Generalized Krull's Principal Ideal Theorem

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outline

1. Main Theorem
2. Definitions
3. Supporting results
4. Proof of theorem
5. Consequences

Main Theorems

- **Krull's Principal Ideal Theorem**

  Let R be a Noetherian Ring and P be a minimal prime of an ideal (a) of R. Then $ht\ P \leq 1$. Consequently, $ht\ (a) \leq 1$.

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- **Krull's Principal Ideal Theorem**

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- **Generalized Krull's Principal Ideal Theorem**

  Let R be a Noetherian Ring and P be a minimal prime of an ideal $(a_1, \ldots, a_r)$ of R. Then $ht\ P \leq r$. Consequently, $ht\ (a_1, \ldots, a_r) \leq r$. This theorem is also known as Krull's Height Theorem.

Definitions

- **Primary ideal**
  A proper ideal I of a ring R is said to be primary if
  $ab \in I$ *and* $a \notin I \Rightarrow b \in \sqrt{I}$.
  Clearly if I is a primary ideal then $\sqrt{I}$ is a prime ideal. Let $\sqrt{I} = P$, then the ideal I is called P-primary.

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  A Prime ideal P is said to be a minimal prime ideal of a ring R if it is a minimal prime ideal over the zero ideal.

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  Let P be a prime idea. We define the height of P as follows. $ht\ P = Sup\{r|\exists\ a\ chain\ P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq\ P_r\ where\ P_i\ is\ a\ prime\ ideal\}$.

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Remarks:
1) The height of a prime ideal P is equal to the dimension of the ring R_P.
2) If the height of a prime ideal is zero then it is a minimal prime ideal of the ring R.

Primary Decomposition Theorem for ideals

Let R be a Noetherian ring and I be a proper ideal of R.

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- Let $P_i = \sqrt{Q_i}$. The primary ideals above can be chosen such that $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$ and P_i's are distinct. Then $I = Q_1 \cap Q_2 \cdots \cap \ Q_n$ is called an irredundant primary decomposition of R.

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- If P_i, Q_i are chosen as above then $P_1, P_2, \ldots, P_n$ are unique.
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Remark:

- For a Noetherian ring R, minimal associated primes of an ideal I are precisely the minimal prime ideals over I.

Symbolic powers of a prime ideal

Let $I = P^n$ where P is a prime ideal of a Noetherian ring R.
Let $I = Q_1 \cap Q_2 \cdots \cap Q_k$ be a irredundant primary decomposition of I with $P_i = \sqrt{Q_i}$.

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- Thus $P = P_j$ is the minimal associated prime ideal of I. So, by the primary decomposition theorem the corresponding primary ideal $Q_J = P^n R_P \cap R$. We denote the ideal $P^n R_P \cap R$ by $P^{(n)}$ called the nth symbolic power of the prime ideal P.
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- Notice that $P^{(1)} \supseteq P^{(2)} \supseteq P^{(3)} \ldots$.

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- If I is an ideal of a Noetherian local ring (R, m) such that m is minimal ideal of I then I is m-primary.
- If the maximal ideal of a local ring is nilpotent then it is the unique prime ideal of the local ring.

Supporting results

Nakayama Lemma

Let M be a finitely generated R-module and I be an ideal contained in $J(R)$, the Jacobson radical of R,such that $IM = M$. Then $M = 0$.

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Prime Avoidance Lemma
Let I be an ideal with $I \subseteq \cup_{i=1}^{n} P_i$, P_i prime ideal. Then $I \subseteq P_i$ for some i. The result holds even if the first two ideals P_1 and P_2 are not prime ideals.

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Proof:

$ht\ P \leq 1 \Leftrightarrow dim(R_P) \leq 1$

By using the correspondence of ideals and prime ideals of a ring and its localization we can replace R by a local ring. Thus we have a Noetherian local ring R and maximal ideal P is the minimal prime of the ideal (a).

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By using the correspondence of prime and maximal ideal in a ring and its quotient it follows that $R/(a)$ has a unique prime/maximal ideal. Thus $dim\ R/(a) = 0$ and $R/(a)$ is Noetherian hence $R/(a)$ is Artinian.

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Let P_1 be any prime ideal of R properly contained in P. Then $ht\ P_1 = 0 \Leftrightarrow ht\ P \leq 1$.

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This corresponds to a descending chain of ideals in the ring $R/(a)$. Since $R/(a)$ is Artinian so the corresponding descending of ideals in $R/(a)$ becomes stationary.Hence there exists a positive integer n such that $(a) + P_1^{(n)} = (a) + P_1^{(n+1)}$.

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Since $P_1^{(n+1)} \subseteq P_1^{(n)}$ so $ab \in P_1^{(n)}$.
$P_1^{(n)}$ is P_1-Primary and We have that $ab \in P_1^{(n)}$ and $a \notin P_1$. It follows that $b \in P_1^{(n)}$
This gives that $P_1^{(n)} = (a)P_1^{(n)} + P_1^{(n+1)}$.

Proof of Principal Ideal Theorem

$P_1^{(n)} = (a)P_1^{(n)} + P_1^{(n+1)} \Rightarrow P_1^{(n)}/P_1^{(n+1)} = (a)P_1^{(n)}/P_1^{(n+1)}$.

By applying Nakayama's Lemma to the R-module $P_1^{(n)}/P_1^{(n+1)}$, we get that $P_1^{(n)} = P_1^{(n+1)}$.

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We know that $P_1^{(n)} = P_1^n R_{P_1} \cap R$ and $P_1^{(n+1)} = P_1^{(n+1)} R_{P_1} \cap R$. Using the bijection between contracted and extended ideals we get that $P_1^n R_{P_1} = P_1^{n+1} R_{P_1}$.

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$P_1^{n+1} = P_1 P_1^n$ and localisation preserves finite product of ideals so we get $P_1^{n+1} R_{P_1} = P_1 R_{P_1} P_1^n R_{P_1}$, hence $P_1^n R_{P_1} = P_1 R_{P_1} P_1^n R_{P_1}$.

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Applying Nakayama's Lemma to the R_{P_1}-module $P_1^n R_{P_1}$, we get $(P_1 R_{P_1})^n = P_1^n R_{P_1} = 0$.

Proof of Principal Ideal Theorem

$P_1^{(n)} = (a)P_1^{(n)} + P_1^{(n+1)} \Rightarrow P_1^{(n)}/P_1^{(n+1)} = (a)P_1^{(n)}/P_1^{(n+1)}$.

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So the maximal ideal of the local ring is nilpotent which means that $P_1 R_{P_1}$ is the unique prime ideal of R_{P_1}.
Thus, $dim\ R_{P_1} = 0$ which means that $ht\ P_1 = 0$. This implies that $ht\ P \leq 1$.

Krull's Height Theorem

Let R be a Noetherian Ring and P be a minimal prime of an ideal $(a_1, \ldots, a_r)$ of R. Then $ht\ P \leq r$. Consequently, $ht\ (a_1, \ldots, a_r) \leq r$.

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Thus, we have a Noetherian local ring R and its maximal ideal P is the minimal prime of the ideal $(a_1, a_2, \ldots, a_r)$. It follows that $(a_1, a_2, \ldots, a_r)$ is P-Primary.

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Let P_1 be any prime ideal properly contained in P and is maximal among the prime ideals of R properly contained in P.
$ht\ P_1 \leq r-1 \Leftrightarrow ht\ P \leq r$. We will show that $ht\ P_1 \leq r-1$.

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To show that $ht\ P_1 \leq r-1$ we will try to produce an ideal generated by $(r-1)$ elements for which P_1 is a minimal prime ideal and then use the induction hypothesis.

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Hence, for $i = 2, 3, \ldots, r$ we have that $a_i^n = y_i + ax_i$ where $y_i \in P_1$ and $x_i \in R$.This means that there exist a large enough positive integer l such that $(a_1, a_2, \ldots, a_r)^l \subseteq (a_1, y_2, \ldots, y_r)$.

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$(a_1, a_2, \ldots, a_r)$ is P-Primary. So, as before there exist an integer m such that $P^m \subseteq (a_1, a_2, \ldots, a_r)$.This means that $P^{ml} \subseteq (a_1, y_2, \ldots, y_r)$.

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By taking quotient with $(y_2, \ldots, y_r)$ it follows that $P/(y_2, \ldots, y_r)$ is the minimal prime of $(a_1, y_2, \ldots, y_r)/(y_2, \ldots, y_r)$. Hence, by the Principal Ideal Theorem we get that $ht\ (P/(y_2, \ldots, y_r)) \leq 1$.

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This implies that P_1 is a minimal prime ideal of $(y_2, \ldots, y_r)$.By the induction hypothesis, $ht\ P_1 \leq r - 1$. Hence, we can conclude that $ht\ P \leq r$.

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  Let R be a Noetherian Ring and P be a minimal prime of an ideal (a) of R. Then $ht\ P \leq 1$. Consequently, $ht\ (a) \leq 1$. Moreover, if a is a nonzero divisor then $ht\ P = 1$ and hence $ht\ (a) = 1$.

Regular Sequence

A sequence of elements $a_1, a_2, \ldots, a_r$ in a ring R is said to be regular if $(a_1, a_2, \ldots, a_r) \neq R$, a_1 is a nonzero divisor of R and a_i is a nonzero divisor of $R/(a_1, a_2, \ldots, a_{i-1})$ for $i = 2, \ldots, r$.

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Let R be a Noetherian Ring and P be a minimal prime of an ideal $(a_1, \ldots, a_r)$ of R. Then $ht\ P \leq r$. Consequently, $ht\ (a_1, \ldots, a_r) \leq r$. Moreover, if $a_1, a_2, \ldots, a_r$ is a regular sequence then $ht\ P = r$ and hence $ht\ (a_1, a_2, \ldots, a_r) = r$.

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- Let K be a field. Consider the ideal $m = (X_1, X_2, \ldots, X_n)$ of $K[X_1, X_2, \ldots, X_n]$. Notice that $X_1, X_2, \ldots, X_n$ is a regular sequence hence it follows that $ht\ m = n$.

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- Let K be a field and K^a be an algebraic closure of K. Then, $K^a[X_1, X_2, \ldots, X_n]$ is integral over $K[X_1, X_2, \ldots, X_n]$. Hence, $dim\ K[X_1, X_2, \ldots, X_n] = dim\ K^a[X_1, X_2, \ldots, X_n]$.

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  As a consequence we can conclude that the dimension of any finitely generated K-algebra is finite.

Consequences

Let R be a Noetherian ring. Then, R is a UFD if and only if every height one prime ideal is principal.

Proof :

Suppose R is a UFD and let P be a prime ideal such that $ht\ P = 1$. Then, clearly P is a non-zero ideal. So, by using unique factorization property and that P is a prime ideal it follows that there exists an irreducible element $a \in P$. Hence, $(a) \subseteq P$.

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Let a an irreducible element of R and P be a minimal prime over (a). By Krull's Height Theorem, $ht\ P \leq 1$. Since R is a domain hence $ht\ P = 1$. So P is a principal ideal. Let $P = (b)$. Then we have that $(a) \subseteq (b)$ and (a) is an irreducible element, hence $(a) = (b)$. So, (a) is a prime ideal, hence a is a prime element.

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Let R be a Noetherian ring and I an ideal of R such that $ht\ I = n$. Then there exists $a_1, a_2, \ldots, a_n \in I$ such that $ht(a_1, a_2, \ldots, a_r) = r\ \forall\ 1 \leq r \leq n$.

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Choose $a \in I - (P_1 \cup P_2 \cdots \cup P_l)$. It follows that $ht\ (a) \geq 1$. But, by Principal Ideal Theorem $ht(a) \leq 1$. Hence, we get that $ht\ (a) = 1$. This proves the result for $r = 1$.

Consequences

Let R be a Noetherian ring and I an ideal of R such that $ht\ I = n$. Then there exists $a_1, a_2, \ldots, a_n \in I$ such that $ht(a_1, a_2, \ldots, a_r) = r\ \forall\ 1 \leq r \leq n$.

Proof :
We will use induction on r. Let $ht\ I \geq 1$. Let $P_1, P_2, \ldots, P_l$ be the minimal prime ideals of R.

Since $ht\ I \geq 1$ and height of a minimal prime ideal of R is zero, it follows that $I \nsubseteq P_j$ for $1 \leq j \leq l$. By prime avoidance, it follows that $I \nsubseteq P_1 \cup P_2 \cdots \cup P_l$.

Choose $a \in I - (P_1 \cup P_2 \cdots \cup P_l)$. It follows that $ht\ (a) \geq 1$. But, by Principal Ideal Theorem $ht(a) \leq 1$. Hence, we get that $ht\ (a) = 1$. This proves the result for $r = 1$.

Proof Continued

Suppose $i > 1$ and we have chosen $a_1, a_2, \ldots, a_{i-1} \in I$ such that $ht\ (a_1, a_2, \ldots, a_{i-1}) = i - 1$.

Proof Continued

Suppose $i > 1$ and we have chosen $a_1, a_2, \ldots, a_{i-1} \in I$ such that $ht\ (a_1, a_2, \ldots, a_{i-1}) = i - 1$.

Let $Q_1, Q_2, \ldots, Q_k$ be the minimal primes over $(a_1, a_2, \ldots, a_{i-1})$. By Krull's Height Theorem, $ht\ Q_j \leq i - 1 < i = ht\ I$. Hence, it follows that $I \nsubseteq Q_j$ for $1 \leq j \leq k$. By prime avoidance, $I \nsubseteq Q_1 \cup Q_2 \cdots \cup Q_k$.

Proof Continued

Suppose $i > 1$ and we have chosen $a_1, a_2, \ldots, a_{i-1} \in I$ such that $ht\ (a_1, a_2, \ldots, a_{i-1}) = i - 1$.

Let $Q_1, Q_2, \ldots, Q_k$ be the minimal primes over $(a_1, a_2, \ldots, a_{i-1})$. By Krull's Height Theorem, $ht\ Q_j \leq i - 1 < i = ht\ I$. Hence, it follows that $I \nsubseteq Q_j$ for $1 \leq j \leq k$. By prime avoidance, $I \nsubseteq Q_1 \cup Q_2 \cdots \cup Q_k$.

Choose $a_i \in I - (Q_1 \cup Q_2 \cdots \cup Q_k)$. By, Height Theorem, $ht\ (a_1, a_2, \ldots, a_i) \leq i$. Let P' be any minimal prime ideal over $(a_1, a_2, \ldots, a_i)$. Since P' contains a_i, so it cannot be a minimal prime over $(a_1, a_2, \ldots, a_{i-1})$ and since $ht\ (a_1, a_2, \ldots, a_{i-1}) = i - 1$. Thus $ht\ P' > i - 1$. It follows that $ht\ (a_1, a_2, \ldots, a_i) = i$.
This completes the induction and the result holds true for $1 \leq r \leq n$.

References


Sudhir R. Ghorpade, *Lectures on Commutative Algebra* http://www.math.iitb.ac.in/ srg/Lecnotes/AfsPuneLecNotes.p


N S Gopala Krishnan, *Commutative Algebra*, University Press.


I. Kaplansky, *Commutative Rings*, The University of Chicago Press,1974.