# Ramanujan Mathematical Society 

## Lecture Notes Series

Number 18

## Combinatorial Topology and Algebra

## Volume Editors

Sudhir R. Ghorpade, Anant R. Shastri, Murali K. Srinivasan and Jugal K. Verma

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## Lecture Notes Series

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## Preface

This volume contains the lecture notes prepared for the participants of the Instructional Conference on Combinatorial Topology and Algebra (ICCTA-93) held at IIT Bombay during December 5-24, 1993. These notes include a complete solution, due to R. Stanley, of the Upper Bound Conjecture, and an exposition of topics in combinatorial topology such as triangulations of compact surfaces and minimal triangulation of manifolds. A key feature of these notes is that all the relevant background material from commutative algebra, combinatorics, and topology is developed from scratch. The table of contents should give a more detailed idea of the topics covered in these notes and the interdependence of topics is indicated in the leitfaden appearing after the table of contents. We have also included the text of an introductory talk by one of us, which may provide an overall perspective on the contents of these notes. Finally, there are two appendices at the end that correspond to two of the special talks given at ICCTA-93 and an epilogue by the editors that attempts to briefly provide an update on some of the developments that have occurred since the writing of these notes and their publication here.

We take this opportunity to thank the National Board of Higher Mathematics of the Government of India for sponsoring ICCTA-93, and the Department of Mathematics at IIT Bombay for providing excellent facilities and support not only for ICCTA-93, but also for the year long ACT (=Algebra-Combinatorics-Topology) seminar that preceded ICCTA-93. We are also grateful to the Ramanujan Mathematical Society for its interest in publishing these notes in its Lecture Notes Series even though they are about two decades old.

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## Leitfaden

$$
\begin{array}{r}
V-E+F=2 \\
-L . \text { Euler }
\end{array}
$$



## Opening Remarks ${ }^{1}$

The formula $v-e+f=2$ for a graph drawn on a sheet of paper was apparently implicit in the works of Descartes in the 17th century. His work was lost and it was Euler who discovered this formula independently a century later in 1752. Following the work of Riemann and Betti, the desire to have such a formula for arbitrary surfaces was felt by many mathematicians of 19th century. The credit here goes to Jordan who found and proved the formula $v-e+f=3-P_{1}$. Here $P_{1}$ denotes the "order of connectivity" of the surface on which the graph is drawn; this notion was introduced by Betti. The name Betti number was given by Lefschetz to $P_{1}-1$. Around the mid 19th century, Schläfli formulated the following problem for convex polytopes of higher dimensions: show that the alternate sum of the face numbers is equal to 1 . Schläfli published a 'proof' of this only around 1905. But meanwhile, this result had captured the imagination of several contemporary mathematicians and as many "proofs" of this result appeared (all these "proofs" implicitly assumed in some form or the other, the so called shellability property of convex polytopes, which was proved much later in 1971). Finally, this problem caught the attention of the indomitable Poincaré. He realized that the problem is really topological in nature, and hence needs to be reformulated in the more general setting of a smooth manifold. In the process, Poincaré laid down the foundations of a branch of topology that we now call PLtopology. (See the last section of his classic work Analysis Situs of 1895.) The creation of PL-topology was a gem of an idea that was directly motivated by the combinatorial problem of Schläfli. For a vivid account of the history of algebraic topology, see the book of Dieudonné [D].

Alternating sum of numbers of chains of different lengths occur in the study of Möbius inversion formula. Rota (1964) pointed out that instead of counting merely the numbers, one should look at the homology groups of the order-complexes. He conjectured that geometric lattices have the "Cohen-Macaulay property". This was proved by Folkman (1966) and the proof is rather complicated. Shellability seems to be an important passage between various combinatorial questions. Here again, motivated by MacMahon's work (1916) on permutation enumerations and Stanley's work (1972) on R-labelling, Björner (1980) introduced the notion of EL-labelling and CL-labelling. These notions give effective methods of determining shellability. On the other hand, inspired possibly by Macaulay's work on the study of Hilbert polynomial of ordercomplexes, Hochster and Stanley independently introduced the powerful notion of the "face ring" of a simplicial complex. Reisner (1974) gave a brilliant combinatorial characterization of Cohen-Macaulay property of the face ring of a simplicial complex. This led Baclawsky (1976) and Stanley (1977) to introduce the notion of Cohen-Macaulayness in a purely combinatorial setting. Once we have the result that a shellable complex is Cohen-Macaulay in this combinatorial sense, the result of Folkman quoted above becomes transparent.

Another important achievement of the notion of face ring is the solution of the Upper Bound Conjecture by Stanley (1975). This will be dealt with in full detail in this conference. And I believe that this seems to justify allotting as much time as is done here to the study of commutative algebra per se.

It is the aim of this instructional conference to bring out some of the aspects outlined above to potential researchers interested in working in this area. We have preferred to concentrate on a few central themes than to discuss a large number of topics peripherally. One of the obvious reasons is the time constraint. Upon specific instructions from the sponsors, we have tried to keep the level as elementary as these concepts permit.

Exactly an year ago, I was attending a conference on Novikov conjecture at the Indian Statistical Institute (ISI) Calcutta. Due to the prevalent disturbed atmosphere in the city, we were forced to stay within the ISI campus and enjoy the natural surroundings it offered. Parameswaran Sankaran of Madras was my constant companion during several hours of walks that we took around many beautiful ponds therein. The germ of the idea to do something like the present conference was born during these walks. It got ready support from some of the other participants there such as Basudev Datta and Himadri Mukherjee. Back home at Bombay, I met M. S. Raghunathan at the Tata Institute of Fundamental Research (TIFR) and was discussing some other matters. As if reading my mind, he suggested that I should organize an instructional conference on some theme related to PL-topology. The present form of the conference is a result of the detailed discussion with colleagues in the department, friends at Bombay University and TIFR.

It gives me great pleasure to place these pre-lecture notes in your hand on the very first day of the conference. From the point of view of the organizers, preparation of such notes has many advantages, the details of which I need not go into at present. We believe that it would also be welcomed by the participants. The notes contain almost everything that is going to be presented to you during the conference. The material is the outcome of the year long weekly seminars we had in the department. Though the notes are generally written by the respective speakers, it would be more appropriate to call the entire notes a single piece of team-work involving many others apart from the speakers. I am confident that these notes will prove to be useful to you during as well as after the conference.

I conclude with some of the key words for this conference: Eulerian Property, Shallability, Cohen-Macaulayness. During the course of this conference and through these notes, you will learn what these words mean and much more. We hope that you will enjoy this journey.

