Pro C*-algebras

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Overview

1. C*-algebra : Definition and Examples

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3. C*-algebra VS Pro C*-algebra

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6. Spectrum of Pro C*-algebra

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7. Pro C*-algebra $\overset{?}{\Longleftrightarrow}$ C*-algebra

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Topological *-algebra

- An algebra is a vector space with a ring structure.
- An involutive algebra A with Hausdorff topology such that
  1. $(a, b) \mapsto a + b$
  2. $(a, b) \mapsto ab$
  3. $(\alpha, a) \mapsto \alpha a$
  4. $a \mapsto a^*$ $\qquad \forall a, b \in A, \alpha \in \mathbb{C}$

  are continuous.

C*-algebra

Complete Hausdorff topological*-algebra A whose topology is induced by the C*- norm.

C*-algebra

- It has a norm $||.|| : A \to \mathbb{R}$

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Examples

1. $(\mathbb{C}^n, ||.||_\infty)$ under coordinatewise operations and conjugation .
2. $(C(X), ||.||_\infty)$ for compact Hausdorff space X, under pointwise operations and conjugation.
3. $(B(\mathcal{H}), ||.||_{op})$, the space of bounded linear operators on a Hilbert space.

Pro C*-algebra

Complete Hausdorff topological*-algebra A whose topology is induced by a family of C*- seminorms $(p_i)_{i\in I}$.

Pro C*-algebra

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- $p_i(a^*a) = p_i(a)^2$ (C*-condition)
- A has the norm topology which is the weakest topology such that each p_i function is continuous.

Examples of Pro C*-algebra

1. $\mathbb{C}^{\mathbb{N}}$ with the family of C*- seminorms $(p_n)_{n\in\mathbb{N}}$ given by
$$p_n(x_1, x_2, \dots) = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

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2. $C(\mathbb{R})$ with the family of C*- seminorms $(p_K)_K$ (K is compact in $\mathbb{R}$) given by
$$p_K(f) = \sup\{|f(x)| : x \in K\}$$

Spectrum of an algebra with unity

For an algebra A with unity e, the spectrum of $a \in A$ is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} \ : \ \lambda e - a \text{ has no inverse in } A\}$$

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Example: The algebra, $C(X)$. Let, $f \in C(X)$ and $\lambda \notin f(X) \iff \frac{1}{\lambda - f(x)}$ is continuous $\iff \lambda - f$ is invertible. Hence,

$$\sigma(f) = f(X)$$

$C[0,1]$ VS $C(\mathbb{R})$

$C[0,1]$

- Spectrum
  $\sigma(id) = [0,1]$
- Spectrum is compact in $\mathbb{C}$

$C(\mathbb{R})$

- Spectrum
  $\sigma(id) = \mathbb{R}$
- Spectrum is not compact in $\mathbb{C}$

$C(\mathbb{R})$ **is not a C*-algebra** as the spectrum fails to be compact.

$\mathbb{C}^n$ VS $\mathbb{C}^{\mathbb{N}}$

$\mathbb{C}^n$

- Spectrum
  $\sigma(1, 2, \ldots, n) = \{1, 2, \ldots, n\}$
- Spectrum is compact in $\mathbb{C}$
- Invertible elements : $(\mathbb{C} - \{0\})^n$
- Set of invertible elements is open in $\mathbb{C}^n$.

$\mathbb{C}^{\mathbb{N}}$

- Spectrum
  $\sigma(1, 2, \ldots) = \{1, 2, \ldots\}$
- Spectrum is not compact in $\mathbb{C}$
- Invertible elements : $(\mathbb{C} - \{0\})^{\mathbb{N}}$
- Set of invertible elements is not open in $\mathbb{C}^{\mathbb{N}}$.

C*-algebra and Pro C*-algebra

- Every C*-algebra is a Pro C*-algebra.

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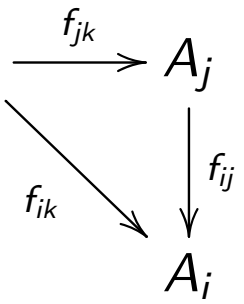
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Under what conditions a Pro C*-algebra is (isomorphic to) a C*-algebra?

Projective System of C*-algebras

1. $(I, \leq)$ is a directed set[1].
2. $\{A_i \mid i \in I\}$ is a family of C*-algebras.
3. Continuous * morphisms $\{f_{ij} : A_j \to A_i \mid i \leq j \text{ in } I\}$ with
   - $f_{ij} \circ f_{jk} = f_{ik} \quad \forall\ i \leq j \leq k \text{ in } I$



   - $f_{ii} = id_{A_i} \quad \forall\ i \in I$

(A_i, f_{ij}, I) is called **Projective System of C*- Algebras**.

[1] Poset such that $\forall i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$

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1. Directed set : $\mathbb{N}$ with natural order.

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3. Connecting maps : $f_{nm} : \mathbb{C}^m \xrightarrow{(m \geq n)} \mathbb{C}^n$ as $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n)$

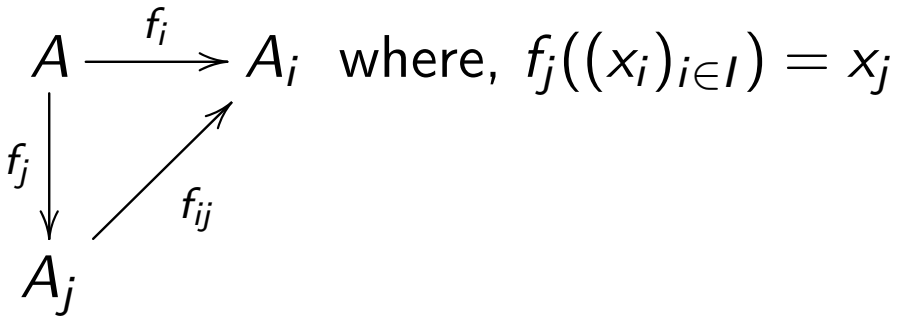
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$\boxed{\left(\mathbb{C}^n, f_{nm}, \mathbb{N}\right) \text{ is a projective system of C*-algebras.}}$

Projective limit of C*-algebras

Given a projective system of C*-algebras (A_i, f_{ij}, I).
Consider, $A = \{(x_i) \in \prod_{i \in I} A_i \mid f_{ij}(x_j) = x_i \ \ \forall i \leq j \text{ in } I \ \}$



$A = \varprojlim(A_i, f_{ij}, I)$ is called *the* projective limit of C*-algebras.

Example: $\mathbb{C}^{\mathbb{N}}$

- $\prod_{n\in\mathbb{N}} \mathbb{C}^n = \{((x_{11}), (x_{21}, x_{22}), (x_{31}, x_{32}, x_{33}), \dots) \ : \ x_{ij} \in \mathbb{C}\}$

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$$\varprojlim (\mathbb{C}^n, f_{nm}, \mathbb{N}) = \{x \in \prod_{n\in\mathbb{N}} \mathbb{C}^n : f_{nm}(x_{m1}, \dots, x_{mm}) = (x_{n1}, \dots, x_{nn}) \ \forall n \leq m\}$$

$$= \{x \in \prod_{n\in\mathbb{N}} \mathbb{C}^n : (x_{m1}, \dots, x_{mn}) = (x_{n1}, \dots, x_{nn}) \ \forall n \leq m\}$$

$$= \{((x_1), (x_1, x_2), (x_1, x_2, x_3), \dots) \ : \ x_i \in \mathbb{C}\}$$

$$\cong \mathbb{C}^{\mathbb{N}}$$

Conclusion: $\boxed{\varprojlim \mathbb{C}^n \cong \mathbb{C}^{\mathbb{N}}}$

$\varprojlim (A_i, f_{ij}, I)$ as Pro C*-algebra

1. Topology : Product topology inherited as a subspace of $\prod_{i \in I} A_i$
2. Involution : Pointwise
3. Family of C*-seminorms : $p_i(x_i)_{i \in I} = ||x_i||_{A_i}$

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Remark:
Local Sub-basis under projective topology w.r.t. $(p_i)_{i \in I}$
$= \{x_i \in \varprojlim A_i \mid ||x_i||_{A_i} < \epsilon\}$
=Local Sub-basis under product topology

Characterization of Pro C*-algebra

Theorem

A is a pro C-algebra* $\iff$ *A is projective limit of C*-algebras.*

Sketch of the proof

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   - Connecting maps : $f_{ij}(x + N_j) = x + N_i$
2. (Isomorphism) $\phi : A \to \varprojlim A_i$ as $\phi(x) = x + N_i$

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Let, $x \in A$. Spectrum of x turns out to be

$$\boxed{\sigma_A(x) = \bigcup_{i \in I} \sigma_{A_i}(x + N_i) \neq \phi}$$

When is Pro C*-algebra isomorphic to a C*-algebra?

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Pro C-algebra A is a C*-algebra* $\iff$ *The set of invertible elements $G(A)$ is open in A.*

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Let, $G(A)$ be open in $A = \varprojlim (A_i, f_{ij}, I)$.

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4. f_i is an isomorphism $\implies A \cong A_i$

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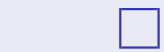
Thank You!

$p_i^{-1}\{0\} = \{0\}$

Proof.

Consider, $U = \{x \in A \mid p_i(x-1) < \epsilon\} \subseteq G(A)$ and $0 \neq x \in p_i^{-1}\{0\}$

1. A is Hausdorff, so $\exists\ j \in I : p_j(x) \neq 0$. Then
$$x_j = x + p_j^{-1}\{0\} \neq 0 \implies x_j^* \neq 0$$
2. $x_j^* x_j \neq 0$ because if , $x_j^* x_j = 0$ then
$$x^*x \in p_j^{-1}\{0\} \implies p_j(x^*x) = 0 \implies p_j(x)^2 = 0 \implies p_j(x) = 0$$
which is a contradiction.
3. As A_j is a C*-algebra then spectral radius of $x_j^* x_j = ||x_j^* x_j|| > 0$ Since, $\sigma(x_j^* x_j) \subseteq [0, \infty)$, therefore, $\exists\ \lambda > 0$ in $\sigma(x_j^* x_j)$. Since $\sigma(x_j^* x_j) \subseteq \sigma(x^*x) \implies 0 < \lambda \in \sigma(x^*x)$
4. $1 - \lambda^{-1}x^*x$ is not invertible in E which can not be true as $x^*x \in U$ because $p_i((1 - \lambda^{-1}x^*x)) - 1) = p_i(\lambda^{-1}x^*x) = 0 < \epsilon$. Hence, $p_i^{-1}\{0\} = \{0\}$.



$\forall\, j \geq i, A_j \cong A_i \implies i = j$

Let, j in $I : j \geq i$ and consider, $f_{ij} : A_j \to A_i$ given by $f_{ij}(x_j) = x_i$

1. f_{ij} is continuous and surjective *-homomorphism.
2. f_{ij} is injective. Take $x_i = 0 \implies x + p_i^{-1}\{0\} = 0 \implies x \in p_i^{-1}\{0\} \implies x = 0 \implies x_j = 0$.
3. Since continuous bijection between C*-algebras is an isometry, hence, Homeomorphism. So, f_{ij} is a topological *-isomorphism isometry.

Isometry $\implies ||x_i|| = ||x_j|| \; \forall j \geq i \implies p_j(x) = p_i(x) \implies i = j$
Hence, i is the upper bound of all the elements in I.

$A \cong A_i$

We have that $f_i : A \to A_i$ is a continuous *-isomorphism.
Assume, U be open in A. Then $\exists j$ in $I : p_j^{-1}[0, \epsilon) \subseteq U$ But $j \leq i$
$\implies p_i^{-1}[0, \epsilon) \subseteq U \implies ||.||^{-1}[0, \epsilon) \subseteq f_i(U) \implies f_i$ is an open map.