies \lambda \in \sigma(\rho_i(x)) \implies \sigma(x) \subseteq \cup_{i\in I}\ \mathrm{sp}(\rho_i(x))$

2. Conversely, $\lambda \in \cup_{i\in I}\ \mathrm{sp}(\rho_i(x)) \implies \lambda \in \mathrm{sp}(\rho_i(x))$ for some $i \in I$.
   $\implies \rho_i(\lambda - x)$ is not invertible in E_i
   3.1.5 (2) $\implies \lambda - x$ is not invertible in $E \implies \lambda \in \mathrm{sp}(x)$
   $\implies \mathrm{sp}(x) \supseteq \cup_{i\in I}\ \mathrm{sp}(\rho_i(x))$

□

Proposition 3.2.4 A pro C*-algebra E is isomorphic to C*-algebra if and only if $G(E)$ is open in E.

Proof. Assume that E is unital pro C*-algebra endowed with *projective topology* w.r.t C*-seminorms $(p_i)_{i\in I}$.

1. Let, $G(E)$ is open in E. Then $\exists\ i \in I, \epsilon > 0$:

$$U = \{x \in E \mid p_i(x - 1) < \epsilon\} \subseteq G(E).$$

2. Consider, $x \in p_i^{-1}\{0\}$ such that $x \neq 0$.

   (a) E is Hausdorff, so $\exists\ j \in I : p_j(x) \neq 0$. Then

$$\rho_j(x) = x_j = x + p_j^{-1}\{0\} \neq 0 \implies x_j^* \neq 0$$

(b) $x_j^* x_j \neq 0$ because if , $x_j^* x_j = 0$ then

$x^*x \in p_j^{-1}\{0\} \implies p_j(x^*x) = 0 \implies p_j(x)^2 = 0 \implies p_j(x) = 0$

which is a contradiction.

(c) As E_j is a C*-algebra then

$$\text{spectral radius of } x_j^* x_j = ||x_j^* x_j|| > 0$$

Since, $\sigma(x_j^* x_j) \subseteq [0, \infty)$, therefore,

$$\exists\ \lambda > 0 \text{ in } \sigma(x_j^* x_j).$$

Since $\sigma(x_j^* x_j) \subseteq \sigma(x^*x) \implies 0 < \lambda \in \sigma(x^*x)$

(d) $1 - \lambda^{-1} x^*x$ is not invertible in E which can not be true as $x^*x \in U$ because $p_i((1 - \lambda^{-1} x^*x)) - 1) = p_i(\lambda^{-1} x^*x) = 0 < \epsilon$

Hence, $p_i^{-1}\{0\} = 0$

3. Let j in $I : j \geq i$ and consider, $f_{ij} : E_j \to E_i$ given by $f_{ij}(x_j) = x_i$

(a) f_{ij} is a well defined continuous surjective *-homomorphism.

(b) f_{ij} is injective. Take $x_i = 0$
$\implies x + p_i^{-1}\{0\} = 0 \implies x \in p_i^{-1}\{0\} \implies x = 0 \implies x_j = 0.$

(c) Since continuous bijection between C*-algebras is an isometry, hence, Homeomorphism. So, f_{ij} is a topological *-isomorphism.

4. Isometry implies that $||f_{ij}(x_j)|| = ||x_i||$. Hence ,

$$\forall x \in E, p_i(x) = p_j(x) \implies i = j$$

So, i is the upper bound of elements in I, i.e. $j \leq i\ \forall j \in I$

5. We have that $f_i : E \to E_i$ is a continuous *-isomorphism.
Assume, U be open in E. Then $\exists j$ in $I : p_j^{-1}[0, \epsilon) \subseteq U$ But $j \leq i$
$\implies p_i^{-1}[0, \epsilon) \subseteq U \implies ||.||^{-1}[0, \epsilon) \subseteq f_i(U) \implies f_i$ is an open map.

$\implies E \cong E_i \implies E$ is isomorphic to a C*-algebra.

Let, E be non-unital pro C*-algebra with the set of units open in E^+.
$\implies E^+$ a C*-algebra. Hence, E is also a C*-algebra as E is closed in E^+.

Conversely, In every C*-algebra, group of invertible elements is open. □

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