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Higher Weights of Affine Grassmann Codes and Their Duals

Mrinmoy Datta and Sudhir R. Ghorpade

Abstract. We consider the question of determining the higher weights or the generalized Hamming weights of affine Grassmann codes and their duals. Several initial as well as terminal higher weights of affine Grassmann codes of an arbitrary level are determined explicitly. In the case of duals of these codes, we give a formula for many initial as well as terminal higher weights. As a special case, we obtain an alternative simpler proof of the formula of Beelen et al for the minimum distance of the dual of an affine Grasmann code.

1. Introduction

A q-ary linear code of length n and dimension k, or in short, a $[n,k]_q$-code, is simply a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements. A basic example is that of a (generalized) Reed-Muller code $\mathrm{RM}(\nu,\delta)$ of order ν and length $n := q^\delta$, given by the image of the evaluation map

$$\mathrm{Ev} : \mathbb{F}_q[X_1, \dots X_\delta]_{\le \nu} \to \mathbb{F}_q^n \quad \text{defined by} \quad \mathrm{Ev}(f) = (f(P_1), \dots, f(P_n)),$$

where $\mathbb{F}_q[X_1, \dots X_\delta]_{\le \nu}$ denotes the space of polynomials in δ variables of (total) degree $\le \nu$ with coefficients in $\mathbb{F}_q$ and $P_1, \dots, P_n$ is an ordered listing of the points of the affine space $\mathbb{A}^\delta(\mathbb{F}_q) = \mathbb{F}_q^\delta$. A useful variant of this is the projective Reed-Muller code $\mathrm{PRM}(\nu,\delta)$ of order ν and length $n := (q^{\delta+1}-1)/(q-1)$, which is obtained by evaluating homogeneous polynomials in $\delta+1$ variables of degree ν with coefficients in $\mathbb{F}_q$ at points of the projective space $\mathbb{P}^\delta = \mathbb{P}^\delta(\mathbb{F}_q)$ or rather at suitably normalized representatives in $\mathbb{F}_q^{\delta+1}$ of an ordered listing of the points of $\mathbb{P}^\delta$.

From a geometric viewpoint, projective Reed-Muller codes $\mathrm{PRM}(\nu,\delta)$ correspond (at least when $\nu < q$) to the Veronese variety given by the image of $\mathbb{P}^\delta$ in $\mathbb{P}^{k-1}$ under the Veronese map of degree ν, where $k := \binom{\nu+\delta}{\nu}$. In this set-up, $\mathrm{RM}(\nu,\delta)$ corresponds to the image of this Veronese map when restricted to an $\mathbb{A}^\delta$ inside $\mathbb{P}^\delta$ (for instance, the set of points $(x_0 : x_1 : \cdots : x_\delta)$ of $\mathbb{P}^\delta$ with $x_0 = 1$).

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Reed-Muller codes are classical objects and in the generalized setting above, their study goes back at least to Kasami, Lin, and Peterson [**10**] as well as Delsarte, Goethals, and MacWilliams [**4**]. One may refer to [**2**, Prop. 4] for a summary of several of the basic properties of $\mathrm{RM}(\nu, \delta)$. Projective Reed-Muller codes appeared explicitly in the work of Lachaud [**11, 12**] and Sørensen [**17**]. Around the same time, a new class of codes called Grassmann codes were studied by Ryan [**14, 15**], and later by Nogin [**13**] and several others (see, e.g., [**6, 7, 8, 5**]). These correspond geometrically to the Grassmann variety $G_{\ell,m}$ formed by the ℓ-dimensional subspaces of $\mathbb{F}_q^m$ together with the Plücker embedding $G_{\ell,m} \hookrightarrow \mathbb{P}^{k-1}$, where $k = \binom{m}{\ell}$. In effect, the Grassmann code $C(\ell, m)$ is a linear code whose generator matrix has as its columns certain fixed representatives in $\mathbb{F}_q^k$ of the Plücker coordinates of all $\mathbb{F}_q$-rational points of $G_{\ell,m}$. Affine Grassmann codes were introduced in [**1**] and further studied in [**2**] and [**5**]. Given positive integers ℓ, ℓ' with $\ell \le \ell'$, upon letting $m = \ell + \ell'$ and $\delta = \ell\ell'$, the affine Grassmann code $C^{\mathbb{A}}(l, m)$ is defined, like a Reed-Muller code, as the q-ary linear code of length $n = q^\delta$ given by the image of the evaluation map

$$\text{(1)} \qquad \mathrm{Ev} : \mathcal{F}(\ell, m) \to \mathbb{F}_q^n \quad \text{defined by} \quad \mathrm{Ev}(f) = (f(P_1), \dots, f(P_n)),$$

where $\mathcal{F}(\ell, m)$ is the space of linear polynomials in the minors of a generic $\ell \times \ell'$ matrix X and $P_1, \dots, P_n$ is an ordered listing of the δ-dimensional affine space of all $\ell \times \ell'$ matrices with entries in $\mathbb{F}_q$. The relationship between affine Grassmann codes $C^{\mathbb{A}}(l, m)$ and Grassmann codes $C(l, m)$ is akin to that between Reed-Muller codes $\mathrm{RM}(\nu, \delta)$ and projective Reed-Muller codes $\mathrm{PRM}(\nu, \delta)$.

The notion of higher weight, also known as generalized Hamming weight, of a linear code is a natural and useful generalization of the basic notion of minimum distance. If C is a $[n, k]_q$-code, then for $r = 0, 1, \dots, k$, the r^{th} *higher weight* of C is defined by

$$d_r = d_r(C) = \min\{\mathrm{w_H}(D) : D \text{ is a subspace of } C \text{ with } \dim D = r\},$$

where $\mathrm{w_H}(D)$ denotes the support weight of D [see Section 2 below for a definition]. Clearly, $d_1(C)$ is the minimum distance $d(C)$ of C. It is well-known and easy to see that $0 = d_0 < d_1 < \cdots < d_k$ and moreover $d_k = n$ provided C is nondegenerate. It is, in general, an interesting and difficult question to determine the weight hierarchy, i.e., all the higher weights, of a given class of codes. For example, in a significant piece of work, Heijnen and Pelikaan [**9**] completely determined the higher weights of Reed-Muller codes $\mathrm{RM}(\nu, \delta)$. In the case of projective Reed-Muller codes, the minimum distance was determined by Lachaud [**12**] and independently by Sørrensen [**17**]. In fact, Lachaud derives it as a consequence of an affirmative answer given by Serre [**16**] to a question of Tsfasman concerning the maximum number of $\mathbb{F}_q$-rational points on a projective hypersurface of a given degree. The second higher weight was determined by Boguslavsky [**3**], while the determination of $d_r(\mathrm{PRM}(\nu, \delta))$ is still open for $r > 2$. In the case of Grassmann codes, the r^{th} higher weight is known for the first few and the last few values of r, thanks to Nogin [**13**] (see also [**6**]) and Hansen, Johnsen and Ranestad [**8**] (see also [**7**]). More precisely, for $r = 0, 1, \dots \mu$, where $\mu := 1 + \max\{\ell, m - \ell\}$, we have

$$d_r(C(\ell, m)) = q^\delta + q^{\delta-1} + \cdots + q^{\delta-r+1} \quad \text{and} \quad d_{k-r}(C(\ell, m)) = n - (1 + q + \cdots + q^{r-1}),$$

where n denotes the length of $C(\ell, m)$ or in other words, the number of $\mathbb{F}_q$-rational points of $G_{\ell,m}$, and it is given by the Gaussian binomial coefficient $\left[{m \atop \ell}\right]_q$. In case $\ell = 2$, we know a little more (cf. [**7**]), but the general case is still open.

We consider in this paper the problem of determining the higher weights of affine Grassmann codes and their duals. Our main result is an explicit formula for $d_r\left(C^{\mathbb{A}}(l, m)\right)$ for the first few and the last few values of r, or more precisely, for $0 \le r \le \mu'$ and for $k - \mu \le r \le k$, where

$$\mu' = 1 + \max\{\ell,\, \ell' - \ell\} \quad \text{ and } \quad \mu := 1 + \max\{\ell,\, \ell'\} = \ell' + 1. \tag{2}$$

In the case of the result for the first μ' higher weights, we have to make an additional mild assumption that $\ell < \ell'$. The result for the last μ higher weights can be deduced from the corresponding results for Grassmann codes using a geometric approach. However, we give here self-contained proofs in the spirit of [**1, 2**] and this has the advantage that analogous results are also obtained for affine Grassmann codes of arbitrary level introduced in [**2**]. As for the duals, we can in fact go much farther, and determine many more higher weights of the duals of affine Grassmann codes except that the result we give here is best described recursively. As a corollary, we obtain a new and simpler proof of [**2**, Theorem 17], which states that if $\ell' > 1$, then the minimum distance of $C^{\mathbb{A}}(l, m; h)^{\perp}$ is 3 or 4 according as $q > 2$ or $q = 2$. The geometric approach and an alternative proof of the result about the last μ higher weights is also outlined in an appendix for the convenience of the reader.

2. Initial Higher Weights

For any q-ary linear code C of length n, and any $D \subseteq C$, we let

$$\mathrm{Supp}(D) := \{i \in \{1, \dots, n\} : c_i \neq 0 \text{ for some } c \in D\} \quad \text{ and } \quad \mathrm{w_H}(D) = |\mathrm{Supp}(D)|$$

denote, respectively, the *support* and the *support weight* of D. For a codeword $c = (c_1, \dots, c_n) \in C$, we write $\mathrm{w_H}(c) = \mathrm{w_H}(\{c\})$ and note that this is simply the *Hamming weight* of c.

Fix, throughout this paper, positive integers h, ℓ, ℓ' with $h \le \ell \le \ell'$ and an $l \times l'$ matrix $X = (X_{ij})$ whose entries are algebraically independent indeterminates over $\mathbb{F}_q$. Let $\mathbb{F}_q[X]$ denote the ring of polynomials in the $\ell\ell'$ variables X_{ij}'s with coefficients in $\mathbb{F}_q$. As in [**2**], we let $\Delta(\ell, m; h)$ denote the set of all minors of X of degree $\le h$. Note that $\Delta(\ell, m; h)$ is a subset of $\mathbb{F}_q[X]$ that contains the constant polynomial 1, which corresponds to the 0×0 minor of X. Further let

$$\mathcal{F}(\ell, m; h) := \text{the } \mathbb{F}_q\text{-linear subspace of } \mathbb{F}_q[X] \text{ generated by } \Delta(\ell, m; h).$$

Note that the space $\mathcal{F}(\ell, m)$ defined in the Introduction contains $\mathcal{F}(\ell, m; h)$ and the equality holds when $h = \ell$. The *affine Grassmann code of level* h, denoted $C^{\mathbb{A}}(\ell, m; h)$, is defined to be the image of $\mathcal{F}(\ell, m; h)$ under the evaluation map Ev given by (1). Evidently, $C^{\mathbb{A}}(\ell, m; \ell) = C^{\mathbb{A}}(\ell, m)$ and $C^{\mathbb{A}}(\ell, m; h)$ is a subcode of $\mathrm{RM}(h, \delta)$. Now here is a slightly refined version of a basic result proved in [**2**].

Proposition 2.1. *The minimum distance $d(\ell, m; h)$ of $C^{\mathbb{A}}(\ell, m; h)$ is*

$$d(\ell, m; h) = q^{\delta} \prod_{i=1}^{h} \left(1 - \frac{1}{q^i}\right) = q^{\delta - h^2} \left|\mathrm{GL}_h(\mathbb{F}_q)\right|. \tag{3}$$

Moreover, if $\mathcal{M}$ is any $h \times h$ minor of X, then $\mathrm{w_H}\left(\mathrm{Ev}\left(\mathcal{M}\right)\right) = d(\ell, m; h)$.

Proof. The first equality (3) is proved in [**2**, Theorem 5], while the second is easily deduced. Also it is shown that in [**2**, Theorem 5] if $\mathcal{L}_h = \det(X_{ij})_{1\le i,j\le h}$ is the h^{th} leading principal minor of X, then $\mathrm{w_H}(\mathrm{Ev}(\mathcal{L}_h)) = d(\ell, m; h)$. Now if $\mathcal{M}$ is any $h \times h$ minor of X, then there are positive integers $p_1, \dots, p_h$, $q_1, \dots, q_h$ with $p_1 < \cdots < p_h \le \ell$ and $q_1 < \cdots < q_h \le \ell'$ such that $\mathcal{M} = \det(X_{p_i q_j})_{1\le i,j\le h}$. Let $\sigma \in S_\ell$ be a permutation such that $\sigma(i) = p_i$ for $1 \le i \le \ell$ and $P \in \mathrm{GL}_\ell(\mathbb{F}_q)$ be the permutation matrix corresponding to σ so that for $1 \le i, j \le \ell$, the $(i, j)^{\text{th}}$ entry of P is 1 is $j = \sigma(i)$ and 0 otherwise. Likewise, let $\tau \in S_{\ell'}$ be such that $\tau(i) = q_i$ for $1 \le i \le \ell'$ and $Q \in \mathrm{GL}_{\ell'}(\mathbb{F}_q)$ be the permutation matrix corresponding to τ. Then it is easily seen that $\mathcal{M}$ is the the h^{th} leading principal minor of PXQ^{-1}. Moreover, we know from [**2**, §IV] that $X \mapsto PXQ^{-1}$ induces a permutation automorphism of $C^{\mathbb{A}}(\ell, m; h)$. It follows that $\mathrm{w_H}(\mathrm{Ev}(\mathcal{M})) = \mathrm{w_H}(\mathrm{Ev}(\mathcal{L}_h)) = d(\ell, m; h)$. □

The following general observation about the support weights of linear codes will be useful in the sequel.

Lemma 2.2. *Let C be an $[n, k]_q$-code and for $i = 1, \dots, n$, let $\pi_i : C \to \mathbb{F}_q$ denote the i^{th} projection map defined by $\pi_i(c_1, \dots, c_n) = c_i$. Also let D be a subcode of C and $\{y_1, \dots, y_r\}$ be a generating set of D. Then*

$$\mathrm{Supp}(D) = \bigcup_{j=1}^{n} A_j \quad \text{where for } 1 \le j \le n, \quad A_j := \{i \in \{1, \dots, n\} : \pi_i(y_j) \neq 0\}.$$

Proof. Clearly, $\cup_{j=1}^n A_j \subseteq \mathrm{Supp}(D)$. On the other hand, suppose $i \in \{1, \dots, n\}$ is such that $i \notin \cup_{j=1}^n A_j$. Then $\pi_i(y_j) = 0$ for all $j = 1, \dots, r$. Now for any $x \in D$, we can write $x = \sum_{j=1}^r c_j y_j$ for some $c_1, \dots, c_r \in \mathbb{F}_q$; hence $\pi_i(x) = \sum_{j=1}^r c_j \pi_i(y_j) = 0$. Thus $i \notin \mathrm{Supp}(D)$. This shows that $\mathrm{Supp}(D) \subseteq \cup_{j=1}^n A_j$. □

The next two lemmas extend Proposition 2.1 and show that for a judicious choice of a family $\{\mathcal{M}_1, \dots, \mathcal{M}_r\}$ of minors of X, the support weight of the product of any nonempty subfamily is given by a formula analogous to (3).

Lemma 2.3. *Let r be a positive integer such that $r \le \ell' - h + 1$ and let Y be any $h \times (h + r - 1)$ submatrix of X. Also for $j = 1, \dots, r$, let $\mathcal{M}_j$ denote the $h \times h$ minor of Y (and hence of X) corresponding to the first $h - 1$ columns of Y together with the $(h + j - 1)^{\text{th}}$ column of Y, and let $A_j = \{P \in \mathbb{A}^\delta : \mathcal{M}_j(P) \neq 0\}$. Then for any positive integer s with $s \le r$ and any $j_1, \dots, j_s \in \{1, \dots, r\}$ with $j_1 < \cdots < j_s$,*

$$|A_{j_1} \cap \cdots \cap A_{j_s}| = d(\ell, m; h)\left(1 - \frac{1}{q}\right)^{s-1}. \tag{4}$$

Proof. Given any $\ell \times \ell'$ matrix $P \in \mathbb{A}^\delta$ with entries in $\mathbb{F}_q$, let Q denote the $h \times (h + r - 1)$ submatrix of P formed in exactly the same way as Y, and let $Q_1, \dots Q_{h+r-1}$ denote the column vectors of Q. For any positive integer s with $s \le r$ and any $j_1, \dots, j_s \in \{1, \dots, r\}$ with $j_1 < \cdots < j_s$, the condition $P \in A_{j_1} \cap \cdots \cap A_{j_s}$ is equivalent to the condition that the column vectors $Q_1, \dots, Q_{h-1}, Q_{h+j-1}$ in $\mathbb{F}_q^h$ are linearly independent for each $j \in \{j_1, \dots, j_s\}$. This will hold when the submatrix of Q formed by its first $h-1$ columns is chosen in exactly $(q^h - 1)(q^h - q) \cdots (q^h - q^{h-2})$ ways, while each of $Q_{h+j_1-1}, \dots, Q_{h+j_s-1}$ are chosen in $(q^h - q^{h-1})$ ways. The remaining $r - s$ columns of Q may be chosen arbitrarily in $q^{h(r-s)}$ ways. Since P

has $\ell\ell' - h(h+r-1)$, i.e., $\delta - h^2 - h(r-1)$, entries outside Q, it follows that

$$\begin{aligned} |A_{j_1} \cap \cdots \cap A_{j_s}| &= (q^h-1)(q^h-q)\cdots(q^h-q^{h-2})(q^h-q^{h-1})^s q^{\delta-h^2-h(s-1)} \\ &= d(\ell,m;h)\left(1-\frac{1}{q}\right)^{s-1}, \end{aligned}$$

where the last equality follows from (3). □

Lemma 2.4. *Assume that $h < \ell'$. Let r be a positive integer such that $r \le h+1$ and let Y be any $h \times (h+1)$ submatrix of X. Also for $j = 1, \dots, r$, let $\mathcal{M}_j$ denote the determinant of the $h \times h$ submatrix of Y formed by all except the $(h-r+j+1)^{\text{th}}$ column of Y, and let $A_j = \{P \in \mathbb{A}^\delta : \mathcal{M}_j(P) \neq 0\}$. Then (4) holds for any positive integer s with $s \le r$ and any $j_1, \dots, j_s \in \{1, \dots, r\}$ with $j_1 < \cdots < j_s$.*

Proof. Given $P \in \mathbb{A}^\delta$, let Q be the $h \times (h+1)$ submatrix of P corresponding to Y, and let $Q_1, \dots, Q_{h+1}$ denote its column vectors. Fix any positive integer s with $s \le r$ and $j_1, \dots, j_s \in \{1, \dots, r\}$ with $j_1 < \cdots < j_s$. Now $\mathcal{M}_{j_1}(P) \neq 0$ implies that Q has rank h and in particular, Q_{h-r+j_1+1} is a $\mathbb{F}_q$-linear combination of the remaining h column vectors of Q. Moreover, for $2 \le t \le s$, if $\mathcal{M}_{j_t}(P) \neq 0$, then the coefficients of Q_{h-r+j_t+1} in this $\mathbb{F}_q$-linear combination must be nonzero. Conversely, if all except the $(h-r+j_1+1)^{\text{th}}$ column of Q are linearly independent (and these columns can thus be chosen in $|\mathrm{GL}_h(\mathbb{F}_q)|$ ways), while Q_{h-r+j_1+1} is a $\mathbb{F}_q$-linear combination of the remaining h column vectors of Q with a nonzero coefficient for the $s-1$ columns $Q_{h-r+j_2+1}, \dots, Q_{h-r+j_s+1}$, then $\mathcal{M}_{j_t}(P) \neq 0$ for each $t = 1, \dots, s$. The h coefficients in this $\mathbb{F}_q$-linear combination can thus be chosen in $q^{h-s+1}(q-1)^{s-1}$ ways. Since P has $\delta - h(h+1)$ entries outside of Q, it follows that

$$|A_{j_1} \cap \cdots \cap A_{j_s}| = |\mathrm{GL}_h(\mathbb{F}_q)|\, q^{h-s+1}(q-1)^{s-1} q^{\delta-h(h+1)} = d(\ell,m;h)\left(1-\frac{1}{q}\right)^{s-1},$$

where the last equality follows once again from (3). □

Theorem 2.5. *Let r be a positive integer such that $r \le \max\{\ell'-h, h\}+1$. Assume that $h < \ell'$ in case $\max\{\ell'-h, h\} = h$, i.e., $\ell' \le 2h$. Then the r^{th} higher weight $d_r(\ell,m;h)$ of $C^{\mathbb{A}}(\ell,m;h)$ is*

$$d_r(\ell,m;h) = q^{\delta-r+1}\frac{(q^r-1)}{(q-1)}\prod_{i=1}^{h}\left(1-\frac{1}{q^i}\right) = q^{\delta-h^2-r+1}(q^r-1)\frac{|\mathrm{GL}_h(\mathbb{F}_q)|}{q-1}. \tag{5}$$

Moreover, the r^{th} higher weight of $C^{\mathbb{A}}(\ell,m;h)$ attains the Griesmer-Wei bound.

Proof. The hypotheses on r and h together with Lemmas 2.3 and 2.4 ensure that there exist minors $\mathcal{M}_1, \dots, \mathcal{M}_r \in \Delta(\ell,m;h)$ with supports $A_1, \dots, A_r$ respectively, such that (4) holds for any positive integer s with $s \le r$ and any

$j_1, \dots, j_s \in \{1, \dots, r\}$ with $j_1 < \cdots < j_s$. Consequently,

$$\begin{aligned}\left|\bigcup_{j=1}^{r} A_j\right| &= \sum_{s=1}^{r}(-1)^{s-1} \sum_{1\le j_1<\cdots<j_s\le r} |A_{j_1} \cap \cdots \cap A_{j_s}| \\ &= \sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s} d(\ell, m; h)\left(1-\frac{1}{q}\right)^{s-1} \\ &= \frac{d(\ell, m; h)}{1-q^{-1}}\left(1-\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(1-q^{-1}\right)^s\right) \\ &= \frac{d(\ell, m; h)}{1-q^{-1}}\left(1-\left[1-\left(1-q^{-1}\right)\right]^r\right).\end{aligned}$$

Hence, in view of Lemma 2.2 and Proposition 2.1, we see that if D_r is the subspace of $\mathcal{F}(\ell, m; h)$ spanned by $\mathcal{M}_1, \dots, \mathcal{M}_r$, then

$$\mathrm{w_H}(D_r) = d(\ell, m; h)\frac{(q^r-1)}{q^{r-1}(q-1)} = q^{\delta-r+1}\frac{(q^r-1)}{(q-1)}\prod_{i=1}^{h}\left(1-\frac{1}{q^i}\right). \tag{6}$$

Moreover, by [**1**, Lemma 3], we see that D_r is of dimension r. Thus,

$$d_r(\ell, m; h) := d_r\left(C^{\mathbb{A}}(\ell, m; h)\right) \le d(\ell, m; h)\frac{(q^r-1)}{q^{r-1}(q-1)} = \sum_{i=0}^{r-1}\frac{d(\ell, m; h)}{q^i}. \tag{7}$$

On the other hand, the Griesmer-Wei bound (cf. [**20**]) and Proposition 2.1 yields

$$d_r\left(C^{\mathbb{A}}(\ell, m; h)\right) \ge \sum_{i=0}^{r-1}\left\lceil\frac{d(\ell, m; h)}{q^i}\right\rceil \ge \sum_{i=0}^{r-1}\frac{d(\ell, m; h)}{q^i}. \tag{8}$$

Using (6), (7) and (8), we obtain the desired result. □

Remark 2.6. The only case in which the above theorem does not give any higher weights of $C^{\mathbb{A}}(\ell, m; h)$ beyond the minimum distance is when $h = \ell = \ell'$. We believe that in this case even the second higher weight does not meet the Griesmer-Wei bound. In fact, it seems plausible that for $C^{\mathbb{A}}(\ell, 2\ell) = C^{\mathbb{A}}(\ell, 2\ell; \ell)$,

$$d_2\left(C^{\mathbb{A}}(\ell, 2\ell)\right) = q^\ell\left(1+\frac{1}{q}-\frac{1}{q^\ell}\right)\prod_{i=0}^{\ell-1}\left(q^\ell-q^i\right) = d(\ell, 2\ell; \ell)\left(1+\frac{q^{\ell-1}}{q^\ell-1}\right).$$

Note that the expression on the right is strictly greater than $d(\ell, 2\ell; \ell)\left(1+q^{-1}\right)$. We remark also that the expression on the right is an upper bound for $d_2\left(C^{\mathbb{A}}(\ell, 2\ell)\right)$. This can be seen, for example, by considering the 2-dimensional subspace D of $\mathcal{F}(\ell, 2\ell)$ spanned by the ℓ^{th} leading principal minor $\mathcal{M}_1 = \det(X)$ and the $(\ell-1)^{\text{th}}$ leading principal minor $\mathcal{M}_2$ of X, and using Lemma 2.2 to show that

$$\begin{aligned}|\mathrm{Supp}(D)| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ &= |\mathrm{GL}_\ell(\mathbb{F}_q)| + |\mathrm{GL}_{\ell-1}(\mathbb{F}_q)|q^{2\ell-1} - \mathrm{GL}_{\ell-1}(\mathbb{F}_q)|q^{\ell-1}(q^\ell-q^{\ell-1}) \\ &= d(\ell, 2\ell; \ell)\left(1+\frac{q^{\ell-1}}{q^\ell-1}\right),\end{aligned}$$

where, as before, $A_j = \{P \in \mathbb{A}^\delta : \mathcal{M}_j(P) \ne 0\}$ for $j = 1, 2$.

3. Terminal Higher Weights

As in the case of Grassmann codes, determining some of the terminal higher weights is simpler than determining some of the initial higher weights. In fact, we go a little farther than what we could do with the initial higher weights. Thus, as opposed to finding explicitly the first μ' higher weights of say $C^{\mathbb{A}}(\ell, m)$, where $\mu' = \max\{\ell, \ell' - \ell\} + 1$, we are able find explicitly the last $\ell' + 1$ higher weights of not just $C^{\mathbb{A}}(\ell, m)$, but any $C^{\mathbb{A}}(\ell, m; h)$.

The first step is a simple observation that holds, in fact, for any functional code defined by means of an evaluation map on a space of (polynomial) functions. However, we will just restrict to the case of affine Grassmann codes of a given level.

LEMMA 3.1. *Let D be a subcode of $C^{\mathbb{A}}(\ell, m; h)$ and t a positive integer such that there exist t linearly independent polynomials $g_1, \ldots, g_t \in \mathbb{F}_q[X]$ with the property that $\deg g_i \leq 1$ and $\mathrm{Ev}(g_i) \in D$ for all $i = 1, \ldots, t$. Then $\mathrm{w_H}(D) \geq q^\delta - q^{\delta - t}$.*

PROOF. As noted in [**1**, §II], the evaluation map Ev given by (1) is injective. Hence D is in bijection with $W := \mathrm{Ev}^{-1}(D) = \{f \in \mathcal{F}(\ell, m; h) : \mathrm{Ev}(f) \in D\}$. Moreover, if we let $Z(W) := \{P \in \mathbb{A}^\delta : f(P) = 0 \text{ for all } P \in W\}$ denote the corresponding affine variety. then it is clear that

$$\mathrm{w_H}(D) = |\mathrm{Supp}(D)| = |\mathbb{A}^\delta \setminus Z(W)| = q^\delta - |Z(W)|.$$

Now $g_1, \ldots, g_t \in W$ and the number of common zeros in $\mathbb{A}^\delta$ of $g_1, \ldots, g_t$ corresponds to the number, say N, of solutions of a system of t linearly independent nonhomogeneous linear equations in δ variables with coefficients in $\mathbb{F}_q$. Hence $N = 0$ or $N = q^{\delta - t}$ according as the system is inconsistent or consistent. Consequently, $|Z(W)| \leq N \leq q^{\delta - t}$ and so $\mathrm{w_H}(D) \geq q^\delta - q^{\delta - t}$. □

Let k_h denote the dimension of $C^{\mathbb{A}}(\ell, m; h)$. We know from [**2**, Prop. 2] that

$$k_h = \sum_{i=0}^{h} \binom{\ell}{i}\binom{\ell'}{i} \quad \text{and} \quad k_\ell = \binom{m}{\ell}. \tag{9}$$

In the remainder of this paper, we fix an ordering $\mathcal{N}_1, \ldots, \mathcal{N}_{k_h}$ on $\Delta(\ell, m; h)$ such that the 0×0 minor appears at the end, preceded by the 1×1 minors arranged lexicographically. More precisely, we require $\mathcal{N}_{k_h} = 1$ and $\mathcal{N}_{k_h - (i-1)\ell' - j} = X_{ij}$ for $i = 1, \ldots, \ell$ and $j = 1, \ldots, \ell'$. For instance, $\mathcal{N}_{k_h - 1} = X_{11}$ and $\mathcal{N}_{k_h - \delta} = X_{\ell\ell'}$. Note that $\mathcal{N}_1, \ldots, \mathcal{N}_{k_h}$ gives an ordered $\mathbb{F}_q$-basis of $\mathcal{F}(\ell, m; h)$, thanks to [**2**, Prop. 2].

LEMMA 3.2. $d_{k_h - r}(C^{\mathbb{A}}(\ell, m; h)) \geq q^\delta - q^{r-1}$ *for all* $r = 1, \ldots, \delta + 1$.

PROOF. Fix a positive integer r with $r \leq \delta + 1$ and let $s := k_h - r$. Let D be any s-dimensional subcode of $C^{\mathbb{A}}(\ell, m; h)$. Then $W := \mathrm{Ev}^{-1}(D)$ is a linear subspace of $\mathcal{F}(\ell, m; h)$ with $\dim W = s$. Suppose $f_1, \ldots, f_s$ give an ordered $\mathbb{F}_q$-basis of W. Then there is a unique $s \times k_h$ matrix $\Lambda = (\lambda_{ij})$ of rank s with entries in $\mathbb{F}_q$ such that $\underline{\mathbf{f}} = \Lambda \underline{\mathcal{N}}$, where $\underline{\mathbf{f}} := [f_1, \ldots, f_s]^T$ and $\underline{\mathcal{N}} := [\mathcal{N}_1, \ldots, \mathcal{N}_{k_h}]^T$ denote the column vectors (over $\mathbb{F}_q[X]$) corresponding to the abovementioned ordered $\mathbb{F}_q$-bases of W and $\mathcal{F}(\ell, m; h)$ respectively. Now let $\Lambda^* = \left(\lambda^*_{ij}\right)$ be the reduced row-echelon form of Λ and let $p_1, \ldots, p_s$ with $1 \leq p_1 < \cdots < p_s \leq k_h$ be the column indices in which the pivots occur, so that $\lambda^*_{ip_i} = 1$ and $\lambda^*_{ij} = 0$ if $j < p_i$ and also $\lambda^*_{kp_i} = 0$ if $k \neq i$. It is clear that $p_s \geq s \geq k_h - \delta - 1$. Hence if we let $t := \delta + 1 - r$, then in each of the last t rows of Λ^*, the first $k_h - \delta - 1$ entries are zero. Consequently, the last t rows of the product $\Lambda^*\underline{\mathcal{N}}$ are $\mathbb{F}_q$-linear combinations of the last $\delta+1$ minors among $\mathcal{N}_1, \dots, \mathcal{N}_{k_h}$, and these give rise to linearly independent polynomials $g_1, \dots, g_t \in \mathbb{F}_q[X]$ of degree ≤ 1. Moreover, Λ^* is obtained from Λ by a finite sequence of elementary row operations, and hence there is a nonsingular $s \times s$ matrix P with entries in $\mathbb{F}_q$ such that $\Lambda^* = P\Lambda$. In particular, $g_1, \dots, g_t$ correspond to the last t rows of the product $P\underline{\mathbf{f}}$, and hence they are in W. Now Lemma 3.1 implies that $\mathrm{w_H}(D) \geq q^\delta - q^{\delta-t} = q^\delta - q^{r-1}$. Since D was an arbitrary s-dimensional subscode of $C^{\mathbb{A}}(\ell, m; h)$, we obtain the desired result. $\square$

It can be shown that the lower bound in Lemma 3.2 is attained when $r \leq \ell' + 1$, and this leads to the following result about the terminal higher weights.

Theorem 3.3. *$d_{k_h}(C^{\mathbb{A}}(\ell, m; h)) = q^\delta$ and $d_{k_h - r}(C^{\mathbb{A}}(\ell, m; h)) = q^\delta - q^{r-1}$ for any positive integer r with $r \leq \ell' + 1$.*

Proof. Since $1 \in \mathcal{F}(\ell, m; h)$, we see that $d_{k_h}(C^{\mathbb{A}}(\ell, m; h)) = q^\delta$ or in other words, the code $C^{\mathbb{A}}(\ell, m; h)$ is nondegenerate. Fix a positive integer $r \leq \delta+1$ and let $s := k_h - r$. Consider the linear subspace W of $\mathcal{F}(\ell, m; h)$ spanned by $\{\mathcal{N}_1, \dots, \mathcal{N}_s\}$, i.e., by all the minors in $\Delta(\ell, m; h)$, except $1, X_{11}, \dots, X_{1r-1}$. Observe that the corresponding affine variety $Z(W)$ consists precisely of the $\ell \times \ell'$ matrices $P = (P_{ij}) \in \mathbb{A}^\delta$ satisfying $P_{ij} = 0$ for all $i = 1, \dots, \ell$ and $j = 1, \dots, \ell'$, except when $(i, j) = (1, 1), \dots, (1, r-1)$. Consequently, $|Z(W)| = q^{r-1}$ and thus if we let $D := \mathrm{Ev}(W)$ be the s-dimensional subcode of $C^{\mathbb{A}}(\ell, m; h)$ corresponding to W, then $\mathrm{w_H}(D) = q^\delta - q^{r-1}$. This together with Lemma 3.2 yields the desired result. $\square$

4. Higher Weights of Duals of Affine Grassmann Codes

For determining the higher weights of duals of affine Grassmann code, we use a simple, but powerful, method based on the following key result of Wei [**20**].

Proposition 4.1. *Let C be a $[n, k]_q$-code. Then*

(i) (Monotonicity) *If $k > 0$, then $1 \leq d_1(C) < d_2(C) < \cdots < d_k(C) \leq n$.*

(ii) (Duality) *The higher weights of C and its dual are related by*

$$\left\{d_s(C^\perp) : s = 1, \dots, n-k\right\} = \{1, \dots, n\} \setminus \{n + 1 - d_r(C) : r = 1, \dots, k\}.$$

It is convenient and computationally effective to rephrase the above result for nondegenerate linear codes of positive dimension as follows. We will in fact give two equivalent formulations, the first of which is better suited for the terminal weights while the second is better suited for the initial weights.

Corollary 4.2. *Let C be a nondegenerate $[n, k]_q$-code with $k > 0$. Let $d_0 := 0$ and for $1 \leq r \leq k$, let d_r denote the r^{th} higher weight of C. Also let*

$$e_j := d_j - j \quad \text{and} \quad f_j := n - j - d_{k-j} \quad \text{for} \ \ 0 \leq j \leq k. \tag{10}$$

Then the e-sequence and the f-sequence partition $\{0, 1, \dots, n-k\}$; more precisely,

$$0 = e_0 \leq e_1 \leq \cdots \leq e_k = n - k \quad \text{and} \quad 0 = f_0 \leq f_1 \leq \cdots \leq f_k = n - k. \tag{11}$$

Moreover, for $0 \leq s < n - k$, the last s^{th} higher weight of the dual of C is given by

$$d_{n-k-s}\left(C^\perp\right) = n - s - j \quad \text{if } j \text{ is the unique integer} < k \text{ with } e_j \leq s < e_{j+1}. \tag{12}$$

Equivalently, for $0 < s \leq n - k$, the s^{th} higher weight of the dual of C is given by

$$d_s\left(C^\perp\right) = s + j + 1 \quad \text{if } j \text{ is the unique integer} < k \text{ with } f_j < s \leq f_{j+1}. \tag{13}$$

PROOF. Note that $d_k = n$, since C is nondegenerate. With this in view, part (i) of Proposition 4.1 implies (11). Next, parts (i) and (ii) of Proposition 4.1 together with (11) readily imply (12) and (13) . ☐

REMARK 4.3. The above Corollary shows that the higher weights of the dual of a nondegenerate linear code C of positive dimension k take consecutive values in strings of length $d_{r+1} - d_r - 1$ for $0 \le r < k$, where d_r denotes the r^{th} higher weight of C. Evidently, this phenomenon is prevalent if there are large gaps among the consecutive higher weights of C. In fact, a duality of sorts seems to prevail here: more the number of consecutive strings among the higher weights of a code, the less there are among the higher weights of its dual, and vice-versa. In this connection, it may useful to note the following result of Tsfasman and Vlăduţ [**19**, Cor. 3.5], which states that for $1 \le r \le s \le k$,

$$d_s \ge d_r + \sum_{i=1}^{s-r} \left\lceil \frac{(q-1)d_r}{(q^r-1)q} \right\rceil \quad \text{and in particular,} \quad d_{r+1} - d_r \ge \left\lceil \frac{(q-1)d_r}{(q^r-1)q} \right\rceil.$$

Another special case of Corollary 4.2 worth noting is that $C^\perp$ is nondegenerate if and only if $d_1(C) > 1$.

We now turn to duals of affine Grassmann codes. Recall that we have fixed positive integers h, ℓ, ℓ' with $h \le \ell \le \ell'$ and that the length of the corresponding affine Grassmann code $C^{\mathbb{A}}(\ell, m; h)$ of level h is given by $n := q^\delta$ and the dimension k_h is given by (9). To avoid trivialities we will further assume that $\ell' > 1$. Indeed, it is easy to describe what the affine Grassmann code $C^{\mathbb{A}}(\ell, m; h)$ and its dual is in the trivial case $\ell' = 1$ (or another trivial case $h = 0$ that we have ignored from the beginning) and in fact, this has been done in the paragraph before Theorem 17 in [**2**]. Using the results of Sections 2 and 3, we obtain a more concrete version of Corollary 4.2, which determines several initial and terminal higher weights of the $C^{\mathbb{A}}(\ell, m; h)^\perp$. As a very special case, we also obtain an alternative and simpler proof of [**2**, Theorem 17].

THEOREM 4.4. *Assume that* $\ell' > 1$.

(i) *The minimum distance of* $C^{\mathbb{A}}(\ell, m; h)^\perp$ *is given by*

$$d_1\left(C^{\mathbb{A}}(\ell, m; h)^\perp\right) = \begin{cases} 3 & \text{if } q > 2, \\ 4 & \text{if } q = 2. \end{cases}$$

More generally, upon letting $Q_j = q^j - j$ *for* $j \ge 0$, *the* s^{th} *higher weight of* $C^{\mathbb{A}}(\ell, m; h)^\perp$ *for* $1 \le s < q^{\ell'} - \ell'$ *is given by*

$$d_s\left(C^{\mathbb{A}}(\ell, m; h)^\perp\right) = s + j + 1$$

where j *is the unique positive integer* $\le \ell'$ *such that* $Q_{j-1} \le s < Q_j$.

(ii) *With* $d(\ell, m; h)$ *as in* (3), *we have* $d(\ell, m; h) \ge 2$ *and*

$$d_{n-k_h-s}\left(C^{\mathbb{A}}(\ell, m; h)^\perp\right) = q^\delta - s \quad \text{for } 0 \le s \le d(\ell, m; h) - 2.$$

In particular, $C^{\mathbb{A}}(\ell, m; h)^\perp$ *is nondegenerate. Further if we assume that* $h < \ell'$ *or* $\ell' > 2h$, *and we let* $G_0 = H_0 = 0$ *and* $G_j := \sum_{i=0}^{j-1} q^{-i}$ *and* $H_j := d(\ell, m; h)G_j - j$ *for any positive integer* j, *then the last* s^{th} *higher weight of* $C^{\mathbb{A}}(\ell, m; h)^\perp$ *for* $H_1 \le s \le \max\{H_\ell, H_{\ell'-\ell}\}$ *is given by*

$$d_{n-k_h-s}\left(C^{\mathbb{A}}(\ell, m; h)^\perp\right) = n - s - j,$$

where j *is the unique positive integer* $\le \max\{\ell, \ell' - \ell\}$ *with* $H_{j-1} \le s < H_j$.

Proof. Let $C = C^{\mathbb{A}}(\ell, m; h)$, $n = q^{\delta}$ and $k = k_h$, and let d_j, e_j, f_j be as in Corollary 4.2. By Theorem 3.3, the code C is nondegenerate and $d_{k-j} = n - q^{j-1}$ for $1 \le j \le \ell' + 1$. Consequently, the condition $f_j < s \le f_{j+1}$ translates to $Q_{j-1} \le s < Q_j$, provided $1 \le j \le \ell'$. Thus (13) implies the desired formula in (i) for $d_s\left(C^{\mathbb{A}}(\ell, m; h)^{\perp}\right)$. In the particular case when $s = 1$, we have $Q_1 = 1 < 2 = Q_2$ or $0 = Q_0 < 1 < Q_1$ according as $q = 2$ or $q > 2$, and this yields the formula for the minimum distance of $C^{\mathbb{A}}(\ell, m; h)^{\perp}$.

Next, $d_1 = d(\ell, m; h) = q^{\delta-\ell^2}|\mathrm{GL}\ell(\mathbb{F}_q)|$ and since $\ell' > 1$ we see that $q^{\delta-\ell^2} \ge 2$ when $\ell < \ell'$, whereas $|\mathrm{GL}_{\ell}(\mathbb{F}_q)| \ge (q^2-1)(q^2-q) \ge 6$ when $\ell = \ell'$. Thus in any case $d(\ell, m; h) \ge 2$ and so (12) implies the first assertion in (ii). Further, Theorem 2.5 shows that $e_j = H_j$ for $1 \le j \le 1 + \max\{\ell, \ell' - \ell\}$. Thus (12) implies the remaining assertion in (ii) as well. □

Remark 4.5. With the first of Theorem 4.4 above, we can also describe many of the initial higher weights $d_s^{\perp} := d_s\left(C^{\mathbb{A}}(\ell, m; h)^{\perp}\right)$ by the recursive formula

$$d_s^{\perp} = \begin{cases} d_{s-1}^{\perp} + 2 & \text{if } d_{s-1} \text{ is a power of } q, \\ d_{s-1}^{\perp} + 1 & \text{otherwise,} \end{cases}$$

provided $2 \le s \le q^{\ell'} - \ell'$. Likewise, the last higher weight of $C^{\mathbb{A}}(\ell, m; h)^{\perp}$ is $n = q^{\delta}$, and many terminal higher weights of $C^{\mathbb{A}}(\ell, m; h)^{\perp}$ are given by the recursive formula

$$d_{n-k-s}^{\perp} = \begin{cases} d_{n-k-s+1}^{\perp} - 2 & \text{if } d_{n-k-s+1} = n + 1 - d(\ell, m; h)G_j \text{ for some } j, \\ d_{n-k-s+1}^{\perp} - 1 & \text{otherwise,} \end{cases}$$

provided $1 \le s \le \max\{d(\ell, m; h)G_{\ell} - \ell,\ d(\ell, m; h)G_{\ell'-\ell} - (\ell' - \ell)\}$ and it is assumed that $h < \ell'$ or $\ell' > 2h$.

Using the direct formula in Theorem 4.4 or the recursive formula in Remark 4.5, we can easily write down several of the initial and terminal higher weights of $C^{\mathbb{A}}(\ell, m; h)^{\perp}$. Table 1 illustrates the first few higher weights $d_s^{\perp}$ of the dual of $C^{\mathbb{A}}(\ell, m; h)$, where h, ℓ, ℓ' are sufficiently large, say $\ell' > \ell \ge h \ge 27$.

Appendix A. A Geometric Approach to Higher Weights

Let n, k be positive integers with $k \le n$. A nondegenerate $[n, k]_q$-projective system is simply a (multi)set X of n points in the projective space $\mathbb{P}^{k-1}$ over the finite field $\mathbb{F}_q$. If we write $\mathbb{P}^{k-1} = \mathbb{P}(V)$, where V is a k-dimensional vector space over $\mathbb{F}_q$ and fix some lifts, say $v_1, \dots, v_n$, of these n points to V, then the associated nondegenerate linear code C_X is the image of the evaluation map

$$\mathrm{Ev} : V^* \to \mathbb{F}_q^n \quad \text{defined by} \quad \mathrm{Ev}(\phi) = (\phi(v_1), \dots, \phi(v_n)),$$

where V^* denotes the dual of V, i.e., the space of all linear maps from V to $\mathbb{F}_q$. It is shown in [**18, 19**] that the association $X \rightsquigarrow C_X$ is a one-to-one correspondence, modulo natural notions of equivalence, from the class of nondegenerate $[n, k]_q$-projective systems onto the class of nondegenerate $[n, k]_q$-codes. For $1 \le r \le k$, the r^{th} higher weight of C_X corresponds to maximal sections of X by (projective) linear subspaces of $\mathbb{P}^{k-1}$ of codimension r; more precisely,

$$d_r(C_X) = n - \max\{|X \cap \Pi| : \Pi \text{ linear subspace of } \mathbb{P}^{k-1} \text{ with } \operatorname{codim} \Pi = r\}.$$

For more on this, we refer to [**18, 19**].

TABLE 1. Dual Higher Weights of Affine Grassmann Codes

q	2	3	4	5	7	8	9	11	13	16	17
$d_1^\perp$	4	3	3	3	3	3	3	3	3	3	3
$d_2^\perp$	6	5	4	4	4	4	4	4	4	4	4
$d_3^\perp$	7	6	6	5	5	5	5	5	5	5	5
$d_4^\perp$	8	7	7	7	6	6	6	6	6	6	6
$d_5^\perp$	10	8	8	8	7	7	7	7	7	7	7
$d_6^\perp$	11	9	9	9	9	8	8	8	8	8	8
$d_7^\perp$	12	11	10	10	10	10	9	9	9	9	9
$d_8^\perp$	13	12	11	11	11	11	11	10	10	10	10
$d_9^\perp$	14	13	12	12	12	12	12	11	11	11	11
$d_{10}^\perp$	15	14	13	13	13	13	13	13	12	12	12
$d_{11}^\perp$	16	15	14	14	14	14	14	14	13	13	13
$d_{12}^\perp$	18	16	15	15	15	15	15	15	15	14	14
$d_{13}^\perp$	19	17	16	16	16	16	16	16	16	15	15
$d_{14}^\perp$	20	18	18	17	17	17	17	17	17	16	16
$d_{15}^\perp$	21	19	19	18	18	18	18	18	18	18	17
$d_{16}^\perp$	22	20	20	19	19	19	19	19	19	19	19
$d_{17}^\perp$	23	21	21	20	20	20	20	20	20	20	20
$d_{18}^\perp$	24	22	22	21	21	21	21	21	21	21	21
$d_{19}^\perp$	25	23	23	22	22	22	22	22	22	22	22
$d_{20}^\perp$	26	24	24	23	23	23	23	23	23	23	23
$d_{21}^\perp$	27	25	25	24	24	24	24	24	24	24	24
$d_{22}^\perp$	28	26	26	25	25	25	25	25	25	25	25
$d_{23}^\perp$	29	27	27	27	26	26	26	26	26	26	26
$d_{24}^\perp$	30	29	28	28	27	27	27	27	27	27	27
$d_{25}^\perp$	31	30	29	29	28	28	28	28	28	28	28
$d_{26}^\perp$	32	31	30	30	29	29	29	29	29	29	29
$d_{27}^\perp$	34	32	31	31	30	30	30	30	30	30	30

Now let ℓ, m be positive integers with $\ell \le m$ and as before let $k = \binom{m}{\ell}$, $\ell' := m - \ell$ and $\delta = \ell\ell'$. Assume that $1 < \ell \le \ell'$. Consider the Grassmannian $G_{\ell,m} = G_{\ell,m}(\mathbb{F}_q)$ of ℓ-dimensional subspaces of $\mathbb{F}_q^m$. The Plücker embedding

$$G_{\ell,m} \hookrightarrow \mathbb{P}^{\binom{m}{\ell}-1} = \mathbb{P}(\wedge^\ell \mathbb{F}_q^m) \quad \text{given by} \quad W = \langle w_1, \dots, w_\ell \rangle \mapsto [w_1 \wedge \cdots \wedge w_\ell]$$

is known to be nondegenerate and the corresponding nondegenerate linear code is the Grassmann code $C(\ell, m)$. The formula stated in the Introduction for the last few higher weights of $C(\ell, m)$ follows readily from the structure of linear subvarieties of $G_{\ell,m}$ or, in algebraic parlance, the structure of decomposable subspaces of exterior powers. Indeed, $G_{\ell,m}$ contains a linear subspace Π of dimension $r-1$, provided $r \le \mu$, where $\mu := 1 + \max\{\ell, m-\ell\}$; see, for example, [**7**, Cor. 7] or [**5**, Lemma 3.5]. This subspace Π has codimension $(k-1)-(r-1) = k-r$ in $\mathbb{P}^{k-1}$, and clearly,

$|\Pi \cap G_{\ell,m}| = |\Pi| = |\mathbb{P}^{r-1}| = (1 + q + \cdots + q^{r-1})$. Consequently,

$$d_{k-r}\left(C(\ell, m)\right) = n - (1 + q + \cdots + q^{r-1}) \quad \text{ for } 1 \le r \le \mu.$$

Suppose we fix an ordered basis $\{e_1, \ldots, e_m\}$ of $\mathbb{F}_q^m$ and the corresponding basis $\{e_\alpha : \alpha \in I(\ell, m)\}$ of $\wedge^\ell \mathbb{F}_q^m$, where

$$I(\ell, m) = \{\alpha = (\alpha_1, \ldots, \alpha_\ell) :\in \mathbb{Z}^\ell : 1 \le \alpha_1 < \cdots < \alpha_\ell \le m\}$$

and $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_\ell}$ for $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in I(\ell, m)$. The Plücker coordinates of an ℓ-dimensional subspace $W \in G_{\ell,m}$ spanned by $\{w_1, \ldots, w_\ell\}$ are precisely $p = (p_\alpha)_{\alpha \in I(\ell,m)}$, where $p_\alpha \in \mathbb{F}_q$ are determined by the relation

$$w_1 \wedge \cdots \wedge w_\ell = \sum_{\alpha \in I(\ell,m)} p_\alpha e_\alpha.$$

For $\alpha \in I(\ell, m)$, let H_α denote the hyperplane $\{p \in \mathbb{P}^{k-1} : p_\alpha = 0\}$ in $\mathbb{P}^{k-1}$, and let $U_\alpha := \{p \in \mathbb{P}^{k-1} : p_\alpha \neq 0\}$ be the corresponding basic open set. It is a classical fact that $U_\alpha \cap G_{\ell,m}$ is isomorphic to the affine space $\mathbb{A}^\delta$ of $\ell \times \ell'$ matrices over $\mathbb{F}_q$. This correspondence is given explicitly by the Basic Cell Lemma of [**6**]. For the sake of definitiveness, consider $\theta := (\ell' + 1, \ell' + 2, \ldots, m) \in I(\ell, m)$. Then the Plücker embedding restricted to $U_\theta \cap G_{\ell,m}$ gives a nondegenerate embedding of $\mathbb{A}^\delta$ into $\mathbb{P}^{k-1}$, and the linear code corresponding to this projective system is, in fact, the affine Grassmann code $C^{\mathbb{A}}(\ell, m)$ of length q^δ. If Π is a linear subspace of $\mathbb{P}^{k-1}$ of dimension $r - 1$, then $\Pi \cap H_\theta$ would be a linear subspace of dimension $r - 2$ or $r - 1$ according as $\Pi \not\subseteq H_\theta$ or $\Pi \subseteq H_\theta$. Consequently,

$$|\Pi \cap U_\theta| = |\Pi| - |\Pi \cap H_\theta| \le (1 + q + \cdots + q^{r-1}) - (1 + q + \cdots + q^{r-2}) = q^{r-1}.$$

Consequently, $|\Pi \cap U_\theta \cap G_{\ell,m}| \le q^{r-1}$ and so $d_{k-r}(C^{\mathbb{A}}(\ell, m)) \ge q^\delta - q^{r-1}$ for all $r = 1, \ldots, k$. This proves a stronger version of Lemma 3.2 in the case $h = \ell$. Further, if $r \le \mu = \ell' + 1$ and if Π is a linear subspace of $\mathbb{P}^{k-1}$ of codimension $k - r$ chosen in such a way that $\Pi \subseteq G_{\ell,m}$ and $\Pi \subseteq H_\theta$, then $|\Pi \cap U_\theta \cap G_{\ell,m}| = |\Pi \cap H_\theta| = q^{r-1}$ and so $d_{k-r}(C^{\mathbb{A}}(\ell, m)) = q^\delta - q^{r-1}$ for all $r = 1, \ldots, \mu$. Since $1 < \ell \le \ell'$, choosing such a subspace Π is possible for $1 \le r \le \mu$; for example, we can take $\Pi = \{p \in \mathbb{P}^{k-1} : p_{(1,2,\ldots,\ell-1,j)} = 0 \text{ for } j = \ell, \ell + 1, \ldots, \ell + r - 1\}$ to be the intersection of Plücker coordinate hyperplanes that are "close" to each other. Thus we obtain an alternative proof of Theorem 3.3 when $h = \ell$.

On the other hand, deriving the formulas that we have for initial higher weights of $C^{\mathbb{A}}(\ell, m)$ from the corresponding results for the Grassmann code $C(\ell, m)$ is not so straightforward. To be sure, the optimal linear subspace in $G_{\ell,m}$ of large dimension (or small codimension) are obtained in [**6**] by considering close families in $I(\ell, m)$ and the corresponding linear subsbaces of $\mathbb{P}^{k-1}$ given by the intersections of Plücker coordinate hyperplanes. Recall that $\Lambda \subseteq I(\ell, m)$ is said to be *close* if any two distinct elements of Λ have $\ell - 1$ coordinates in common. However, determining the maximum possible cardinality of the intersection of the corresponding linear subspace Π with $U_\theta \cap G_{\ell,m}$ is not easy. It may be tempting to consider $\Lambda \subseteq I(\ell, m)$ not containing θ such that $\Lambda \cup \{\theta\}$ is close. But this doesn't work even when Λ is singleton (which would correspond to looking at the minimum distance). In fact, it is better to keep the elements of Λ as far away from θ as possible. Thus choosing a close family in $I(\ell, \ell')$ rather than $I(\ell, m)$ is helpful and this has, in fact, motivated the proofs of Lemma 2.3 and 2.4, which paved the way for Theorem 2.5.

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Department of Mathematics, Indian Institute of Technology Bombay,
Powai, Mumbai 400076, India.
E-mail address: `mrinmoy.dat@gmail.com`

Department of Mathematics, Indian Institute of Technology Bombay,
Powai, Mumbai 400076, India.
E-mail address: `srg@math.iitb.ac.in`