Hyperplane sections of determinantal varieties over finite fields and linear codes

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A R T I C L E   I N F O

A B S T R A C T

We determine the number of $\mathbb{F}_q$-rational points of hyperplane sections of classical determinantal varieties defined by the vanishing of minors of a fixed size of a generic matrix, and identify the hyperplane sections giving the maximum number of $\mathbb{F}_q$-rational points. Further we consider similar questions for sections by linear subvarieties of a fixed codimension in the ambient projective space. This is closely related to the study of linear codes associated to determinantal varieties, and the determination of their weight distribution, minimum distance, and generalized Hamming weights. The previously known results about these are generalized and expanded significantly. Connections to eigenvalues of certain association schemes, distance regular graphs, and rank metric codes are also indicated.

1. Introduction

The classical determinantal variety defined by the vanishing of all minors of a fixed size in a generic matrix is an object of considerable importance and ubiquity in algebra, combinatorics, algebraic geometry, invariant theory and representation theory. The defining equations clearly have integer coefficients and as such the variety can be defined over any finite field. The number of $\mathbb{F}_q$-rational points of this variety is classically known. We are mainly interested in a more challenging question of determining the number of $\mathbb{F}_q$-rational points of such a variety when intersected with a hyperplane in the ambient projective space, or more generally, with a linear subvariety of a fixed codimension in the ambient projective space. In particular, we wish to know which of these sections have the maximum number of $\mathbb{F}_q$-rational points. These questions are directly related to determining the complete weight distribution and the generalized Hamming weights (as defined in [21]) of the associated linear codes, which are called *determinantal codes*. In this setting, the problem was considered in [2] and a beginning was made by showing that the determination of the weight distribution is related to the problem of computing the number of generic matrices of a given rank with a nonzero "partial trace". More definitive results were obtained in the special case of varieties defined by the vanishing of all 2×2 minors of a generic matrix. Here we settle the question of determination of the weight distribution and the minimum distance of determinantal codes in complete generality. We also show that the determinantal code is generated by its minimum weight codewords. Further, we determine some initial and terminal generalized Hamming weights of determinantal codes. This is then used to show that the duals of determinantal codes have minimum distance 3. Analogous problems have been considered for other classical projective varieties such as Grassmannians, Schubert varieties, etc., leading to interesting classes of linear codes which have been of some current interest; see, for example, [3,12–14,17,22], and the survey [16].

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As was mentioned in [2] and further explained in the next section and Remark 3.8, the results on the weight distribution of determinantal codes are also related to the work of Delsarte [10] on eigenvalues of association schemes of bilinear forms using the rank metric as distance. This is a P-polynomial association scheme and as such, it can also be viewed as a distance regular graph (cf. [4, Prop. 2.7.1]). Ravagnani [18] has shown that these results by Delsarte can also be obtained using the MacWilliams identities for certain rank-metric codes. We remark that a special case of these results was considered by Buckhiester [7]. However, due to the complex nature of the resulting formulae of the weights, some work is required to determine the smallest one among them. As such the formulas of Delsarte (and Ravagnani) do not seem to readily imply an explicit formula for the minimum distance of determinantal codes. Moreover, as far as we know, no results about the generalized Hamming weights of determinantal codes were known except in a special case that was considered in [2].

A more detailed description of the contents of this paper is given in the next section, while the main results are proved in the two sections that follow the next section. An appendix contains self-contained and alternative proofs of some results that were deduced from the work of Delsarte and this might be of an independent interest.

2. Preliminaries

Fix throughout this paper a prime power q, positive integers ℓ, m, and an $\ell \times m$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over $\mathbb{F}_q$, and a nonnegative integer t. We will denote by $\mathbb{F}_q[X]$ the polynomial ring in the ℓm variables X_{ij} $(1 \le i \le \ell, 1 \le j \le m)$ with coefficients in $\mathbb{F}_q$. As usual, by a *minor of size t* or a $t \times t$ *minor* of X we mean the determinant of a $t \times t$ submatrix of X, where t is a nonnegative integer $\le \min\{\ell, m\}$. As per standard conventions, the only 0×0 minor of X is 1. We will be mostly interested in the class of minors of a fixed size, and this class is unchanged if X is replaced by its transpose. With this in view, we shall always assume, without loss of generality, that $\ell \le m$. Given a field $\mathbb{F}$, we denote by $\mathbb{M}_{\ell \times m}(\mathbb{F})$ the set of all $\ell \times m$ matrices with entries in $\mathbb{F}$. Often $\mathbb{F} = \mathbb{F}_q$ and in this case we may simply write $\mathbb{M}_{\ell \times m}$ for $\mathbb{M}_{\ell \times m}(\mathbb{F}_q)$. Note that $\mathbb{M}_{\ell \times m}$ can be viewed as an affine space $\mathbb{A}^{\ell m}$ over $\mathbb{F}_q$ of dimension ℓm. For $0 \le t \le \ell$, the corresponding classical determinantal variety (over $\mathbb{F}_q$) is denoted by $\mathcal{D}_t(\ell, m)$ and defined as the affine algebraic variety in $\mathbb{A}^{\ell m}$ given by the vanishing of all $(t+1) \times (t+1)$ minors of X; in other words

$$\mathcal{D}_t(\ell, m) = \left\{ M \in \mathbb{M}_{\ell \times m}(\mathbb{F}_q) : \operatorname{rank}(M) \le t \right\}.$$

Note that $\mathcal{D}_0(\ell, m)$ only consists of the zero-matrix. For $t = \ell$, no $(t+1) \times (t+1)$ minors of X exist. This means that $\mathcal{D}_\ell(\ell, m) = \mathbb{M}_{\ell \times m}$, which is in agreement with the above description of $\mathcal{D}_\ell(\ell, m)$ as the set of all matrices of rank at most ℓ. Further, we define

$$\widehat{\mathcal{D}}_t(\ell, m) := \{ P \in \mathbb{P}^{\ell m - 1}(\mathbb{F}_q) : \operatorname{rank}(P) \le t \},$$

where $\mathbb{P}^{\ell m - 1}(\mathbb{F}_q)$ denotes the $(\ell m - 1)$-dimensional projective space over $\mathbb{F}_q$, which can be viewed as the set of equivalence classes of nonzero matrices in $\mathbb{M}_{\ell \times m}$, where two nonzero matrices are equivalent if one is a multiple of another (by a nonzero scalar in $\mathbb{F}_q$). Also, for $P \in \mathbb{P}^{\ell m - 1}(\mathbb{F}_q)$, we denote by $\operatorname{rank}(P)$, the rank of a representative in $\mathbb{M}_{\ell \times m}(\mathbb{F}_q)$ of P. Since the rank of a matrix is unaltered upon multiplication by a nonzero scalar, the set $\widehat{\mathcal{D}}_t(\ell, m)$ is well defined. Indeed, it corresponds to the set of $\mathbb{F}_q$-rational points of the projective algebraic variety defined by the homogeneous ideal I_{t+1} generated by all $(t+1) \times (t+1)$ minors of X.

It will also be convenient to define for $0 \le t \le \ell$,

$$\mathfrak{D}_t(\ell, m) := \left\{ M \in \mathbb{M}_{\ell \times m}(\mathbb{F}_q) : \operatorname{rank}(M) = t \right\} \quad \text{and} \quad \mu_t(\ell, m) := |\mathfrak{D}_t(\ell, m)|.$$

A formula for $\mu_t(\ell, m)$ is classically known and goes back at least to Landsberg [15]. We state it below for ease of reference. A proof is outlined in [2, Prop. 2]; see also [18, Lem. 59].

$$\mu_t(\ell, m) = \begin{bmatrix} m \\ t \end{bmatrix}_q \prod_{i=0}^{t-1} (q^\ell - q^i) = q^{\binom{t}{2}} \prod_{i=0}^{t-1} \frac{\left(q^{\ell-i} - 1 \right) \left(q^{m-i} - 1 \right)}{q^{i+1} - 1}, \tag{1}$$

where $\begin{bmatrix} m \\ t \end{bmatrix}_q$ denotes the Gaussian binomial coefficient defined by

$$\begin{bmatrix} m \\ t \end{bmatrix}_q := \frac{[m]_q!}{[t]_q! [m-t]_q!}, \quad \text{where} \quad [n]_q! := \prod_{i=1}^{n} (q^i - 1), \quad \text{for any } n \ge 0.$$

Observe that $\mu_0(\ell, m) = 1$ and

$$\mu_t(\ell, m) = q^{\binom{t}{2}} \begin{bmatrix} m \\ t \end{bmatrix}_q \frac{[\ell]_q!}{[\ell - t]_q!} = q^{\binom{t}{2}} \frac{[m]_q! [\ell]_q!}{[m-t]_q! [t]_q! [\ell - t]_q!} = q^{\binom{t}{2}} \begin{bmatrix} \ell \\ t \end{bmatrix}_q \frac{[m]_q!}{[m-t]_q!}.$$

Next we define

$$\nu_t(\ell, m) := \sum_{s=0}^{t} \mu_s(\ell, m) \quad \text{and} \quad \widehat{\nu}_t(\ell, m) := \frac{\nu_t(\ell, m) - 1}{q - 1} = \frac{1}{q - 1} \sum_{s=1}^{t} \mu_s(\ell, m).$$

It is clear that

$$|\mathcal{D}_t(\ell, m)| = \nu_t(\ell, m) \quad \text{and} \quad |\widehat{\mathcal{D}}_t(\ell, m)| = \widehat{\nu}_t(\ell, m). \tag{2}$$

We are now ready to define the codes we wish to study. Let $\hat{n} := |\widehat{\mathcal{D}}_t(\ell, m)|$ and choose an ordering $P_1, \ldots, P_{\hat{n}}$ of the elements of $\widehat{\mathcal{D}}_t(\ell, m)$. Further, choose representatives $M_i \in \mathbb{M}_{\ell \times m}(\mathbb{F}_q)$ for P_i ($1 \le i \le \hat{n}$). Then consider the evaluation map

$$\widehat{\text{Ev}} : \mathbb{F}_q[X]_1 \to \mathbb{F}_q^{\hat{n}} \quad \text{defined by} \quad \widehat{\text{Ev}}(f) := \hat{c}_f := (f(M_1), \ldots, f(M_{\hat{n}})),$$

where $\mathbb{F}_q[X]_1$ denotes the space of homogeneous polynomials in $\mathbb{F}_q[X]$ of degree 1 together with the zero polynomial. We define $\widehat{C}_{\det}(t; \ell, m)$ to be the image of $\widehat{\text{Ev}}$. A different choice of representatives or a different ordering of these representatives gives in general a different, but monomially equivalent code. Hence basic quantities like minimum distance, weight distribution, and generalized Hamming weights are independent of these choices. It is easy to see (cf. [2, Prop. 2]) that $\widehat{C}_{\det}(t; \ell, m)$ is a q-ary nondegenerate $[\hat{n}, \hat{k}]$ code, where $\hat{k} = \ell m$. For $t = 1$, it was shown in [2] that $\widehat{C}_{\det}(t; \ell, m)$ has minimum distance $q^{\ell+m-2}$. In fact, the first $m + 1$ generalized Hamming weights were computed in [2, Thms. 2 and 3] for $t = 1$. In this article, we prove analogous results for arbitrary t.

The relation between counting $\mathbb{F}_q$-rational points of linear sections of the determinantal variety $\widehat{\mathcal{D}}_t(\ell, m)$ and the parameters of the determinantal code $\widehat{C}_{\det}(t; \ell, m)$ is simply the following:

$$\text{w}_{\text{H}}(\hat{c}_f) = \hat{n} - |\widehat{\mathcal{D}}_t(\ell, m) \cap H_f|,$$

where H_f is the hyperplane in $\mathbb{P}^{\ell m-1} := \mathbb{P}^{\ell m-1}(\mathbb{F}_q)$ corresponding to a nonzero $f \in \mathbb{F}_q[X]_1$. Hence, denoting by $d(\widehat{C}_{\det}(t; \ell, m))$ (resp. $d_r(\widehat{C}_{\det}(t; \ell, m))$) the minimum distance (resp. rth generalized Hamming weight) of $\widehat{C}_{\det}(t; \ell, m)$:

$$\hat{n} - d(\widehat{C}_{\det}(t; \ell, m)) = \max\{|\widehat{\mathcal{D}}_t(\ell, m) \cap H| : H \text{ hyperplane in } \mathbb{P}^{\ell m-1}(\mathbb{F}_q)\}$$

and

$$\hat{n} - d_r(\widehat{C}_{\det}(t; \ell, m)) = \max\{|\widehat{\mathcal{D}}_t(\ell, m) \cap L| : L \text{ codimension } r \text{ linear subspace of } \mathbb{P}^{\ell m-1}(\mathbb{F}_q)\}.$$

We collate below some basic facts shown in [2, Lem. 1, Cor. 1]:

Fact 2.1. *Let $f = \sum_{i=1}^{\ell} \sum_{j=1}^{m} f_{ij} X_{ij} \in \mathbb{F}_q[X]_1$, and let $F = (f_{ij})$ be the coefficient matrix of f. Then the Hamming weight of the corresponding codeword $\hat{c}_f$ of $\widehat{C}_{\det}(t; \ell, m)$ depends only on $\text{rank}(F)$. Consequently, if $r = \text{rank}(F)$, then*

$$\text{w}_{\text{H}}(\hat{c}_f) = \text{w}_{\text{H}}(\hat{c}_{\tau_r}), \quad \text{where} \quad \tau_r := X_{11} + \cdots + X_{rr}.$$

As a result, the code $\widehat{C}_{\det}(t; \ell, m)$ has at most $\ell + 1$ distinct weights, viz., $\text{w}_{\text{H}}(\hat{c}_{\tau_r})$ for $0 \le r \le \ell$.

We call the polynomial τ_r in Fact 2.1 the rth partial trace. Note that $\tau_0 = 0$. Next, we define

$$\widehat{w}_r(t; \ell, m) := \text{w}_{\text{H}}(\hat{c}_{\tau_r}) \quad \text{for } r = 0, 1, \ldots, \ell.$$

Note that $\widehat{w}_0(t; \ell, m) = 0$. To determine the other weights $\widehat{w}_r(t; \ell, m)$, one would need to count the number of $M \in \mathbb{M}_{\ell \times m}(\mathbb{F}_q)$ of rank at most t with nonzero rth partial trace. Delsarte [10] used the theory of association schemes to solve an essentially equivalent problem of determining the number $\text{w}_r(t; \ell, m)$ of $M \in \mathfrak{D}_t(\ell, m)$ with $\tau_r(M) \ne 0$, and showed:

$$\text{w}_r(t; \ell, m) = \frac{q-1}{q}\left(\mu_t(\ell, m) - \sum_{i=0}^{\ell}(-1)^{t-i}q^{im+\binom{t-i}{2}}\begin{bmatrix}\ell-i\\\ell-t\end{bmatrix}_q\begin{bmatrix}\ell-r\\i\end{bmatrix}_q\right). \tag{3}$$

The case $r = \ell = m$ was already dealt with by Buchhiester in [7]. More recently, an alternative approach to Delsarte's formula (3) was given by Ravagnani [18] using the MacWilliams identities for suitable Delsarte rank metric codes. Thus in [18, Thm. 65], it is shown that the number of $\ell \times m$ matrices M over $\mathbb{F}_q$ of rank t with $\tau_r(M) = 0$ is given by

$$\frac{1}{q}\sum_{i=0}^{\ell}(-1)^{t-i}q^{mi+\binom{t-i}{2}}\begin{bmatrix}\ell-i\\\ell-t\end{bmatrix}_q\left(\begin{bmatrix}\ell\\i\end{bmatrix}_q + (q-1)\begin{bmatrix}\ell-r\\i\end{bmatrix}_q\right).$$

To see that this is equivalent to (3) it suffices to note that

$$\mu_t(\ell, m) = \sum_{i=0}^{\ell}(-1)^{t-i}q^{mi+\binom{t-i}{2}}\begin{bmatrix}\ell-i\\\ell-t\end{bmatrix}_q\begin{bmatrix}\ell\\i\end{bmatrix}_q. \tag{4}$$

This follows, for instance by putting $r = 0$ in (3). An alternative proof of the equivalence of (1) and (4) can be easily obtained using elementary properties of Gaussian binomial coefficients, e.g., [1, Thm. 3.3].

In the Appendix of this paper, we obtain using different methods an alternative formula for $\text{w}_r(t; \ell, m)$, which may be of independent interest. For future use, we define

$$\widehat{\text{w}}_r(t; \ell, m) := \frac{\text{w}_r(t; \ell, m)}{q-1} \quad \text{for } 0 \le r \le \ell \text{ and } 1 \le t \le \ell.$$

From Delsarte's result it follows that

$$\widehat{\mathfrak{w}}_r(t; \ell, m) = \frac{1}{q}\left(\mu_t(\ell, m) - \sum_{i=0}^{\ell}(-1)^{t-i}q^{im+\binom{t-i}{2}}\begin{bmatrix}\ell-i\\\ell-t\end{bmatrix}_q\begin{bmatrix}\ell-r\\i\end{bmatrix}_q \right). \tag{5}$$

Consequently, the nonzero weights of $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ are given by

$$\widehat{w}_r(t; \ell, m) = \sum_{s=1}^{t}\widehat{\mathfrak{w}}_r(s; \ell, m) = \sum_{s=1}^{t}\frac{1}{q}\left(\mu_t(\ell, m) - \sum_{i=0}^{\ell}(-1)^{t-i}q^{im+\binom{t-i}{2}}\begin{bmatrix}\ell-i\\\ell-t\end{bmatrix}_q\begin{bmatrix}\ell-r\\i\end{bmatrix}_q \right), \tag{6}$$

for $r = 1, \ldots, \ell$. However, for a fixed t, it is not obvious how $\widehat{w}_1(t; \ell, m), \ldots, \widehat{w}_\ell(t; \ell, m)$ are ordered or even which among them is the least. It is also not clear whether or not $\widehat{w}_1(t; \ell, m), \ldots, \widehat{w}_\ell(t; \ell, m)$ are distinct. Nonetheless, we will show in this paper that the minimum distance, say d_t and the number, say N_t, of minimum weight codewords of $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ are given by

$$d_t = \begin{cases} q^{\ell+m-2}v_{t-1}(\ell-1, m-1) & \text{if } 1 \le t < \ell, \\ q^{\ell m - 1} & \text{if } t = \ell. \end{cases} \quad \text{and} \quad N_t = \begin{cases} \frac{(q^\ell-1)(q^m-1)}{q-1} & \text{if } 1 \le t < \ell, \\ q^{\ell m} - 1 & \text{if } t = \ell. \end{cases}$$

Further, we show that if $1 \le t < \ell$, then $\widehat{w}_1(1; \ell, m) < \widehat{w}_r(1; \ell, m)$ for all $r = 2, \ldots, \ell$. The ordering among $\widehat{w}_r(1; \ell, m)$ for $2 \le r \le \ell$ is not clear, in general, but we make a conjecture here, which is partly motivated by the examples given below.

Example 2.2. If $t = 0$ the code $C_{\mathrm{det}}(t; \ell, m)$ is trivial (containing only the zero word), while the code $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ is not defined. Therefore the easiest nontrivial case occurs for $t = 1$. This case was considered in [2], where it was shown that

$$\widehat{w}_r(1; \ell, m) = q^{\ell+m-2} + q^{\ell+m-3} + \cdots + q^{\ell+m-r-1} = q^{\ell+m-r-1}\frac{q^r - 1}{q - 1}. \tag{7}$$

These formulae also follow fairly directly from (5) and (6). It follows directly that $\widehat{w}_1(1; \ell, m) < \widehat{w}_2(1; \ell, m) < \cdots < \widehat{w}_\ell(1; \ell, m)$ and that $\widehat{w}_1(1; \ell, m) = q^{\ell+m-2}$ is the minimum distance of $\widehat{C}_{\mathrm{det}}(1; \ell, m)$.

Example 2.3. In this example we consider the determinantal code $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ in case $q = 2$, $(\ell, m) = (4, 5)$, and $1 \le t \le 4$. Using the formulae in (5) and (6), we find the following values of $\widehat{w}_r(t; \ell, m)$. For ease of reference, we also mention the lengths of $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ for $1 \le t \le 4$.

$\widehat{w}_r(t; \ell, m)$ r	1	2	3	4
$\widehat{w}_r(1; \ell, m)$	128	192	224	240
$\widehat{w}_r(2; \ell, m)$	13568	16256	16576	16480
$\widehat{w}_r(3; \ell, m)$	201728	212480	211712	211840
$\widehat{w}_r(4; \ell, m)$	524288	524288	524288	524288

t	$\widehat{n} = \widehat{v}_t(\ell, m)$.
1	465
2	33015
3	423615
4	1048575

One sees that it is not true in general that $\widehat{w}_r(t; \ell, m) < \widehat{w}_s(t; \ell, m)$ whenever $r < s$. However, in this example it is true that for a given t, the weight $\widehat{w}_1(t; \ell, m)$ is the smallest among all nonzero weights $\widehat{w}_r(t; \ell, m)$.

Example 2.4. In case $t = \ell$ in the previous example, all weights $\widehat{w}_1, \ldots, \widehat{w}_\ell$ were the same. This holds in general: If $t = \ell$, then $\widehat{\mathcal{D}}_t = \mathbb{P}^{\ell m - 1}$ and $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ is the simplex code of dimension ℓm. All nonzero codewords in this code therefore have weight $q^{\ell m - 1}$. Note that combining this with (5) and (6), we obtain for $1 \le r \le \ell$ the following identity

$$q^{\ell m - 1} = \sum_{s=1}^{\ell}\frac{1}{q}\left(\mu_s(\ell, m) - \sum_{i=0}^{\ell}(-1)^{s-i}q^{im+\binom{s-i}{2}}\begin{bmatrix}\ell-i\\\ell-s\end{bmatrix}_q\begin{bmatrix}\ell-r\\i\end{bmatrix}_q \right).$$

Using (2) with $t = \ell$, we see that for $1 \le r \le \ell$ apparently the following identity holds

$$\sum_{s=1}^{\ell}\sum_{i=0}^{\ell}(-1)^{s-i}q^{im+\binom{s-i}{2}}\begin{bmatrix}\ell-i\\\ell-s\end{bmatrix}_q\begin{bmatrix}\ell-r\\i\end{bmatrix}_q = -1.$$

This identity may readily be shown using for example [1, Thm 3.3] after interchanging the summation order. In any case, it is clear that (5) and (6) may not always give the easiest possible expression for the weights.

While for $t = 1$ and $t = \ell$ all weights $\widehat{w}_r(t; \ell, m)$ are easy to compare with one another, the same cannot be said in case $1 < t < \ell$. The following seems plausible.

Conjecture 2.5. *Let $\ell \leq m$ be positive integers and t an integer satisfying $1 < t < \ell$. Then:*

1. *All weights $\widehat{w}_1(t; \ell, m), \dots, \widehat{w}_\ell(t; \ell, m)$ are mutually distinct.*
2. *$\widehat{w}_1(t; \ell, m) < \widehat{w}_2(t; \ell, m) < \dots < \widehat{w}_{\ell-t+1}(t; \ell, m)$.*
3. *For all $\ell - t + 2 \leq r \leq \ell$, the weight $\widehat{w}_r(t; \ell, m)$ lies between $\widehat{w}_{r-2}(t; \ell, m)$ and $\widehat{w}_{r-1}(t; \ell, m)$.*

3. Minimum distance of determinantal codes

Recall that in general for a linear code C of length n, i.e., for a linear subspace C of $\mathbb{F}_q^n$, the *Hamming weight* of a codeword $c = (c_1, \dots, c_n)$, denoted $w_H(c)$, is defined by

$$w_H(c) := |\{i : c_i \neq 0\}|.$$

The minimum distance of C, denoted $d(C)$, is defined by

$$d(C) := \min\{w_H(c) : c \in C, \ c \neq 0\}.$$

A consequence of Conjecture 2.5 would also be that $\widehat{w}_1(t; \ell, m)$ is the minimum distance of $\widehat{C}_{\det}(t; \ell, m)$. We will now show that this is indeed the case. We start by giving a rather compact expression for $\widehat{w}_1(t; \ell, m)$.

Proposition 3.1. *Let t, ℓ, and m be integers satisfying $1 \leq t \leq \ell \leq m$. Then*

$$\widehat{w}_1(t; \ell, m) = q^{\ell+m-2} v_{t-1}(\ell - 1, m - 1).$$

Proof. First suppose that $t = 1$. In this case Example 2.2 implies that $\widehat{w}_1(1; \ell, m) = q^{m+\ell-2}$. On the other hand, using (1) and (2), we see that $|\mathcal{D}_{t-1}(\ell - 1, m - 1)| = \mu_0(\ell - 1, m - 1) = 1$. So the proposition holds for $t = 1$.

From now on, we assume that $t > 1$ and consequently $\ell > 1$. We will show that

$$\widehat{w}_1(t; \ell, m) = q^{\ell+m-2} \mu_{t-1}(\ell - 1, m - 1) = q^{\ell+m-2+\binom{t-1}{2}} \prod_{i=1}^{t-1} \frac{\left(q^{\ell-i} - 1\right)\left(q^{m-i} - 1\right)}{q^i - 1}. \tag{8}$$

Once we have shown this, the proposition follows using (2) and (6). Let $M = (m_{ij}) \in \mathcal{D}_t(\ell, m)$ and suppose $\tau_1(M) = m_{11} \neq 0$. In that case, we may find uniquely determined square matrices

$$A = \begin{pmatrix} 1 & & & \mathbf{0} \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_{\ell-1} & \mathbf{0} & & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & b_1 & \cdots & b_{m-1} \\ & 1 & & \mathbf{0} \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{pmatrix},$$

such that

$$AMB = \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ 0 & & & \\ 0 & & \widetilde{M} & \\ 0 & & & \end{pmatrix}. \tag{9}$$

The matrices A and B are indeed uniquely determined, since for $2 \leq i \leq \ell$ and $2 \leq j \leq m$,

$$0 = (AMB)_{i1} = a_{i-1}(MB)_{11} + (MB)_{i1} = a_{i-1}m_{11} + m_{i1}$$

and

$$0 = (AMB)_{1j} = (AM)_{11}b_{j-1} + (AM)_{1j} = m_{11}b_{j-1} + m_{1j}.$$

These equations determine the values of $a_1, \dots, b_m$ given the matrix M. The map

$$\phi : \{M \in \mathfrak{D}_t(\ell, m) \mid m_{11} \neq 0\} \to \mathfrak{D}_{t-1}(\ell - 1, m - 1) \quad \text{given by} \quad \phi(M) := \widetilde{M}$$

is therefore well-defined. Moreover, ϕ is clearly surjective (one can for example choose M as in the right hand side of (9)), while the preimage of any matrix $\widetilde{M} \in \mathfrak{D}_{t-1}(\ell - 1, m - 1)$ consist of the $(q-1)q^{\ell+m-2}$ matrices of the form $A^{-1}MB^{-1}$, with A and B as above and again M chosen as in the right-hand-side of (9). Considering the fibers of the map ϕ, we see that

$$\mathfrak{w}_1(t; \ell, m) = |\{M \in \mathfrak{D}_t(\ell, m) \mid m_{11} \neq 0\}| = \sum_{\widetilde{M}} |\phi^{-1}(\widetilde{M})| = |\mathfrak{D}_{t-1}(\ell - 1, m - 1)|(q - 1)q^{\ell+m-2}.$$

Eq. (8), and hence the proposition, follows directly from this. $\square$

Note that the expression for $\widehat{w}_1(t; \ell, m)$ from (5) is considerably more involved than that in (8). We now turn our attention to proving that $\widehat{w}_1(t; \ell, m)$ actually is the minimum distance of the code $\widehat{C}_{\text{det}}(t; \ell, m)$. The proof involves several identities concerning $\widehat{w}(t; \ell, m)$ and $\widehat{w}(t; \ell, m)$. The key is an alternative expression for $\widehat{w}_r(t; \ell, m)$ involving the quantity $A(r, t)$ which is defined as follows. For $1 \le r \le \ell \le m$, set $A(r, 0) := 0$ and further, for $0 \le t < \ell$, set

$$A(r, t) := q^t \widehat{w}_{r-1}(t; \ell - 1, m - 1) + q^{t-1} \left(\mu_t(\ell - 1, m) - \mu_t(\ell - 1, m - 1) \right).$$

Theorem 3.2. *Suppose $1 \le r \le \ell \le m$ and $1 \le t < \ell$. Then*

$$\widehat{w}_r(t; \ell, m) = A(r, t) - A(r, t - 1) + q^{m-1} \mu_{t-1}(\ell - 1, m).$$

Proof. If $t = 1$, the theorem follows easily using (7). Now assume that $1 < t < \ell$. Given a matrix $M = (m_{ij}) \in \mathfrak{D}_t(\ell, m)$, we denote by $\psi(M)$ the matrix obtained from M by deleting its rth row. Since either $\psi(M) \in \mathfrak{D}_t(\ell - 1, m)$ or $\psi(M) \in \mathfrak{D}_{t-1}(\ell - 1, m)$, this defines a map $\psi : \mathfrak{D}_t(\ell, m) \to \mathfrak{D}_t(\ell - 1, m) \coprod \mathfrak{D}_{t-1}(\ell - 1, m)$. It is not hard to see that ψ is surjective. In fact:

$$|\psi^{-1}(N)| = \begin{cases} q^t & \text{if } N \in \mathfrak{D}_t(\ell - 1, m), \\ q^m - q^{t-1} & \text{if } N \in \mathfrak{D}_{t-1}(\ell - 1, m), \end{cases} \tag{10}$$

because if $N \in \mathfrak{D}_t(\ell - 1, m)$, then we obtain all elements of $\psi^{-1}(N)$ by adding a row from the rowspace of N, whereas if $N \in \mathfrak{D}_{t-1}(\ell - 1, m)$, then we obtain all elements of $\psi^{-1}(N)$ by adding any row not from the rowspace of N.

We will now prove the theorem by carefully counting the number of matrices $M \in \mathfrak{D}_t(\ell, m)$ in fibers of the map ψ such that $\tau_r(M) \ne 0$, thus computing $w_r(t; \ell, m)$. Thus, fix $N \in \mathfrak{D}_t(\ell - 1, m) \coprod \mathfrak{D}_{t-1}(\ell - 1, m)$ and consider $M = (m_{ij}) \in \psi^{-1}(N)$ such that $\tau_r(M) \ne 0$. We distinguish four mutually exclusive cases:

Case 1: $N \in \mathfrak{D}_t(\ell - 1, m)$ and the rth column of N is zero.

In this case $m_{rr} = 0$, since otherwise the rth row of M is not in the rowspace of N, contradicting that $\text{rank}(N) = \text{rank}(M)$. Thus M is effectively an $\ell \times (m - 1)$ matrix and N and $(\ell - 1) \times (m - 1)$ matrix. Also, $\tau_r(M) \ne 0$ if and only if $\tau_{r-1}(N) \ne 0$. Hence, by (10), we find the following contribution to $w_r(t; \ell, m)$:

$$q^t w_{r-1}(t; \ell - 1, m - 1). \tag{11}$$

Case 2: $N \in \mathfrak{D}_{t-1}(\ell - 1, m)$ and the rth column of N is zero.

First, assume that $m_{rr} = 0$. Then by a similar reasoning as in Case 1, we find a contribution to $w_r(t; \ell, m)$ of magnitude

$$(q^{m-1} - q^{t-1}) w_{r-1}(t - 1; \ell - 1, m - 1). \tag{12}$$

Next, assume that $m_{rr} \ne 0$. To begin with, suppose $\tau_{r-1}(N) = 0$. Then $\tau_r(M) \ne 0$ if and only if $m_{rr} \ne 0$. Hence, determining $M \in \psi^{-1}(N)$ with $\tau_r(M) \ne 0$ amounts to assigning arbitrary values for m_{rj} with $m_{rr} \ne 0$. This gives a contribution to $w_r(t; \ell, m)$ of magnitude

$$q^{m-1}(q - 1) \left(\mu_{t-1}(\ell - 1, m - 1) - w_{r-1}(t - 1; \ell - 1, m - 1) \right). \tag{13}$$

Suppose on the other hand, $\tau_{r-1}(N) \ne 0$. Then $\tau_r(M) \ne 0$ if and only if $m_{rr} \ne -\tau_{r-1}(N)$. Since we already assumed that $m_{rr} \ne 0$, we find a contribution to $w_r(t; \ell, m)$ of magnitude

$$q^{m-1}(q - 2) w_{r-1}(t - 1; \ell - 1, m - 1). \tag{14}$$

Case 3: $N \in \mathfrak{D}_t(\ell - 1, m)$ and the rth column of N is nonzero.

Since the rth column of N is nonzero, the rth coordinates of elements from the row space of N are distributed evenly over the elements of $\mathbb{F}_q$. This implies that regardless of the value of $\tau_{r-1}(N)$, a $(q - 1)/q$th fraction of the matrices in $\psi^{-1}(N)$ contribute to $w_r(t; \ell, m)$. In total we find the contribution:

$$q^{t-1}(q - 1) \left(\mu_t(\ell - 1, m) - \mu_t(\ell - 1, m - 1) \right). \tag{15}$$

Case 4: $N \in \mathfrak{D}_{t-1}(\ell - 1, m)$ and the rth column of N is nonzero.

Just as in Case 3, since the rth column of N is nonzero, the rth coordinates of elements from the row space of N are distributed evenly over the elements of $\mathbb{F}_q$. Therefore also the rth coordinates of elements not from the row space of N are distributed evenly over the elements of $\mathbb{F}_q$. By a similar reasoning as in Case 3, we find a contribution to $w_r(t; \ell, m)$ of magnitude:

$$(q^{m-1} - q^{t-2})(q - 1) \left(\mu_{t-1}(\ell - 1, m) - \mu_{t-1}(\ell - 1, m - 1) \right). \tag{16}$$

Adding all contributions to $w_r(t; \ell, m)$ from (11), (12), (13), (14), (15), and (16), and noting that $w_r(t; \ell, m) = (q - 1)\widehat{w}_r(t; \ell, m)$, the theorem follows. $\square$

Corollary 3.3. *Let $1 \le r \le \ell \le m$ and $1 \le t < \ell$. Then*

$$\widehat{w}_r(t; \ell, m) = A(r, t) + q^{m-1} v_{t-1}(\ell - 1, m).$$

Proof. By (6) and Theorem 3.2, we see that

$$\widehat{w}_r(t; \ell, m) = \sum_{s=1}^{t} \widehat{\mathfrak{w}}_r(s; \ell, m)$$

$$= \sum_{s=1}^{t} \left(A(r, s) - A(r, s-1) + q^{m-1} \mu_{s-1}(\ell - 1, m) \right)$$

$$= A(r, t) - A(r, 0) + q^{m-1} \sum_{s=1}^{t} \mu_{s-1}(\ell - 1, m).$$

The corollary now follows from (2), since $A(r, 0) = 0$. $\quad\square$

Corollary 3.4. *Suppose $1 \le s \le r \le \ell$ and $1 \le t < \ell$. Then*

$$\widehat{w}_r(t; \ell, m) - \widehat{w}_s(t; \ell, m) = q^t \left(\widehat{\mathfrak{w}}_{r-1}(t; \ell - 1, m - 1) - \widehat{\mathfrak{w}}_{s-1}(t; \ell - 1, m - 1) \right).$$

In particular,

$$\widehat{w}_r(t; \ell, m) - \widehat{w}_1(t; \ell, m) = q^t \widehat{\mathfrak{w}}_{r-1}(t; \ell - 1, m - 1).$$

Proof. Using the previous corollary, we see that

$$\widehat{w}_r(t; \ell, m) - \widehat{w}_s(t; \ell, m) = A(r, t) - A(s, t)$$

$$= q^t \left(\widehat{\mathfrak{w}}_{r-1}(t; \ell - 1, m - 1) - \widehat{\mathfrak{w}}_{s-1}(t; \ell - 1, m - 1) \right).$$

This yields the first part of the corollary. The second part follows directly by choosing $s = 1$. $\quad\square$

We are now ready to prove our main theorem on the minimum distance.

Theorem 3.5. *Suppose $1 \le r \le \ell$ and $1 \le t \le \ell$. Then the minimum distance $\hat{d}$ of the code $\widehat{C}_{\det}(t; \ell, m)$ is given by*

$$\hat{d} = q^{\ell+m-2} v_{t-1}(\ell - 1, m - 1).$$

Proof. We already know that the only ℓ nonzero weights occurring in code $\widehat{C}_{\det}(t; \ell, m)$ are $\widehat{w}_1(t; \ell, m), \ldots, \widehat{w}_\ell(t; \ell, m)$. Moreover, in case $t = \ell$, we already know from Example 2.4 that the minimum distance is given by

$$\widehat{w}_1(\ell; \ell, m) = q^{\ell m - 1} = q^{\ell+m-2} q^{(\ell-1)(m-1)} = q^{\ell+m-2} v_{\ell-1}(\ell - 1, m - 1).$$

Therefore we may assume $t < \ell$. However, in this case the second part of Corollary 3.4 implies that $\widehat{w}_1(t; \ell, m)$ cannot be larger than any of the other weights, since

$$\widehat{w}_r(t; \ell, m) - \widehat{w}_1(t; \ell, m) = q^t \widehat{\mathfrak{w}}_{r-1}(t; \ell - 1, m - 1) \ge 0.$$

The theorem then follows from Proposition 3.1. $\quad\square$

Remark 3.6. Comparing the formulae for $\hat{d} = \widehat{w}_1(t; \ell, m)$ given by Corollary 3.3 and Theorem 3.5, we obtain the following curious identity:

$$q^{\ell+m-2} v_{t-1}(\ell - 1, m - 1) = q^{m-1} v_{t-1}(\ell - 1, m) + q^{t-1} \left(\mu_t(\ell - 1, m) - \mu_t(\ell - 1, m - 1) \right).$$

This is trivial if $t = 1$, whereas it can be verified directly for $t \ge 2$ by noting that

$$v_{t-1}(\ell - 1, m) = q^{\ell-1} v_{t-2}(\ell - 1, m - 1) + q^{t-1} \mu_{t-1}(\ell - 1, m - 1).$$

The last identity follows by counting fibers of the map $\mathcal{D}_{t-1}(\ell - 1, m) \to \mathcal{D}_{t-1}(\ell - 1, m - 1)$ that associates to a matrix, the matrix obtained by deleting its last column.

Exploring the methods used in proving Theorem 3.5, we can also gain some information about codewords of minimum weight in $\widehat{C}_{\det}(t; \ell, m)$.

Theorem 3.7. *Suppose $1 < r \le \ell$ and $1 \le t < \ell$. Then $\widehat{w}_1(t; \ell, m) < \widehat{w}_r(t; \ell, m)$. Moreover, the code $\widehat{C}_{\det}(t; \ell, m)$ has exactly $\mu_1(\ell, m) = (q^\ell - 1)(q^m - 1)/(q - 1)$ codewords of minimum weight and these codewords generate the entire code. More precisely, any codeword in $\widehat{C}_{\det}(t; \ell, m)$ is the sum of at most ℓ minimum weight codewords.*

If $t = \ell$, then all nonzero codewords of $\widehat{C}_{\det}(t; \ell, m)$ have the same weight and in particular, the number of codewords of minimum weight is $q^{\ell m} - 1$ and these codewords generate the code.

Proof. Choosing $s = 1$ in Corollary 3.4 and $r \geq 2$, we obtain

$$\widehat{w}_r(t; \ell, m) - \widehat{w}_1(t; \ell, m) = q^t \widehat{\mathfrak{w}}_{r-1}(t; \ell - 1, m - 1),$$

so the first part of the theorem follows once we have shown that $\widehat{\mathfrak{w}}_{r-1}(t; \ell - 1, m - 1) > 0$. In order to do this, it is sufficient to produce one $(\ell - 1) \times (m - 1)$ matrix M of rank t such that $\tau_{r-1}(M) \neq 0$. However, this is easy to do: Let $P = (p_{ij})$ be a $t \times t$ permutation matrix corresponding to a permutation of $\{1, \ldots, t\}$ that fixes 1, but does not have other fixed points. Then $p_{11} = 1$, while any other diagonal element is zero. Now take $M = (m_{ij})$ to be the $(\ell - 1) \times (m - 1)$ matrix such that $m_{ij} = p_{ij}$ if $i < \ell$ and $j < m$, while $m_{ij} = 0$ otherwise. Then $\mathrm{rank}(M) = t$ and $\tau_{r-1}(M) = 1$, which is exactly what we wanted to show.

Now that we know that $\widehat{w}_1(t; \ell, m)$ is strictly smaller than all other nonzero weights, the minimum weight codewords are exactly those $\hat{c}_f$ such that f has a coefficient matrix of rank 1. This gives exactly $\mu_1(\ell, m)$ possibilities for f and hence for $\hat{c}_f$. Now let $\hat{c} \in \widehat{C}_{\det}(t; \ell, m)$ be given. Then $\hat{c} = \hat{c}_g$ for some $g \in \mathbb{F}_q[X]_1$. Assume that g has coefficient matrix of rank r. Since any matrix of rank r can be written as the sum of r matrices of rank 1, we can write $g = g_1 + \cdots + g_r$ for certain $g_1, \ldots, g_r \in \mathbb{F}_q[X]_1$ all having a coefficient matrix of rank 1. This implies that $\hat{c}_g = \hat{c}_{g_1} + \cdots + \hat{c}_{g_r}$, implying the second part of the theorem.

The assertions in the case $t = \ell$ follow directly from Example 2.4. □

Remark 3.8. If Conjecture 2.5 is true, then Corollary 3.4 implies that the quantities $\widehat{\mathfrak{w}}_r(t; \ell, m)$ would have a behavior similar to that of $\widehat{w}_r(t; \ell, m)$. More precisely, let $1 \leq t \leq \ell$, then it would hold that:

   (i) All weights $\widehat{\mathfrak{w}}_1(t; \ell, m), \ldots, \widehat{\mathfrak{w}}_\ell(t; \ell, m)$ are mutually distinct.
   (ii) $\widehat{\mathfrak{w}}_1(t; \ell, m) < \widehat{\mathfrak{w}}_2(t; \ell, m) < \cdots < \widehat{\mathfrak{w}}_{\ell-t+1}(t; \ell, m)$.
   (iii) For $\ell - t + 2 \leq r \leq \ell$, the weight $\widehat{\mathfrak{w}}_r(t; \ell, m)$ lies between $\widehat{\mathfrak{w}}_{r-2}(t; \ell, m)$ and $\widehat{\mathfrak{w}}_{r-1}(t; \ell, m)$.

We remark that these assertions have a bearing on the eigenvalues of the association scheme of bilinear forms (using the rank metric as distance) [4, Section 9.5.A]. Indeed, the eigenvalues of this association scheme are precisely given by the expressions

$$P_t(r) := \sum_{i=0}^{\ell} (-1)^{t-i} q^{im + \binom{t-i}{2}} \begin{bmatrix} \ell - i \\ \ell - t \end{bmatrix}_q \begin{bmatrix} \ell - r \\ i \end{bmatrix}_q \tag{17}$$

occurring in (5). For a general association scheme, it is not known how its eigenvalues are ordered or if they are all distinct. See [5] for a study of the nondistinctness of some such eigenvalues. It is known in general that the eigenvalues exhibit sign changes (see for example [6, Prop. 11.6.2]), which is in consonance with the conjectured behavior of the $\widehat{\mathfrak{w}}_r(t; \ell, m)$ in part (iii) above.

4. Generalized Hamming weights of determinantal codes

We now turn our attention to the computation of several of the generalized Hamming weights of the determinantal code $\widehat{C}_{\det}(t; \ell, m)$. Given that it was not trivial to compute the minimum distance, this may seem ambitious, but it turns out that we can use the work carried out in the previous section to compute the first m generalized Hamming weights.

For a linear code C of length n and dimension k the *support weight* of any $D \subseteq C$, denoted $\|D\|$, is defined by

$$\|D\| := |\{i : \text{there exists } c \in D \text{ with } c_i \neq 0\}|.$$

For $1 \leq s \leq k$ the sth *generalized Hamming weight* of C, denoted $d_s(C)$, is defined by

$$d_s(C) := \min\{\|D\| : D \text{ is a subcode of } C \text{ with } \dim D = s\}.$$

Clearly, $d_1(C) = d(C)$, the minimum distance of the code C, while $d_k(C) = n$ if the code C is nondegenerate.

Theorem 4.1. *For* $1 \leq s \leq m$, *the* sth *generalized Hamming weight* $\hat{d}_s$ *of* $\widehat{C}_{\det}(t; \ell, m)$ *is given by*

$$\hat{d}_s = \frac{q^s - 1}{q^s - q^{s-1}} \widehat{w}_1(t; \ell, m) = \frac{q^{\ell+m-1} - q^{\ell+m-s-1}}{q - 1} \nu_{t-1}(\ell - 1, m - 1). \tag{18}$$

Proof. Fix $s \in \{1, \ldots, m\}$ and let L_s be the s-dimensional subspace of $\mathbb{F}_q[X]_1$ generated by $X_{11}, \ldots, X_{1s}$. Also let $D_s = \mathrm{Ev}(L_s)$ be the corresponding subcode of $\widehat{C}_{\det}(t; \ell, m)$. Since Ev is injective and linear, $\dim D_s = s$. Moreover, since the coefficient matrix of any $f \in L_s$ different from zero has rank one, it follows from Fact 2.1 that $w_H(\hat{c}_f) = \widehat{w}_1(t; \ell, m)$. Using the formula for the support weight of an s-dimensional subcode given in for example [11, Lemma 12], we obtain

$$\|D_s\| = \frac{1}{q^s - q^{s-1}} \sum_{c \in D_s} w_H(c) = \frac{q^s - 1}{q^s - q^{s-1}} \widehat{w}_1(t; \ell, m).$$

On the other hand, since $\widehat{w}_1(t; \ell, m)$ is the minimum distance of $\widehat{C}_{\det}(t; \ell, m)$, it holds for any subspace $D \subseteq \widehat{C}_{\det}(t; \ell, m)$ of dimension s that

$$\|D\| = \frac{1}{q^s - q^{s-1}} \sum_{c \in D_s} w_H(c) \geq \frac{q^s - 1}{q^s - q^{s-1}} \widehat{w}_1(t; \ell, m).$$

This yields the first equality in (18). The second equality in (18) follows from Theorem 3.5. $\square$

The first equality in (18) shows that the generalized Hamming weights $\hat{d}_s$ of $\widehat{C}_{\det}(t; \ell, m)$ meet the Griesmer–Wei bound [20, Cor. 3.3], provided $1 \leq s \leq m$. If $s \geq m + 1$, then the Griesmer–Wei bound is not attained in general. This can be seen from the following.

Proposition 4.2. Suppose that $\ell \geq 2$, then the $(m + 1)$th generalized Hamming weight $\hat{d}_{m+1}$ of $\widehat{C}_{\det}(t; \ell, m)$ is given by

$$\hat{d}_{m+1} = \hat{d}_m + q^{\ell-2} v_{t-1}(\ell - 1, m - 1) + (q^{m-1} - 1)q^{\ell+t-1}\mu_{t-1}(\ell - 1, m - 1).$$

Proof. Let $L_{m+1} \subseteq \mathbb{F}_q[X]_1$ be the $m + 1$-dimensional space generated by $X_{11}, \ldots, X_{1m}, X_{21}$. Write $D_{m+1} = \text{Ev}(L_{m+1})$. As in the proof of [2, Lem. 2] one readily sees that L_{m+1} contains one function with coefficient matrix of rank 0 (namely the zero function), $q^m + q^2 - q - 1$ functions with coefficient matrix of rank 1, and $(q - 1)(q^m - q) = q^{m+1} - q^m + q^2 + q$ functions with coefficient matrix of rank 2. It follows that

$$\begin{aligned}
\hat{d}_{m+1} &\leq \frac{1}{q^{m+1} - q^m} \sum_{c \in D_{m+1}} w_H(c) \\
&= \frac{1}{q^{m+1} - q^m} \left((q^m + q^2 - q - 1)\widehat{w}_1(t; \ell, m) + (q - 1)(q^m - q)\widehat{w}_2(t; \ell, m) \right) \\
&= \hat{d}_m + \frac{\widehat{w}_1(t; \ell, m)}{q^m} + \frac{q^{m-1} - 1}{q^{m-1}}(\widehat{w}_2(t; \ell, m) - \widehat{w}_1(t; \ell, m)) \\
&= \hat{d}_m + q^{\ell-2} v_{t-1}(\ell - 1, m - 1) + (q^{m-1} - 1)q^{\ell+t-1}\mu_{t-1}(\ell - 1, m - 1),
\end{aligned}$$

where the penultimate equality follows from (18) and an elementary calculation, whereas the last equality follows from Proposition 3.1, Corollary 3.4, and Eq. (8). On the other hand, in [2, Lem. 4] it is shown that any $(m + 1)$-dimensional subspace of $\mathbb{M}_{\ell \times m}$ contains at most $q^m + q^2 - q - 1$ matrices of rank 1 and at least $(q^m - q)(q - 1)$ matrices of rank ≥ 2. This implies the desired result. $\square$

Finally, we will determine the last tm generalized Hamming weights. While before, we have mainly used the description of $\widehat{C}_{\det}(t; \ell, m)$ as an evaluation code, it turns out to be more convenient now to use the geometric description of $\widehat{C}_{\det}(t; \ell, m)$ as a projective system coming from $\widehat{\mathcal{D}}_t$. The approach is similar to the one given Appendix A in [8], though there a completely different class of codes was considered. The following lemma holds the key:

Lemma 4.3. The projective variety $\widehat{\mathcal{D}}_t(\ell, m) \subseteq \mathbb{P}^{\ell m - 1}$ contains the projective space $\mathbb{P}^{tm - 1}$.

Proof. Since any matrix in $\mathbb{M}_{\ell \times m}$ with at most t nonzero rows is in $\mathcal{D}_t(\ell, m)$, we see that

$$\{(m_{ij}) \in \mathbb{M}_{\ell \times m} \mid m_{ij} = 0 \text{ for } 1 \leq i \leq \ell - t \text{ and } 1 \leq j \leq m\} \subseteq \mathcal{D}_t(\ell, m).$$

Passing to homogeneous coordinates, the lemma follows. $\square$

In the language of projective systems, the sth generalized Hamming weight can be described rather elegantly. If C is a code of length n and dimension k described by a projective system $X \subseteq \mathbb{P}^{k-1}$, then

$$d_s(C) = n - \max_{\text{codim} L = s} |X \cap L|, \tag{19}$$

where the maximum is taken over all projective linear subspaces $L \subseteq \mathbb{P}^{k-1}$ of codimension s (see [19,20] for more details). This description, combined with the previous lemma, gives the following result.

Theorem 4.4. Let t, s be positive integers such that $t \leq \ell$ and $(\ell - t)m \leq s \leq \ell m$. Then $\hat{d}_s$, the sth generalized Hamming weight of $\widehat{C}_{\det}(t; \ell, m)$, is given by

$$\hat{d}_s = \hat{n} - \sum_{i=0}^{\ell m - s - 1} q^i.$$

Proof. First of all note that if $s = \ell m$, then $\hat{d}_s = \hat{n}$, since the code $\widehat{C}_{\det}(t; \ell, m)$ is nondegenerate (see Fact 2.1). Therefore, we assume that $(\ell - t)m \leq s < \ell m$. Then $0 \leq \ell m - s - 1 \leq tm - 1$ and so by Lemma 4.3, there exists a projective linear

subspace L_s of dimension $\ell m - s - 1$ contained in $\widehat{\mathcal{D}}_t(\ell, m)$. Clearly, codim$L_s = s$ and

$$\max_{\text{codim}L=s} |\widehat{\mathcal{D}}_t(\ell, m) \cap L| = |\widehat{\mathcal{D}}_t(\ell, m) \cap L_s| = |L_s| = \sum_{i=0}^{\ell m - s - 1} q^i.$$

In view of (19), this yields the desired expression for $\hat{d}_s$. □

Corollary 4.5. *The minimum distance of $\widehat{C}_{\det}(t; \ell, m)^\perp$ equals 3.*

Proof. From Theorem 4.4, we see that $d_{\ell m - 2} = \hat{n} - q - 1$, $d_{\ell m - 1} = \hat{n} - 1$, and $d_{\ell m} = \hat{n}$. By Wei duality [21, Thm. 3], this implies that the first generalized Hamming weight of $\widehat{C}_{\det}(t; \ell, m)^\perp$ (that is to say, its minimum distance) is given by 3. □

Remark 4.6. Taking $t = \ell$ in Theorem 4.4, we obtain all the generalized Hamming weights of $\widehat{C}_{\det}(t; \ell, m)$, or in other words ,we recover the well-known result about the complete weight hierarchy of simplex codes. Similarly, Corollary 4.5 may be viewed as a generalization of the well-known fact that the q-ary Hammming code has minimum distance 3.

We can also determine the complete weight hierarchy of determinantal codes corresponding to determinantal ideals generated by the maximal minors of X.

Corollary 4.7. *In case $t = \ell - 1$ all generalized Hamming weights of $\widehat{C}_{\det}(t; \ell, m)$ are known and given by*

$$d_s = \begin{cases} q^{\ell + m - s - 1} \nu_{\ell - 2}(\ell - 1, m - 1), & \text{if } 1 \leq s \leq m, \\ \hat{n} - \sum_{i=0}^{\ell m - s - 1} q^i, & \text{otherwise.} \end{cases}$$

Proof. This follows by combining Theorems 4.1 and 4.4. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Appendix

In this appendix we give a self-contained computation of the quantity $\mathfrak{w}_r(t; \ell, m)$. The method we use is different from the one Delsarte used in [10] and consequently gives rise to an alternative formula to the one Delsarte obtained. Essentially our method concerns the study of a refined description of the sets $\mathfrak{D}_t(\ell, m)$ as the union of disjoint subsets. For $M \in \mathbb{M}_{\ell \times m}$, and $1 \leq r \leq \ell$, we denote by $\underline{M}_r$ the $r \times m$ matrix obtained by taking the first r rows of M. We use this to define the following quantities:

Definition A.1. Let r, s, t, ℓ, and m be integers satisfying $1 \leq \ell \leq m$, $1 \leq r \leq \ell$, and $0 \leq s \leq t \leq \ell$. Then we define

$$\mathfrak{D}_t(\ell, m; r, s) := \{M \in \mathfrak{D}_t(\ell, m) \mid \text{rank}(\underline{M}_r) = s\}.$$

Further we define

$$\mathfrak{w}_r^{(s)}(t; \ell, m) := w_H((\tau_r(M))_{M \in \mathfrak{D}_t(\ell, m; r, s)}),$$

with as before $\tau_r = X_{11} + \cdots + X_{rr}$.

Note that $\mathfrak{w}_r^{(0)}(t;\ell,m)=0$ and

$$\mathfrak{w}_r(t;\ell,m)=\sum_{s=0}^r \mathfrak{w}_r^{(s)}(t;\ell,m).\tag{20}$$

Proposition A.2. *Let $r,s,t,\ell,$ and m be integers satisfying $1\le t\le\ell\le m$, $1\le r\le\ell$, and $0\le s\le t$. Then*

$$|\mathfrak{D}_t(\ell,m;r,s)|=\frac{[m]_q!}{[m-t]_q!}q^{s(\ell-r)}q^{\binom{s}{2}}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q.$$

Proof. Fix $r\ge1$. We will prove the proposition with induction on ℓ for values $\ell\ge r$.

Induction basis: If $\ell=r$, then $\mathfrak{D}_t(\ell,m;r,s)=\mathfrak{D}_t(\ell,m)$ if $s=t$, while otherwise $\mathfrak{D}_t(\ell,m;r,s)=\emptyset$. In the latter case the proposed formula gives the correct value 0, while if $s=t$ also the correct value from (1) is recovered. This completes the induction basis.

Induction step: Suppose $\ell>r$. Let $A\in\mathfrak{D}_t(\ell,m;r,s)$. Then $\underline{A}_{\ell-1}$ is an element of $\mathfrak{D}_t(\ell-1,m;r,s)$ or of $\mathfrak{D}_{t-1}(\ell-1,m;r,s)$. Conversely, a matrix from $\mathfrak{D}_t(\ell-1,m;r,s)$ can be extended (by adding a row from the rowspace of the matrix) to an element of $\mathfrak{D}_t(\ell,m;r,s)$ in exactly q^t ways, while a matrix from $\mathfrak{D}_{t-1}(\ell-1,m;r,s)$ can be extended (by adding a row not from the rowspace of the matrix) to an element of $\mathfrak{D}_t(\ell,m;r,s)$ in exactly q^m-q^{t-1} ways. Therefore

$$|\mathfrak{D}_t(\ell,m;r,s)|=q^t|\mathfrak{D}_t(\ell-1,m;r,s)|+(q^m-q^{t-1})|\mathfrak{D}_{t-1}(\ell-1,m;r,s)|.$$

Using the induction hypothesis, this equation implies:

$$\begin{aligned}|\mathfrak{D}_t(\ell,m;r,s)|=&\ \frac{[m]_q!}{[m-t]_q!}q^{s(\ell-r)}q^{\binom{s}{2}}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q\\&\times\left(q^tq^{-s}\frac{q^{\ell-r-t+s}-1}{q^{\ell-r}-1}+(q^m-q^{t-1})\frac{1}{q^{m-t+1}}q^{-s}q^{-(t-1-s)}\frac{q^{t-s}-1}{q^{\ell-r}-1}\right).\end{aligned}$$

However, the term inside the large parentheses is easily seen to be equal to 1, concluding the inductive proof. □

The key argument in the induction step above can also be used to prove the following.

Lemma A.3. *Let $r,s,t,\ell,$ and m be integers satisfying $1\le t\le\ell\le m$, $1\le r<\ell$, and $0\le s\le t$. Then*

$$\mathfrak{w}_r^{(s)}(t;\ell,m)=q^t\mathfrak{w}_r^{(s)}(t;\ell-1,m)+(q^m-q^{t-1})\mathfrak{w}_r^{(s)}(t-1;\ell-1,m)$$

and

$$\mathfrak{w}_r(t;\ell,m)=q^t\mathfrak{w}_r(t;\ell-1,m)+(q^m-q^{t-1})\mathfrak{w}_r(t-1;\ell-1,m).$$

Proof. In the proof of Proposition A.2, we have seen that if $\ell>r$, then any matrix from $\mathfrak{D}_t(\ell-1,m;r,s)$ can be extended to an element of $\mathfrak{D}_t(\ell,m;r,s)$ in exactly q^t ways, while a matrix from $\mathfrak{D}_{t-1}(\ell-1,m;r,s)$ can be extended to an element of $\mathfrak{D}_t(\ell,m;r,s)$ in q^m-q^{t-1} ways. Since $\ell>r$, the value of τ_r is the same for the original matrix and its extension. This immediately implies the first equation in the lemma. The second one follows from the first one using (20). □

Remark A.4. By interchanging the roles of rows and columns, one can also show that if $r<m$, then

$$\mathfrak{w}_r^{(s)}(t;\ell,m)=q^t\mathfrak{w}_r^{(s)}(t;\ell,m-1)+(q^\ell-q^{t-1})\mathfrak{w}_r^{(s)}(t-1;\ell,m-1)$$

and

$$\mathfrak{w}_r(t;\ell,m)=q^t\mathfrak{w}_r(t;\ell,m-1)+(q^\ell-q^{t-1})\mathfrak{w}_r(t-1;\ell,m-1).$$

We will now derive a closed expression for the quantities $\mathfrak{w}_s(r,t;\ell,m)$. Like in the proof of Proposition A.2, we will use an inductive argument with base $r=\ell$. This explains why we first settle this case separately.

Proposition A.5. *Let $s,t,\ell,$ and m be integers satisfying $1\le t\le\ell\le m$ and $0\le s\le t$. Then*

$$\mathfrak{w}_\ell^{(s)}(t;\ell,m)=0,\quad\text{if }t\ne s,$$

while

$$\mathfrak{w}_\ell^{(t)}(t;\ell,m)=\mathfrak{w}_\ell(t;\ell,m)=\frac{q-1}{q}\left(\mu_t(\ell,m)-(-1)^tq^{\binom{t}{2}}\begin{bmatrix}\ell\\t\end{bmatrix}_q\right).$$

Proof. We have already seen that $\mathfrak{D}_t(\ell, m; r, s) = \mathfrak{D}_t(\ell, m)$ if $s = t$, while otherwise $\mathfrak{D}_t(\ell, m; r, s) = \emptyset$. Therefore the first part of the proposition follows, as well as the identity $\mathfrak{w}_\ell^{(t)}(t; \ell, m) = \mathfrak{w}_\ell(t; \ell, m)$. Now we prove that

$$\mathfrak{w}_\ell(t; \ell, m) = \frac{q-1}{q}\left(\mu_t(\ell, m) - (-1)^t q^{\binom{t}{2}}\begin{bmatrix}\ell\\t\end{bmatrix}_q\right)$$

with induction on ℓ.

Induction basis: If $\ell = 1$ (implying that $t = 1$ as well), Proposition 3.1 (or a direct computation) implies that $\mathfrak{w}_1(t; 1, m) = (q-1)q^{m-1}$, which fits with the formula we wish to show.

Induction step: Assume that $\ell > 1$ and that the formula holds for $\ell - 1$. Using Theorem 3.2 in the special case that $r = \ell$, we see that

$$\mathfrak{w}_\ell(t; \ell, m) = q^t \mathfrak{w}_{\ell-1}(t; \ell-1, m-1) - q^{t-1}\mathfrak{w}_{\ell-1}(t-1; \ell-1, m-1) + A,$$

where A is easily seen to be equal to

$$A = \frac{q-1}{q}\left(\mu_t(\ell, m) - q^t \mu_t(\ell-1, m-1) + q^{t-1}\mu_{t-1}(\ell-1, m-1)\right),$$

using the identity $\mu_t(\ell, m) = q^t \mu_t(\ell-1, m) + (q^m - q^{t-1})\mu_{t-1}(\ell-1, m)$. The induction hypothesis now implies that

$$\begin{aligned}
\mathfrak{w}_\ell(t; \ell, m) &= \frac{q-1}{q}\left(\mu_t(\ell, m) - q^t(-1)^t q^{\binom{t}{2}}\begin{bmatrix}\ell-1\\t\end{bmatrix}_q + q^{t-1}(-1)^{t-1}q^{\binom{t-1}{2}}\begin{bmatrix}\ell-1\\t-1\end{bmatrix}_q\right)\\
&= \frac{q-1}{q}\left(\mu_t(\ell, m) - (-1)^t q^{\binom{t}{2}}\left(q^t\begin{bmatrix}\ell-1\\t\end{bmatrix}_q + \begin{bmatrix}\ell-1\\t-1\end{bmatrix}_q\right)\right)\\
&= \frac{q-1}{q}\left(\mu_t(\ell, m) - (-1)^t q^{\binom{t}{2}}\begin{bmatrix}\ell\\t\end{bmatrix}_q\right),
\end{aligned}$$

which is what we wanted to show. $\square$

Now that the case $r = \ell$ is settled, we deal with the general case.

Theorem A.6. *Let $r, s, t, \ell,$ and m be integers satisfying $1 \le t \le \ell \le m$, $1 \le r \le \ell$, and $0 \le s \le t$. Then*

$$\mathfrak{w}_r^{(s)}(t; \ell, m) = \frac{q-1}{q}q^{\binom{s}{2}}\left(\frac{[m]_q!}{[m-t]_q!} - (-1)^s\frac{[m-s]_q!}{[m-t]_q!}\right)q^{s(\ell-r)}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q.$$

Proof. Fix $r \ge 1$. We prove the theorem by induction on ℓ for values $\ell \ge r$. If $\ell = r$, we have $\mathfrak{w}_r^{(s)}(t; \ell, m) = 0$ if $s \ne t$ and $\mathfrak{w}_r^{(s)}(t; \ell, m) = \mathfrak{w}_r(t; \ell, m)$ if $s = t$. Using Proposition A.5 we see that the case $\ell = r$ of the theorem is valid.

Now suppose $\ell > r$. We may then apply Lemma A.3 and apply the induction hypothesis. Performing very similar computations as in the proof of Proposition A.2, the induction step follows. $\square$

We can now state our alternative formula for $\mathfrak{w}_r(t; \ell, m)$.

Theorem A.7. *Let $r, t, \ell,$ and m be integers satisfying $1 \le t \le \ell \le m$ and $1 \le r \le \ell$. Then*

$$\begin{aligned}
\mathfrak{w}_r(t; \ell, m) &= \frac{q-1}{q}\sum_{s=0}^{r}q^{\binom{s}{2}}\left(\frac{[m]_q!}{[m-t]_q!} - (-1)^s\frac{[m-s]_q!}{[m-t]_q!}\right)q^{s(\ell-r)}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q\\
&= \frac{q-1}{q}\left(\mu_t(\ell, m) - \sum_{s=0}^{r}q^{\binom{s}{2}}(-1)^s\frac{[m-s]_q!}{[m-t]_q!}q^{s(\ell-r)}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q\right).
\end{aligned}$$

Proof. The first equation is a direct consequence of (20) and Theorem A.6. For the second equation, note that

$$\sum_{s=0}^{r}q^{\binom{s}{2}}\frac{[m]_q!}{[m-t]_q!}q^{s(\ell-r)}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q = \sum_{s=0}^{r}|\mathfrak{D}_t(\ell, m; r, s)| = |\mathfrak{D}_t(\ell, m)|,$$

since $\mathfrak{D}_t(\ell, m)$ is the disjoint union of the sets $\mathfrak{D}_t(\ell, m; r, s)$, $0 \le s \le r$. $\square$

The above theorem in particular implies that

$$P_t(r) = \sum_{s=0}^{r}q^{\binom{s}{2}}(-1)^s\frac{[m-s]_q!}{[m-t]_q!}q^{s(\ell-r)}q^{\binom{t-s}{2}}\begin{bmatrix}r\\s\end{bmatrix}_q\begin{bmatrix}\ell-r\\t-s\end{bmatrix}_q, \tag{21}$$

where $P_t(r)$ is the expression from (17). It is not immediately clear that these two expressions for $P_t(r)$ are in fact equal. However, in [9, Eq. (15)] the generalized Krawtchouk polynomial $F(x, k, n)$ is defined (involving parameters x, k, n as well as a parameter c). If one chooses $c = q^{m-\ell}$, $n = \ell$, $k = t$, and $x = r$ one obtains the polynomial $P_t(r)$ from (17). An alternative expression for $F(x, k, n)$ is then given in [9, p. 167]. Choosing the parameters n, k, x, and c as before in this expression, we obtain (21).

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