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Hilbert functions of ladder determinantal varieties ☆

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Abstract

We consider algebraic varieties defined by the vanishing of all minors of a fixed size of a rectangular matrix with indeterminate entries such that the indeterminates in these minors are restricted to lie in a ladder shaped region of the rectangular array. Explicit formulae for the Hilbert function of such varieties are obtained in (i) the rectangular case by Abhyankar (Rend. Sem. Mat. Univers. Politecn. Torino 42 (1984) 65), and (ii) the case of 2×2 minors in one-sided ladders by Kulkarni (Semigroup of ordinary multiple point, analysis of straightening formula and counting monomials, Ph.D. Thesis, Purdue University, West Lafayette, USA, 1985). More recently, Krattenthaler and Prohaska (Trans. Amer. Math. Soc. 351 (1999) 1015) have proved a 'remarkable formula', conjectured by Conca and Herzog (Adv. Math. 132 (1997) 120) for the Hilbert series in the case of arbitrary sized minors in one-sided ladders. We describe here an explicit, albeit complicated, formula for the Hilbert function and the Hilbert series in the case of arbitrary sized minors in two-sided ladders. From a combinatorial viewpoint, this is equivalent to the enumeration of certain sets of 'indexed monomials'. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let K be a field and $X=(X_{ij})$ be an $m(1)\times m(2)$ matrix whose entries are variables over K. Given a subset Y of the integral rectangle

$$[1,m(1)]\times[1,m(2)]=\{(i,j)\in\mathbb{Z}^2\colon\ 1\leqslant i\leqslant m(1)\text{ and }1\leqslant j\leqslant m(2)\},$$

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let $K[Y]$ denote the ring of all polynomials in $\{X_{ij}: (i,j) \in Y\}$ with coefficients in K. Also, let $I_p(Y)$ denote the ideal of $K[Y]$ generated by all $p \times p$ minors of X with entries in Y and $\mathscr{V}_p(Y)$ denote the projective variety corresponding to $I_p(Y)$. We call Y a *generalized ladder* or a *saturated set* if

$$(i_1,i_2),\ (j_1,j_2) \in Y \quad \text{with } i_1 < j_1,\ i_2 < j_2 \ \Rightarrow\ (i_1,j_2) \in Y \text{ and } (i_2,j_1) \in Y.$$

In effect, a generalized ladder looks like a ladder (see Fig. 1) or a biladder with or without overlap (see Fig. 2 (a) and (b)). In the literature, generalized ladders, ladders and biladders are sometimes referred to as ladders, one-sided ladders and two-sided ladders, respectively.

If $\mathscr{L}$ is a generalized ladder, then $I_p(\mathscr{L})$ is called a ladder determinantal ideal and $\mathscr{V}_p(\mathscr{L})$ a ladder determinantal variety. These varieties were introduced by Abhyankar [2] in connection with his study of singularities of Schubert varieties in flag manifolds. Viewed as affine varieties (i.e., as cones over $\mathscr{V}_p(\mathscr{L})$), these are essentially the 'opposite big cells' of Schubert varieties in flag manifolds (see [26,27] for details). Using the connection with Schubert varieties or otherwise, it is now known that ladder determinantal varieties have a number of nice properties such as irreducibility, Cohen–Macaulayness, normality, etc.; for details, we refer to the papers [2,6–9,16,17,24,25,28] which directly deal with the ladder determinantal varieties, and also to the papers [4,14,15,18,22,23,29–31] which study the related Schubert varieties.

In this paper, we consider the problem of finding an explicit formula for the Hilbert function of $\mathscr{V}_p(\mathscr{L})$ or equivalently, of $K[\mathscr{L}]/I_p(\mathscr{L})$, where p is any positive integer and $\mathscr{L}$ is any biladder. A formula in the first nontrivial case of $p=2$ and $\mathscr{L}$ a (one-sided) ladder was obtained by Kulkarni in his 1985 thesis [20] (see also [21]). Subsequently, in 1989 it was shown by Abhyankar and Kulkarni [3] that the ideals $I_p(\mathscr{L})$ are Hilbertian (which means that the Hilbert function of $K[\mathscr{L}]/I_p(\mathscr{L})$ coincides with its Hilbert polynomial at *all* nonnegative integers), for any $p > 1$ and any generalized ladder $\mathscr{L}$. Hilbertianness of $I_p(\mathscr{L})$ also follows from the later work of Herzog and Trung [17], who showed that the Hilbert function of $K[\mathscr{L}]/I_p(\mathscr{L})$ can be described in terms of the f-vector of an associated simplicial complex. Nevertheless, there still remains the problem of finding explicitly the Hilbert function of $K[\mathscr{L}]/I_p(\mathscr{L})$. To this end, Conca and Herzog [5] conjectured a 'remarkable formula' for the Hilbert series in the case of (one-sided) ladders and any $p > 1$. Recently, Krattenthaler and Prohaska [19] have established this conjecture using the so called 'two-rowed arrays'. It may be noted that in the degenerate case when $\mathscr{L}$ is the entire rectangle $[1,m(1)] \times [1,m(2)]$, the ideal $I_p(\mathscr{L})$ reduces to the classical determinantal ideal $I_p(X)$ and its Hilbert function is explicitly known from the work of Abhyankar [1,2] (see also [10]). Short proofs of Abhyankar's formula as well as formulae for the Hilbert series of $I_p(X)$ are described in [5,12].

In this paper, we aim at giving an explicit formula for the Hilbert function of $K[\mathscr{L}]/I_p(\mathscr{L})$ for any biladder $\mathscr{L}$ and any $p > 1$. This result was announced in [10] and an outline of the proof was described in [13]. In this paper, we also show that even though this 'explicit' formula is complicated, it can be used to derive fairly simple estimates for some useful geometric invariants such as the degree of the Hilbert polynomial.

Our starting point, as in Kulkarni [20], is a theorem of Abhyankar [2], which describes bases for the graded components of $K[\mathscr{L}]/I_{p+1}(\mathscr{L})$ in terms of monomials of 'index' $\leqslant p$. (With this in view, we shall find it convenient to consider $I_{p+1}(\mathscr{L})$ rather than $I_p(\mathscr{L})$ and henceforth we shall do so.) In [20] (see also [21]), Kulkarni enumerated the monomials of index $\leqslant 1$ in a (one-sided) ladder L by counting certain related objects, called radicals and skeletons. Roughly speaking, a *radical* is like a subset of a lattice path in L whereas a skeleton is like a set of nodes (or South–West corners) in a radical. The main idea behind our formula in the general case is simply as follows. First, generalize Kulkarni's computation of radicals and skeletons from ladders to biladders, and then use induction! The technical details, however, seem rather long and tedious. This is partly due to the fact that for a smooth passage in the inductive step, it is necessary to consider biladders with possible horizontal and/or vertical overlap (see Fig. 2(b)). With such configurations in the fray, it was felt prudent that we give complete (and seemingly pedantic) proofs of the auxiliary results needed for the main result even though in some cases it appears tempting to dismiss them as (pictorially) obvious. An overview of the main steps in the proof is given in [13] and it may be advisable to read that before proceeding with the details given here.

This paper is organized as follows. In the next section, we describe most of the notation and terminology used in the paper, and also some preliminary results. The relation between radicals and skeletons is established in Section 2. In Section 3, we give a formula to enumerate the skeletons in a biladder. Next, in Section 4, we take up the enumeration of p-fold radicals as well as of monomials of index $\leqslant p$ in a biladder $\mathscr{L}$. This enables us to determine the Hilbert function of $\mathscr{V}_{p+1}(\mathscr{L})$. Finally, in Section 5, we describe some applications to the computation of the Hilbert series as well as the dimension of ladder determinantal varieties.

1. Preliminaries

The notation and terminology introduced in this section will be used throughout this paper.

1.1. Intervals and matrices

By $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$, and $\mathbb{N}^+$, we denote the sets of all rational numbers, integers, nonnegative integers, and positive integers, respectively. Given any $\alpha, \beta \in \mathbb{Z}$, we let

$$[\alpha,\beta] = \{\gamma \in \mathbb{Z}\colon\ \alpha \leqslant \gamma \leqslant \beta\}$$

and we also let

$$[\alpha,\beta) = \{\gamma \in \mathbb{Z}\colon\ \alpha \leqslant \gamma < \beta\}, \qquad (\alpha,\beta] = \{\gamma \in \mathbb{Z}\colon\ \alpha < \gamma \leqslant \beta\}.$$

The corresponding notation for open integral intervals will never be used so as to avoid confusion with the elements (e.g., (a,b)) of the direct product of two sets (e.g., $A \times B$). The cardinality of a set A will be denoted by $|A|$. Let h and h' be any positive integers. For any nonnegative integer l, we let

$M_{h\,h'}(\mathbb{N}, l) =$ the set of all $h \times h'$ matrices $\alpha = (\alpha_{kk'})$ such that

$$\alpha_{kk'} \in \mathbb{N} \quad \text{for } 1 \leqslant k \leqslant h,\ 1 \leqslant k' \leqslant h' \quad \text{and} \quad \sum_{k=1}^{h} \sum_{k'=1}^{h'} \alpha_{kk'} = l.$$

Note that $M_{h\,h'}(\mathbb{N}, l)$ is a finite set. We can define the set $M_{h\,h'}(\mathbb{N})$ of all $\mathbb{N}$-valued $h \times h'$ matrices by putting

$$M_{h\,h'}(\mathbb{N}) = \coprod_{l \in \mathbb{N}} M_{h\,h'}(\mathbb{N}, l),$$

where $\coprod$ denotes disjoint union. Given any $\mathbb{N}$-valued $h \times h'$ matrix $\alpha = (\alpha_{kk'})$ and any $p \in [1, h]$ and $q \in [1, h']$, we define

$$\sigma_p(\alpha) = \sum_{k=1}^{p} \sum_{k'=1}^{h'} \alpha_{kk'} \quad \text{and} \quad \tau_q(\alpha) = \sum_{k=1}^{h} \sum_{k'=q}^{h'} \alpha_{kk'}.$$

Note that for any $\alpha \in M_{h\,h'}(\mathbb{N})$ and $l \in \mathbb{N}$, we have

$$\alpha \in M_{h\,h'}(\mathbb{N}, l) \quad \Leftrightarrow \quad \sigma_h(\alpha) = \tau_1(\alpha) = l.$$

Given any α, β in $M_{h\,h'}(\mathbb{N})$, we define

$$\sigma(\beta) \leqslant \sigma(\alpha) \text{ to mean that } \sigma_p(\beta) \leqslant \sigma_p(\alpha) \quad \text{for all } p \in [1, h]$$

and

$$\tau(\beta) \leqslant \tau(\alpha) \text{ to mean that } \tau_q(\beta) \leqslant \tau_q(\alpha) \quad \text{for all } q \in [1, h'].$$

1.2. *Index and monomials*

A field K, a pair $m = (m(1), m(2)) \in \mathbb{N}^+ \times \mathbb{N}^+$ of positive integers, and a set $X = \{X_{ij} : 1 \leqslant i \leqslant m(1),\ 1 \leqslant j \leqslant m(2)\}$ of $m(1)m(2)$ independent indeterminates over K will be kept fixed throughout this paper.

Let Y be a subset of the rectangle $[1, m(1)] \times [1, m(2)]$. As in Section 0, let $K[Y]$ denote the ring of polynomials in the variables $\{X_{ij} : (i, j) \in Y\}$ with coefficients in K, and for any $V \in \mathbb{N}$, let $K[Y]_V$ denote its Vth graded component, i.e., the set of all homogeneous polynomials in $K[Y]$ of degree V including the zero polynomial. Given any $p \in \mathbb{N}^+$, we let $I_p(Y)$ denote the ideal of $K[Y]$ generated by all $p \times p$ minors of Y, and let $\mathscr{V}_p(Y)$ denote the corresponding projective variety.

By a *monomial* on Y, we mean a map $\theta : Y \to \mathbb{N}$, and we let

$$\text{mon}(Y) = \text{the set of all monomials on } Y.$$

Note that we have a natural injective map $\theta \mapsto X^\theta$ of $\text{mon}(Y) \to K[Y]$, where

$$X^\theta = \prod_{(i,j) \in Y} X_{ij}^{\theta(i,j)} \quad \text{for any } \theta \in \text{mon } (Y).$$

The *support* of a monomial $\theta \in \text{mon}(Y)$ will be denoted by $\text{supp}(\theta)$; thus,

$$\text{supp}(\theta) = \{(i, j) \in Y : \theta((i, j)) \neq 0\}.$$

To any subset T of $[1,m(1)]\times[1,m(2)]$ we associate the *index* of T which we denote by $\mathrm{ind}(T)$ and which we define by

$$\mathrm{ind}(T)=\max\{p\in\mathbb{N}:\ \exists (i_1,j_1),(i_2,j_2),\ldots,(i_p,j_p) \text{ in } T \text{ with } i_1<i_2<\cdots<i_p \text{ and } j_1<j_2<\cdots<j_p\}.$$

Notice that $\mathrm{ind}(T)=0 \Leftrightarrow T$ is empty. The *index* of a monomial $\theta\in\mathrm{mon}(Y)$ is defined by putting

$$\mathrm{ind}(\theta)=\mathrm{ind}(\mathrm{supp}(\theta)).$$

For every $p\in\mathbb{N}$, we let

$$\mathrm{mon}(Y,p)=\{\theta\in\mathrm{mon}(Y):\ \mathrm{ind}(\theta)\leqslant p\}$$

and, restricting attention to monomials of a specified degree, for every $p\in\mathbb{N}$ and $V\in\mathbb{N}$ we let

$$\mathrm{mon}(Y,p,V)=\left\{\theta\in\mathrm{mon}(Y,p):\ \sum_{y\in Y}\theta(y)=V\right\}.$$

1.3. Radicals and skeletons

Let $p\in\mathbb{N}$ and Y be a subset of $[1,m(1)]\times[1,m(2)]$. Define

$$\mathrm{rad}^p(Y)=\{R:\ R\subseteq Y \text{and } \mathrm{ind}(Y)\leqslant p\} \quad\text{and}\quad \mathrm{rad}(Y)=\mathrm{rad}^1(Y).$$

Elements of $\mathrm{rad}^p(Y)$ may be called the *p-fold radicals* in Y whereas elements of $\mathrm{rad}(Y)$ are simply called the *radicals* in Y. The cardinality of a radical R in Y may be called the *size* of R. Given any $d\in\mathbb{N}$, we define

$$\mathrm{rad}(Y,d)=\{R\in\mathrm{rad}(Y):\ |R|=d\}.$$

A subset E of Y is said to be a *skeleton* in Y if the elements of E can be arranged on an antidiagonal, i.e., for any two distinct elements (x_1,y_1) and (x_2,y_2) of E, we have

$$\text{either}: \quad x_1<x_2 \text{ and } y_1>y_2 \qquad \text{or}: \quad x_1>x_2 \text{ and } y_1<y_2.$$

Let $\mathrm{skel}(Y)$ denote the set of all skeletons in Y. Note that $\mathrm{skel}(Y)\subseteq\mathrm{rad}(Y)$. The cardinality of a skeleton E in Y may be called the *length* of E. Given any $l\in\mathbb{N}$, we define

$$\mathrm{skel}(Y,l)=\{E\in\mathrm{skel}(Y):\ |E|=l\}.$$

1.4. Ladders and biladders

Recall that we have fixed a pair $m=(m(1),m(2))$ of positive integers. Given any $h\in\mathbb{N}^+$, by a *ladder generating bisequence of length h*, we mean a map $S:[1,2]\times[0,h]\to\mathbb{N}$ such that

$$1=S(1,0)\leqslant S(1,1)<S(1,2)<\cdots<S(1,h)=m(1)$$

and

$$m(2)=S(2,0)>S(2,1)>\cdots>S(2,h-1)\geqslant S(2,h)=1.$$

The positive integer h, called the *length* of S, is denoted by $\mathrm{len}(S)$. We shall find it convenient to also consider the empty bisequence, which we declare to be the unique ladder generating bisequence of length 0. Define

$$\mathrm{lad}[m,h]=\text{the set of all ladder generating bisequences of length } h$$

and

$$\mathrm{lad}(m)=\bigcup_{h\in\mathbb{N}}\mathrm{lad}[m,h].$$

Let $S\in\mathrm{lad}(m)$. We define the *ladder* $L(S)$ corresponding to S by

$$L(S)=\bigcup_{k=1}^{\mathrm{len}(S)}[S(1,k-1),S(1,k)]\times[1,S(2,k-1)],$$

and we define its *interior* $L(S)^{\mathrm{o}}$ by

$$L(S)^{\mathrm{o}}=\bigcup_{k=1}^{\mathrm{len}\,(S)}[S(1,k-1),S(1,k))\times[1,S(2,k-1)).$$

Note that if $\mathrm{len}(S)=0$, then $L(S)=L(S)^{\mathrm{o}}=\emptyset$ and if $\mathrm{len}(S)\neq 0$ then

$$L(S)^{\mathrm{o}}\subset L(S)\subseteq[1,m(1)]\times[1,m(2)],$$

where the first inclusion is proper. Also note that if $\mathrm{len}(S)=1$, then $L(S)$ is the full rectangle $[1,m(1)]\times[1,m(2)]$. The *boundary* of S or of the ladder $L(S)$, denoted by ∂S or by $\partial L(S)$, is defined by

$$\partial S=L(S)\backslash L(S)^{\mathrm{o}}.$$

Points $(S(1,k),S(2,k))$ for $1\leqslant k\leqslant h-1$ are called the *nodes* of S, and we let

$$\mathcal{N}(S)=\{(S(1,k),S(2,k)):1\leqslant k\leqslant h-1\}$$

denote the set of all nodes of S. It may be noted that $\mathcal{N}(S)\subseteq\partial S$.

Pictorially, a ladder looks as in Fig. 1. In this picture, we adopt the ‘matrix notation’ rather than that of co-ordinate geometry to represent the points. Thus, the bullet in top left-hand corner indicates the point $(1,1)$ while the other bullets indicate the nodes of this ladder.

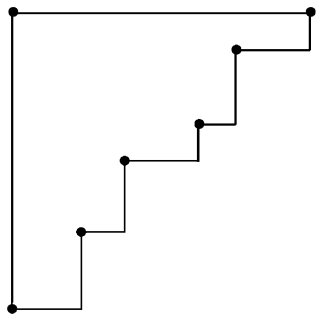
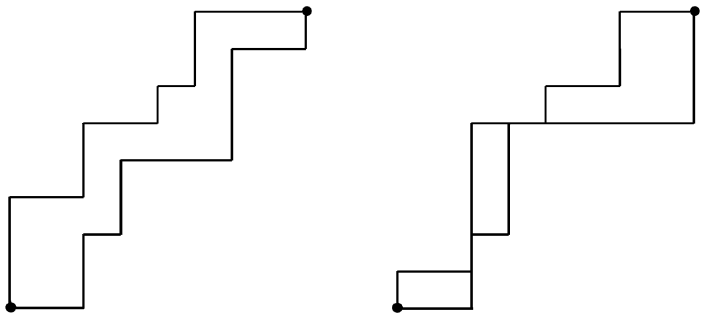


Fig. 1.



(a) (b)

Fig. 2.

For any $S', S \in \mathrm{lad}(m)$, we define

$$S' \leqslant S \quad \Leftrightarrow \quad L(S') \subseteq L(S)$$

and we note that this defines a partial order on $\mathrm{lad}(m)$. Given any $S', S \in \mathrm{lad}\ (m)$ such that $S' \leqslant S$ and $\mathrm{len}(S) \neq 0$, we define the *biladder* $\mathscr{L}(S', S)$ corresponding to (S', S) by

$$\mathscr{L}(S', S) = L(S) \backslash L(S')$$

and its *interior* $\mathscr{L}(S', S)^{\mathrm{o}}$ by

$$\mathscr{L}(S', S)^{\mathrm{o}} = L(S)^{\mathrm{o}} \backslash L(S').$$

The common intersection of the boundaries of S and S' will be denoted by $\varDelta(S', S)$. Note that if $S' \leqslant S$, then

$$\varDelta(S', S) = \partial S \cap \partial S' = \partial S \cap L(S').$$

Typically, a biladder looks as in Fig. 2(a). But there can also be some boundary overlaps and in this case, a biladder looks as in Fig. 2(b). In these pictures, we continue to adopt the 'matrix notation' as in Fig. 1.

We shall now record several elementary observations concerning ladders and biladders. The reader is invited to supply formal and/or pictorial proofs so as to get a better feel of some of the definitions above. We will tacitly use the following observations in the succeeding sections.

Observations. Let $h \in \mathbb{N}$ and $S \in \mathrm{lad}[m,h]$. Let i, j, p, q denote positive integers. Then we have the following.

(1.1) The ladder $L(S)$ and its interior $L(S)^{\mathrm{o}}$ can be alternatively described as

$$L(S) = \bigcup_{k=1}^{h} [1, S(1,k)] \times [1, S(2,k-1)]$$

and

$$L(S)^{\mathrm{o}} = \bigcup_{k=1}^{h} [1, S(1,k)) \times [1, S(2,k-1)).$$

(1.2) If $h = \mathrm{len}(S) \neq 0$, then $\partial S = L(S) \setminus L(S)^{\mathrm{o}} = L_0 \coprod L_1 \coprod L_2$, where

$$L_0 = \{(S(1,0), S(2,0))\}, \qquad L_1 = \coprod_{k=1}^{h} (S(1,k-1), S(1,k)] \times \{S(2,k-1)\},$$

and

$$L_2 = \coprod_{k=1}^{h} \{S(1,k)\} \times [S(2,k), S(2,k-1)).$$

In particular, $|\partial S| = m(1) + m(2) - 1$.

(1.3) $(p,q) \in L(S),\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q \Rightarrow (i,j) \in L(S)$.

(1.4) $(p,q) \in L(S)^{\mathrm{o}},\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q \Rightarrow (i,j) \in L(S)^{\mathrm{o}}$.

(1.5) $(p,q) \in L(S),\ 1 \leqslant i < p,\ 1 \leqslant j < q \Rightarrow (i,j) \in L(S)^{\mathrm{o}}$.

(1.6) $(p,q) \in \partial S,\ i > p,\ j > q \Rightarrow (i,j) \notin L(S)$.

(1.7) $\partial S \in \mathrm{rad}(L(S))$.

(1.8) $(p,q) \in L(S)^{\mathrm{o}} \Rightarrow (p+1, q+1) \in L(S)$.

(1.9) $(p,q) \in L(S)^{\mathrm{o}} \Rightarrow 1 \leqslant p < m(1)$ and $1 \leqslant q < m(2)$.

(1.10) If $(p,q) = (S(1,k), S(2,k))$ for some $k \in [1,h]$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ and $(i,j) \neq (p,q)$, then $(i,j) \in L(S)^{\mathrm{o}}$.

Proof. The identities in (1.1) follow from the fact that the $S(1,k)$'s increase with k whereas the $S(2,k)$'s decrease with k. Next, (1.2) follows just from the definitions of $L(S)$ and $L(S)^{\mathrm{o}}$. The assertions (1.3), (1.4), (1.5), (1.8) and (1.9) are easy consequences of (1.1). From (1.5) we get (1.6). And (1.7) follows from (1.5) and (1.6). Finally, (1.10) follows from (1.1) by noting that if $1 \leqslant i < p$, then $k \neq 0$ and $(i,j) \in [1, S(1,k)) \times [1, S(2,k-1))$, whereas if $1 \leqslant j < q$, then $k \neq h$ and $(i,j) \in [1, S(1,k+1)) \times [1, S(2,k))$. $\square$

1.5. Binomials and monomials

Following Abhyankar [2], we define a variant of the ordinary binomial coefficient, called the *twisted binomial coefficient*, as follows:

$$\begin{bmatrix} V \\ A \end{bmatrix} = \binom{V+A}{A} = \frac{(V+1)(V+2)\cdots(V+A)}{A!}.$$

Here, $A \in \mathbb{Z}$ and V can be an integer or an indeterminate over $\mathbb{Q}$. For $A < 0$, we follow the usual convention that $1/A! = 0$ so that

$$\begin{bmatrix} V \\ A \end{bmatrix} = \binom{V+A}{A} = 0 \quad \text{if } A < 0.$$

We now record some elementary properties of binomial coefficients in the lemma below. Proofs are fairly straightforward, and hence omitted. If necessary, the reader is referred to [2,11] for details.

Lemma 1.1. *Given any integers d, e, f, and V, we have the following*:

(i) $$\binom{d}{V-1} + \binom{d}{V} = \binom{d+1}{V} \quad \textit{and} \quad \begin{bmatrix} d \\ V-1 \end{bmatrix} + \begin{bmatrix} d-1 \\ V \end{bmatrix} = \begin{bmatrix} d \\ V \end{bmatrix}.$$

(ii) $$\sum_{j=0}^{V} \begin{bmatrix} d-1 \\ j \end{bmatrix} = \begin{bmatrix} d \\ V. \end{bmatrix}.$$

(iii) $$\binom{d}{e}\binom{e}{f} = \binom{d}{f}\binom{d-f}{f-e} \quad \textit{and} \quad \begin{bmatrix} d \\ e \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} d \\ f \end{bmatrix}\begin{bmatrix} d-f \\ f-e \end{bmatrix}.$$

(iv) $$\sum_{i \in \mathbb{Z}} \binom{d}{f+i}\binom{e}{V-i} = \binom{d+e}{f+V}$$

and

$$\sum_{i \in \mathbb{Z}} \begin{bmatrix} d \\ f+i \end{bmatrix}\begin{bmatrix} e \\ V-i \end{bmatrix} = \begin{bmatrix} d+e+1 \\ f+V \end{bmatrix}.$$

More generally, *given any $a_1, \ldots, a_p$ in an overfield of $\mathbb{Q}$ and any integers $f_1, \ldots, f_p$*, *we have*

$$\sum \prod_{i=1}^{p} \binom{a_i}{f_i + d_i} = \binom{a}{f+d} \quad \textit{and} \quad \sum \prod_{i=1}^{p} \begin{bmatrix} a_i \\ f_i + d_i \end{bmatrix} = \begin{bmatrix} a+p-1 \\ f+d \end{bmatrix}$$

where $a = a_1 + \cdots + a_p$, $f = f_1 + \cdots + f_p$, and each of the sum above is taken over all p-tuples $(d_1, \ldots, d_p)$ of integers such that $d_1 + \cdots + d_p = d$.

(v) $$\binom{d}{e} = 0 \quad \textit{if and only if } e < 0 \text{ or } e > d \geqslant 0$$

whereas

$$\begin{bmatrix} d \\ e \end{bmatrix} = 0 \quad \textit{if and only if } e < 0 \text{ or } e > (d+e) \geqslant 0.$$

(vi) $$\binom{d}{e} = \binom{d}{d-e} \quad \textit{if and only if } d \geqslant 0 \textit{ or } d < e < 0$$

whereas

$$\begin{bmatrix} d \\ e \end{bmatrix} = \begin{bmatrix} e \\ d \end{bmatrix} \quad \textit{if and only if } (d+e) \geqslant 0 \textit{ or } d < 0 \textit{ and } e < 0.$$

(vii) $$\binom{d}{e} = (-1)^e \begin{bmatrix} -d-1 \\ e \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} d \\ e \end{bmatrix} = (-1)^e \binom{-d-1}{e}.$$

It may be remarked that the summations in (iv) above are *essentially finite*, by which we mean that all except finitely many summands are zero.

Lemma 1.2. *Let Y be a finite set and R be a subset of Y. Let $e = |Y|$ and $d = |R|$. Given any $V \in \mathbb{N}$, let*

$$M_V(Y) = \left\{ \theta \in \mathrm{mon}(Y) \colon \sum_{y \in Y} \theta(y) = V \right\}$$

and

$$M_V[Y,R] = \{\theta \in M_V(Y) \colon \ \mathrm{supp}(\theta) = R\}.$$

Then

$$|M_V(Y)| = \begin{bmatrix} e-1 \\ V \end{bmatrix} \quad \textit{and} \quad |M_V[Y,R]| = \begin{bmatrix} d-1 \\ V-d \end{bmatrix}.$$

Proof. The first asserted equality is a well-known formula for the number of monomials of a fixed degree. The second assertion follows from the first by noting that the map $\theta \mapsto \tilde{\theta}$, where $\tilde{\theta}(y) = \theta(y) - 1$ for $y \in R$, defines a bijection of $M_V[Y,R]$ onto $M_{V-d}(R)$. $\square$

Lemma 1.3. *Let M be a finite set and E be a subset of M. Let $m = |M|$ and $e = |E|$. Then for any $d \in \mathbb{N}$, we have*

$$|\{R \colon E \subseteq R \subseteq M \textit{ and } |R| = d\}| = \binom{m-e}{d-e}.$$

Proof. The map $R \mapsto R \backslash E$ clearly sets up a bijection of the set whose cardinality is desired onto the set of all subsets of $M \backslash E$ of cardinality $d - e$. $\square$

1.6. Generalized ladders and Abhyankar's Theorem

As in Section 0, we call a subset Y of $[1, m(1)] \times [1, m(2)]$ a *generalized ladder* or a *saturated set* if

$$(i_1, i_2),\ (j_1, j_2) \in Y \quad \text{with } i_1 < j_1,\ i_2 < j_2 \Rightarrow (i_1, j_2) \in Y \text{ and } (i_2, j_1) \in Y.$$

Observe that from (1.3) it is easily seen that any ladder as well as any biladder is a generalized ladder. We now recall a basic result of Abhyankar [2, Theorem 20.10; see also 10, Theorem 6.7], which was alluded to in Section 0.

Theorem 1.4. *Let $p \in \mathbb{N}$ and $Y \subseteq [1, m(1)] \times [1, m(2)]$ be any generalized ladder. Given any $V \in \mathbb{N}$, the set (of residue classes of the elements of) $\{X^\theta\colon \theta \in \operatorname{mon}(Y, p, V)\}$ forms a free K-basis of the Vth homogeneous component $K[Y]_V / I_{p+1}(Y)_V$ of the residue class ring $K[Y]/I_{p+1}(Y)$. Consequently, the Hilbert function of the residue class ring $K[Y]/I_{p+1}(Y)$ or of the corresponding projective variety $\mathscr{V}_{p+1}(Y)$ is given by*

$$\mathscr{H}(V) = |\operatorname{mon}(Y, p, V)| \qquad (V \in \mathbb{N}).$$

2. Correspondence between radicals and skeletons in a biladder

Let $S', S \in \operatorname{lad}(m)$ be such that $\operatorname{len}(S) \neq 0$ and $S' \leqslant S$. Let L and L^0 denote the ladder $L(S)$ and its interior $L(S)^0$, respectively, and let $\mathscr{L}$ and $\mathscr{L}^{\mathrm{o}}$ denote the biladder $\mathscr{L}(S', S)$ and its interior $\mathscr{L}(S', S)^0$, respectively. We may denote the boundary ∂S of $L(S)$ by ∂L. The assumption that L is not the empty ladder will be tacitly used in proving some of the results in this section.

Our aim in this section is to prove that there exists a surjective map $\lambda\colon \operatorname{rad}(\mathscr{L}) \to \operatorname{skel}(\mathscr{L}^{\mathrm{o}})$ which is bijective when restricted to the subset $\operatorname{marad}(\mathscr{L})$ of all maximal radicals in $\operatorname{rad}(\mathscr{L})$. Here by a *maximal radical* we mean a radical which is not strictly contained in another radical. Thus, we would obtain the inverse map $\mu\colon \operatorname{skel}(\mathscr{L}^{\mathrm{o}}) \to \operatorname{marad}(\mathscr{L})$. We shall use this correspondence to reduce the problem of counting the radicals in $\mathscr{L}$ to that of counting the skeletons in $\mathscr{L}^{\mathrm{o}}$. These maps may be viewed as variants of Viennot's 'light and shadow procedure' (cf. [32]); however, the definitions given below are self-contained.

Given any radical R in the biladder $\mathscr{L}$, or more generally, any subset R of $\mathscr{L}$, we successively find points $(i_1, j_1), (i_2, j_2), \ldots$ in $R \cap \mathscr{L}^{\mathrm{o}}$ which lie on the corners as we trace the points of $R \cap \mathscr{L}^{\mathrm{o}}$ starting from the top right-hand corner of the integral rectangle $[1, m(1)] \times [1, m(2)]$ traveling left and descending down only if no point of $R \cap \mathscr{L}^{\mathrm{o}}$ occurs further to the left; $\lambda(R)$ is then defined as the set $\{(i_1, j_1),\ (i_2, j_2), \ldots\}$ consisting of the 'corners', and it is easily seen to be a well defined member of $\operatorname{skel}(\mathscr{L}^{\mathrm{o}})$.

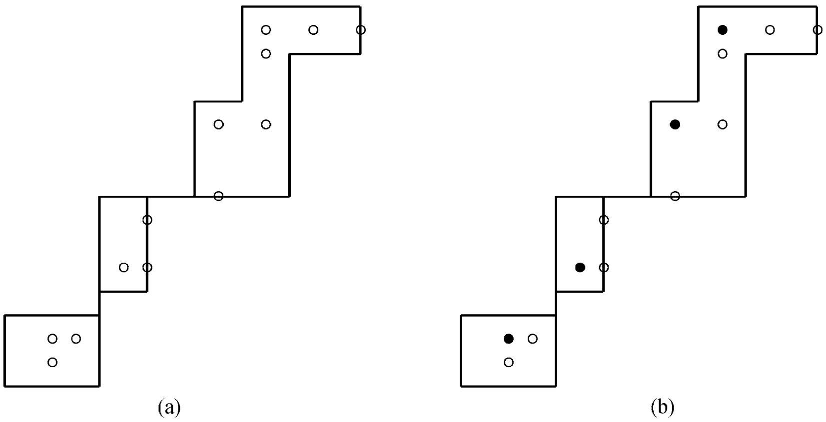


Fig. 3.

Construction of the map λ may be illustrated by Figs. 3(a) and (b). The set R, which happens to be a radical here, is described by hollow circles in Fig. 3(a), while $\lambda(R)$ is described by bullets or thick circles in Fig. 3(b).

More precisely, we make the following definition.

Definition 2.1. For any $R \subseteq \mathscr{L}$, we define

$$\lambda(R) = \{(i_1, j_1),\ (i_2, j_2), \ldots, (i_l, j_l)\},$$

where $l \in \mathbb{N}$ and $(i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l)$ in $R \cap \mathscr{L}^{\mathrm{o}}$ are the unique elements such that upon letting $i_0 = 0$ and $j_0 = m(2)$, we have

$$i_s = \min\{i \in (i_{s-1}, m(1)] : (i, j) \in R \cap \mathscr{L}^{\mathrm{o}} \text{ for some } j \in [1, j_{s-1})\}$$

and

$$j_s = \min\{ j \in [1, j_{s-1}) \colon (i_s, j) \in R \cap \mathscr{L}^{\mathrm{o}}\}; \text{ for } s \in [1, l]$$

and

$$\{i \in (i_l, m(1)] \colon (i, j) \in R \cap \mathscr{L}^{\mathrm{o}} \text{ for some } j \in [1, j_l)\} = \emptyset.$$

Analogously, for any $T \subseteq L$, we define

$$\hat{\lambda}(T) = \{(i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l)\},$$

where $l \in \mathbb{N}$ and $(i_1, j_1), (i_2, j_2), \ldots, (i_l, j_l)$ in $T \cap L^0$ are the unique elements obtained by replacing R, $\mathscr{L}$ and $\mathscr{L}^{\mathrm{o}}$ by T, L and L^0, respectively, in Definition 2.1.

We record below some elementary observations concerning the above definitions.

Lemma 2.2. *Given any $R \subseteq \mathscr{L}$ and $T \subseteq L$ we have the following*:

(i) $\lambda(R) \in \mathrm{skel}(\mathscr{L}^{\mathrm{o}})$ *and* $\hat{\lambda}(T) \in \mathrm{skel}(L^0)$.

(ii) $\lambda(R)=\emptyset \Leftrightarrow R\cap\mathscr{L}^{\circ}=\emptyset$ *and* $\hat{\lambda}(T)=\emptyset \Leftrightarrow T\cap L^0=\emptyset$.
(iii) $\lambda(R)\subseteq R\cap\mathscr{L}^{\circ}$ *and* $\hat{\lambda}(T)\subseteq T\cap L^0$.
(iv) $\lambda(R)=R$ if $R\in\mathrm{skel}(\mathscr{L}^{\circ})$ *and* $\hat{\lambda}(T)=T$ if $T\in\mathrm{skel}(L^0)$.
(v) $\lambda(R)=\lambda(R\cup L^*)$ *for any* $L^*\subseteq\mathscr{L}\backslash\mathscr{L}^{\circ}$ *and* $\hat{\lambda}(T)=\hat{\lambda}(T\cup L^{**})$ *for any* $L^{**}\subseteq\partial L$.
(vi) $\hat{\lambda}(R)=\lambda(R)$.

Proof. Obvious. □

In this section, we are mainly interested in $\lambda(R)$ and $\hat{\lambda}(T)$ when $R\in\mathrm{rad}(\mathscr{L})$ and $T\in\mathrm{rad}(L)$. At any rate we have defined the maps λ: $\mathrm{rad}(\mathscr{L})\to\mathrm{skel}(\mathscr{L}^{\circ})$, and $\hat{\lambda}$: $\mathrm{rad}(L)\to\mathrm{skel}(L^0)$ which equal the identity maps when restricted to the subsets $\mathrm{skel}(\mathscr{L}^{\circ})$ and $\mathrm{skel}(L^0)$, respectively. Hence in particular, both the maps are surjective. We will now proceed to see how far these maps are injective and whether we can somehow obtain their inverses.

First, we define the map μ on $\mathrm{skel}(\mathscr{L}^0)$ which, vaguely speaking, would give the 'inverse' of λ. The definition of μ will be given using that of the map $\hat{\mu}$ on $\mathrm{skel}(L^0)$. The construction for $\hat{\mu}$ may be briefly described as follows.

Given a skeleton E in the interior L^0 of L, we can canonically associate a (maximal) radical $\hat{\mu}(E)$ in L by drawing a path in L, with the points of E as its 'corners', traveling in a manner analogous to that described in the discussion before defining λ and $\hat{\lambda}$. It may be necessary to include additional points in ∂L so that the path remains within L; these points can be characterized as those nodes of L such that no point of E lies in the rectangles having these as their bottom rightmost corner points. In the case of skeletons $\mathscr{E}$ in the interior of the biladder $\mathscr{L}$, we shall obtain a radical $\mu(\mathscr{E})$ by considering the intersection of $\hat{\mu}(\mathscr{E})$ with $\mathscr{L}$.

Construction of the maps $\hat{\mu}$ and μ can be illustrated by the two pictures in Fig. 4. There, we consider a skeleton $\mathscr{E}$ in the interior of a biladder, which is marked by bullets in Fig. 4(a). The corresponding maximal radical $\hat{\mu}(\mathscr{E})$ is given by the dotted path in Fig. 4(b). Note that the points marked by a cross in Fig. 4(b) are the 'additional points' on the boundary, which are needed for the path to remain within the corresponding ladder.

More precisely, we make the following definition.

Definition 2.3. Given any $E\in\mathrm{skel}(L^0)$, we define

$$\tilde{E}=E\cup\{(p,q)\in\mathscr{N}(S): E\cap([1,p]\times[1,q])=\emptyset\}$$

and, upon letting t and $(\alpha_0,\beta_0),(\alpha_1,\beta_1),\ldots,(\alpha_{t+1},\beta_{t+1})$ be the unique elements such that $t\in\mathbb{N}$ and $\tilde{E}=\{(\alpha_1,\beta_1),\ldots,(\alpha_t,\beta_t)\}$ with

$$0=\alpha_0<\alpha_1<\cdots<\alpha_t\leqslant\alpha_{t+1}=m(1)$$

and

$$m(2)=\beta_0\geqslant\beta_1>\beta_2>\cdots>\beta_t\geqslant\beta_{t+1}=1$$

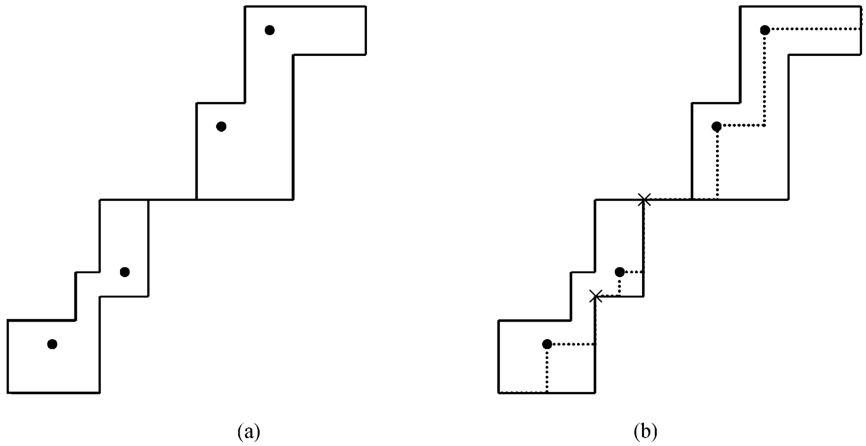


Fig. 4.

we define

$$\hat{\mu}(E) = \bigcup_{r=1}^{t+1} ((\alpha_{r-1}, \alpha_r] \times \{\beta_{r-1}\}) \cup (\{\alpha_r\} \times [\beta_r, \beta_{r-1})).$$

Lastly, given any $\mathscr{E} \in \operatorname{skel}(\mathscr{L}^0)$, we define

$$\mu(\mathscr{E}) = \hat{\mu}(\mathscr{E}) \cap \mathscr{L}.$$

Now pictorially, it may appear obvious that $\hat{\mu}(E)$ is a maximal radical in L for any $E \in \operatorname{skel}(L^0)$; to prove it formally however, seems to require a somewhat lengthy argument which we present below. The reader may wish to skip it depending upon his or her belief in pictures.

Lemma 2.4. *For any* $E \in \operatorname{skel}(L^0)$, *we have* $\hat{\mu}(E) \in \operatorname{marad}(L)$ *and* $|\hat{\mu}(E)| = m(1) + m(2) - 1$.

Proof. Let $E \in \operatorname{skel}(L^0)$ and $t \in \mathbb{N}$. Let $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_{t+1}, \beta_{t+1})$ in L be the unique elements corresponding to E as in Definition 2.3.

We shall first show that $\hat{\mu}(E) \subseteq L$. To this effect, for every $r \in [1, t+1]$ we will find some $k^* \in [1, \operatorname{len}(S)]$ such that

$$S(1, k^*-1) \leqslant \alpha_{r-1} \leqslant \alpha_r \leqslant S(1, k^*) \quad \text{and} \quad \beta_r \leqslant \beta_{r-1} \leqslant S(2, k^*-1) \tag{1}$$

thus showing that

$$(u, \beta_{r-1}) \in [S(1, k^*-1), S(1, k^*)] \times [1, S(2, k^*-1)] \subseteq L \ \forall \ u \in (\alpha_{r-1}, \alpha_r]$$

and

$$(\alpha_r, v) \in [S(1, k^*-1), S(1, k^*)] \times [1, S(2, k^*-1)] \subseteq L \ \forall \ v \in [\beta_r, \beta_{r-1}).$$

For $r=1$, we see that if $\alpha_1 > S(1,1)$ then $1 < \operatorname{len}(S)$ and, since $\alpha_i \geqslant \alpha_1$ for $i \geqslant 1$, we have

$$E \cap ([1,S(1,1)] \times [1,S(2,1)]) \subseteq \tilde{E} \cap ([1,S(1,1)] \times [1,S(2,1)]) = \emptyset,$$

which contradicts the definition of $\tilde{E}$. Thus, $\alpha_1 \leqslant S(1,1)$; also we have $\beta_0 = S(2,0) = m(2)$, and therefore (1) holds with $k^* = 1$ in the case $r = 1$.

Now let us suppose that $r \in [2,t+1]$. Then there exists a unique $k \in [1,\operatorname{len}(S)]$ such that $\alpha_{r-1} \in [S(1,k-1), S(1,k))$ and $\beta_{r-1} \in [1, S(2,k-1)]$. If $k = \operatorname{len}(S)$ then clearly $\alpha_r \leqslant S(1,k)$; also we have $\beta_{r-1} \leqslant S(2,k-1)$, and therefore (1) holds with $k^* = k$. Thus, we will now assume that $k \in [1,\operatorname{len}(S)-1]$ and prove (1) by considering separately the two cases below.

Case (i): $\beta_{r-1} > S(2,k)$: In this case if $\alpha_r > S(1,k)$ then, in view of the fact that $\alpha_{r+i} \geqslant \alpha_r > S(1,k)$ and $\beta_{r-i} \geqslant \beta_{r-1} > S(2,k)$ for $i \geqslant 1$, we find that

$$E \cap ([1,S(1,k)] \times [1,S(2,k)]) \subseteq \tilde{E} \cap ([1,S(1,k)] \times [1,S(2,k)]) = \emptyset,$$

which contradicts the definition of $\tilde{E}$. Thus, $\alpha_r \leqslant S(1,k)$; also we have $\beta_{r-1} \leqslant S(2,k-1)$, and therefore (1) holds with $k^* = k$.

Case (ii): $\beta_{r-1} \leqslant S(2,k)$: In this case, we let k' to be the greatest integer such that $\beta_{r-1} \leqslant S(2,k')$. Then $k' \leqslant k < \operatorname{len}(S)$ and $S(2,k'+1) < \beta_{r-1} \leqslant S(2,k')$. Now using the same arguments as in Case (i) with k' replacing k, we can show that $\alpha_r \leqslant S(1,k')$ and consequently, (1) holds with $k^* = k'$.

Having shown that $\hat{\mu}(E) \subseteq L$, we will now prove that $\operatorname{ind}(\hat{\mu}(E)) \leqslant 1$. Suppose we are given any $(i,j),(p,q)$ in $\hat{\mu}(E)$ such that $i < p$ and $j < q$. Then either (I) $i = \alpha_r$ for some $r \in [1,t+1]$ or (II) $j = \beta_{r-1}$ for some $r \in [1,t+1]$. In the first case, $\beta_r \leqslant j < \beta_{r-1}$ and, since $j < q$, we can find $r' \in [1,t+1]$ such that $r' \leqslant r$ and $\beta_{r'} \leqslant q < \beta_{r'-1}$. But then we must have $p \leqslant \alpha_{r'} \leqslant \alpha_r = i$, which is a contradiction. In the second case, $\alpha_{r-1} < i \leqslant \alpha_r$ and, since $i < p$, we can find $r' \in [1,t+1]$ such that $r' \geqslant r$ and $\alpha_{r'-1} < p \leqslant \alpha_{r'}$. But then we must have $q \leqslant \beta_{r'-1} \leqslant \beta_{r-1} = j$, which is a contradiction. Thus, we have proved that $\hat{\mu}(E) \in \operatorname{rad}(L)$.

Now we show that $\hat{\mu}(E)$ is maximal in $\operatorname{rad}(L)$. Thus suppose $(p,q) \in L \setminus \hat{\mu}(E)$ is such that $\hat{\mu}(E) \cup \{(p,q)\}$ is in $\operatorname{rad}(L)$. We will arrive at a contradiction by exhibiting an element $(i,j) \in \hat{\mu}(E)$ such that $i < p$ and $j < q$. First, we find the unique $r \in [1,t+1]$ such that $\alpha_{r-1} < p \leqslant \alpha_r$. Since $\hat{\mu}(E) \cup \{(p,q)\}$ is in $\operatorname{rad}(L)$ and $(p,q) \notin \hat{\mu}(E)$, we obtain that $q < \beta_{r-1}$. Now the desired $(i,j) \in \hat{\mu}(E)$ is obtained by noting that $(p,q) \neq (\alpha_r, \beta_r)$, and that $r \neq t+1$ if $q < \beta_r$, and by taking

$$(i,j) = \begin{cases} (\alpha_r, \beta_{r-1}) & \text{if } p < \alpha_r, \\ (\alpha_{r+1}, \beta_r) & \text{if } p = \alpha_r \text{ and } q < \beta_r, \\ (\alpha_{r-1}, \beta_r) & \text{if } p = \alpha_r \text{ and } q > \beta_r \text{ and } r > 1, \\ (1, \beta_r) & \text{if } p = \alpha_r \text{ and } q > \beta_r \text{ and } r = 1. \end{cases}$$

Lastly, by the definition of $\hat{\mu}(E)$, we clearly have that

$$|\hat{\mu}(E)| = \sum_{r=1}^{t+1} (\alpha_r - \alpha_{r-1}) + (\beta_{r-1} - \beta_r) = \alpha_{t+1} - \alpha_0 + \beta_0 - \beta_{t+1}$$
$$= m(1) + m(2) - 1. \qquad \square$$

Thus, we have defined the map $\hat{\mu}\colon \operatorname{skel}(L^0) \to \operatorname{marad}(L)$. We describe its relation to the map $\hat{\lambda}$ in the following two lemmas.

Lemma 2.5. *Given any $T \in \operatorname{rad}(L)$, we have $T \subseteq \hat{\mu}(\hat{\lambda}(T))$, and moreover,*

$$T \in \operatorname{marad}(L) \quad \Leftrightarrow \quad T = \hat{\mu}(\hat{\lambda}(T)) \quad \Leftrightarrow \quad |T| = m(1) + m(2) - 1.$$

Proof. First, in view of Lemma 2.4 and (i) of Lemma 2.2, we see that $\hat{\lambda}(T) \in \operatorname{skel}(L^0)$ and that $\hat{\mu}(\hat{\lambda}(T))$ is a well defined member of $\operatorname{marad}(L)$. Let $l, t, (i_0, j_0), \ldots, (i_{l+1}, j_{l+1})$, $(\alpha_0, \beta_0), \ldots, (\alpha_{t+1}, \beta_{t+1})$ be the unique elements such that $l, t \in \mathbb{N}$ and

$$i_0 < i_1 < \cdots < i_l < i_{l+1}, \qquad \alpha_0 < \alpha_1 < \cdots < \alpha_t < \alpha_{t+1},$$
$$j_0 > j_1 > \cdots > j_l \geqslant j_{l+1}, \qquad \beta_0 \geqslant \beta_1 > \cdots > \beta_t \geqslant \beta_{t+1}, \tag{2}$$

and

$$\{(i_1, j_1), \ldots, (i_l, j_l)\} = \hat{\lambda}(T) \subseteq \widetilde{\hat{\lambda}(T)} = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\},$$

where by convention

$$(i_0, j_0) = (\alpha_0, \beta_0) = (0, m(2)) \quad \text{and} \quad (i_{l+1}, j_{l+1}) = (\alpha_{t+1}, \beta_{t+1}) = (m(1), 1).$$

Let $(i,j) \in T$. Then there exist unique integers $r \in [1, t+1]$ and $s \in [1, l+1]$ such that $\alpha_{r-1} < i \leqslant \alpha_r$ and $i_{s-1} < i \leqslant i_s$; further, in view of (2), it follows that

$$i_{s-1} \leqslant \alpha_{r-1} < i \leqslant \alpha_r \leqslant i_s \quad \text{and} \quad j_{s-1} \geqslant \beta_{r-1} \geqslant \beta_r \geqslant j_s. \tag{3}$$

Now if $j > \beta_{r-1}$ then $r > 1$ and $(\alpha_{r-1}, \beta_{r-1})$ cannot be in $\hat{\lambda}(T) \subseteq T$ (because $i > \alpha_{r-1}$ and $T \in \operatorname{rad}(L)$) and consequently $(\alpha_{r-1}, \beta_{r-1}) = (S(1,k), S(2,k))$ for some $k \in [1, \operatorname{len}(S)]$. But then, by observation (1.6), we find that $(i,j) \notin L$, which is a contradiction. Thus, $j \leqslant \beta_{r-1}$; moreover, if $j = \beta_{r-1}$ then

$$(i,j) \in (\alpha_{r-1}, \alpha_r] \times \{\beta_{r-1}\} \subseteq \hat{\mu}(\hat{\lambda}(T)). \tag{4}$$

We shall now assume that $j < \beta_{r-1}$ and prove that

$$(i,j) \in \{\alpha_r\} \times [\beta_r, \beta_{r-1}) \subseteq \hat{\mu}(\hat{\lambda}(T)) \tag{5}$$

by reaching at a contradiction in each of the remaining cases considered below.

Case (i): $i = \alpha_r$ and $j < \beta_r$: Since (α_r, β_r) is either in L^0 or equals $(S(1,k), S(2,k))$ for some $k \in [1, \operatorname{len}(S)]$, it follows from observation (1.10) that $(i,j) \in T \cap L^0$. Now by (3), $i > i_{s-1}$ and hence by the definition of i_s, we see that $i \geqslant i_s$ which, using (3) once again, implies that $i = i_s = \alpha_r$, and consequently $\beta_r = j_s$. But then $(i_s, j) = (i, j) \in T \cap L^0$ and $j < \beta_{r-1} \leqslant j_{s-1}$, and so, by the definition of j_s, we find that $j \geqslant j_s = \beta_r$, which is a contradiction.

Case (ii): $i < \alpha_r$ and $j \leqslant \beta_r$: As in Case (i), it follows from observation (1.10) that $(i, j) \in T \cap L^0$, and that $i \geqslant i_s$, implying that $i = i_s = \alpha_r$, which is a contradiction.

Case (iii): $i < \alpha_r$ and $j > \beta_r$: If $(i, j) \in T \cap L^0$, then as in the two cases above, we would get $i \geqslant i_s$, implying that $i = i_s = \alpha_r$, which is a contradiction. Hence, $(i, j) \in L \backslash L^0$, and so either $i = S(1, k)$ for some $k \in [1, \mathrm{len}(S)]$ or $j = S(2, k-1)$ for some $k \in [1, \mathrm{len}(S)]$.

If $i = S(1, k)$ for some $k \in [1, \mathrm{len}(S)]$ then $S(2, k) \leqslant j \leqslant S(2, k-1)$ and we see that $\hat{\lambda}(T) \cap ([1, S(1, k)] \times [1, S(2, k)]) = \emptyset$, because if (u, v) is in this intersection then $(u, v) = (\alpha_q, \beta_q)$ for some $q \in [1, t]$ and we have

$$\alpha_q = u \leqslant S(1, k) = i < \alpha_r \ \Rightarrow \ q \leqslant r - 1$$

and

$$\beta_q = v \leqslant S(2, k) \leqslant j < \beta_{r-1} \ \Rightarrow \ q \geqslant r,$$

which is a contradiction. Moreover, since $i < \alpha_r$, we have $k < \mathrm{len}(S)$, and hence $(S(1, k), S(2, k)) \in \widetilde{\hat{\lambda}(T)}$ so that $(S(1, k), S(2, k)) = (\alpha_q, \beta_q)$ for some $q \in [1, t]$, which is again seen to yield a contradiction. On the other hand, if $j = S(2, k-1)$ for some $k \in [1, \mathrm{len}(S)]$ then $S(1, k-1) \leqslant i \leqslant S(1, k)$, and in an analogous manner we would find that $k > 1$ and $(S(1, k-1), S(2, k-1)) \in \widetilde{\hat{\lambda}(T)}$, which would again lead to a contradiction.

This completes the proof that $T \subseteq \hat{\mu}(\hat{\lambda}(T))$. Now, in view of Lemma 2.4, we can also conclude that

$$T \in \mathrm{marad}(L) \ \Leftrightarrow \ T = \hat{\mu}(\hat{\lambda}(T)) \ \Leftrightarrow \ |T| = m(1) + m(2) - 1. \qquad \square$$

Lemma 2.6. *Given any* $E \in \mathrm{skel}(L^0)$ *and* $T \subseteq L$ *we have*

$$T \in \mathrm{rad}(L) \text{ and } \hat{\lambda}(T) = E \ \Leftrightarrow \ E \subseteq T \subseteq \hat{\mu}(E).$$

In particular, $\hat{\lambda}(\hat{\mu}(E)) = E$ *and* $\hat{\mu}(E)$ *is the unique maximal element of the set* $\hat{\lambda}^{-1}(E) = \{T' \in \mathrm{rad}(L)\colon \hat{\lambda}(T') = E\}$.

Proof. If $T \in \mathrm{rad}(L)$ and $\hat{\lambda}(T) = E$ then, by Lemma 2.5 and (iii) of Lemma 2.2 we see that $E = \hat{\lambda}(T) \subseteq T \subseteq \hat{\mu}(\hat{\lambda}(T)) = \hat{\mu}(E)$.

Conversely, suppose $E \subseteq T \subseteq \hat{\mu}(E)$. Then, by Lemma 2.4, we see that $\hat{\mu}(E) \in \mathrm{rad}(L)$ and, therefore, $T \in \mathrm{rad}(L)$. We will now show that $\hat{\lambda}(T) \subseteq E$ and $E \subseteq \hat{\lambda}(T)$ to complete the proof. Let $l, t, (i_0, j_0), \ldots, (i_{l+1}, j_{l+1})$, $(\alpha_0, \beta_0), \ldots, (\alpha_{t+1}, \beta_{t+1})$ be the unique elements such that $l \in \mathbb{N}$, $t \in \mathbb{N}$, and

$$0 = i_0 < i_1 < \cdots < i_l < i_{l+1} = m(1), \qquad 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_t < \alpha_{t+1} = m(1),$$

$$m(2) = j_0 > j_1 > \cdots > j_l \geqslant j_{l+1} = 1, \qquad m(2) = \beta_0 \geqslant \beta_1 > \cdots > \beta_t \geqslant \beta_{t+1} = 1,$$

$$\hat{\lambda}(\hat{\mu}(E)) = \{(i_1, j_1), \ldots, (i_l, j_l)\}, \qquad \tilde{E} = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\},$$

and

$$\hat{\mu}(E) = \bigcup_{r=1}^{t+1} ((\alpha_{r-1}, \alpha_r] \times \{\beta_{r-1}\}) \cup (\{\alpha_r\} \times [\beta_r, \beta_{r-1})).$$

Note that $\{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\} \cap L^0 = \tilde{E} \cap L^0 = E \subseteq T \cap L^0$. Also recall that, by observation (1.9), whenever $(i, j) \in L^0$ we have $i < m(1)$ and $j < m(2)$. We shall use these facts tacitly in the proof below.

Given $s \in [1, l]$, since $T \subseteq \hat{\mu}(E)$, we can find a unique $r \in [1, t+1]$ such that either (I) $i_s = \alpha_r$ and $\beta_r \leqslant j_s < \beta_{r-1}$, or (II) $\alpha_{r-1} < i_s \leqslant \alpha_r$ and $j_s = \beta_{r-1}$.

In the first case, since $\alpha_r = i_s < m(1)$ and $(i_s, j_s) \in L^0$, we see that $r \neq t+1$ and $(\alpha_r, \beta_r) \in T \cap L^0$, which contradicts the definition of j_s unless $j_s = \beta_r$. Thus, if (I) holds then $(i_s, j_s) = (\alpha_r, \beta_r) \in E$.

In the second case we note that the r, corresponding to each $s \in [1, l]$, is always different from 1 since $\beta_{r-1} = j_s < m(2)$.

We will now prove that $(i_s, j_s) \in E$ by induction on $s \in [1, l]$. If $s = 1$ and if (II) holds then $(\alpha_{r-1}, \beta_{r-1}) \in T \cap L^0$ and $0 = i_0 < \alpha_{r-1}$, which contradicts the definition of i_1. Thus $(i_1, j_1) \in E$. Next, assume that $s > 1$ and that the assertion is true for all values of s smaller than the given one. If (II) holds and if $i_{s-1} < \alpha_{r-1}$ then $(\alpha_{r-1}, \beta_{r-1}) \in T \cap L^0$ and the definition of i_s would be contradicted. So let us suppose that (II) holds and that $i_{s-1} \geqslant \alpha_{r-1}$. Then, since $r \neq 1$ and $\alpha_{r-1} \leqslant i_{s-1} < i_s \leqslant \alpha_r$, we have that $s \neq 1$ and, by the induction hypothesis, $(i_{s-1}, j_{s-1}) = (\alpha_{r-1}, \beta_{r-1})$. But this contradicts the fact that $\beta_{r-1} = j_s > j_{s-1}$. Hence $(i_s, j_s) \in E$.

Thus, we have proved that $\hat{\lambda}(T) \subseteq E$. Suppose for some $r \in [0, t+1]$, $(\alpha_r, \beta_r) \in E \subseteq T \cap L^0$. Then $r \in [1, t]$ and we can find a unique $s \in [1, l+1]$ such that $i_{s-1} < \alpha_r \leqslant i_s$. Since $\hat{\mu}(E) \in \mathrm{rad}(L)$ we must have $\beta_r \leqslant j_{s-1}$. Now if $\alpha_r = i_s$ then $s \leqslant l$, and since $\hat{\lambda}(T) \subseteq E$, it follows that $j_s = \beta_r$. Whereas if $\alpha_r < i_s$ then the definition of i_s is contradicted unless $\beta_r = j_{s-1}$ in which case $s > 1$, and using $\hat{\lambda}(T) \subseteq E$ once again, it follows that $i_{s-1} = \alpha_r$. Hence in any case $(\alpha_r, \beta_r) \in \hat{\lambda}(T)$. We thus conclude that $\hat{\lambda}(T) = E$. $\square$

Theorem 2.7. *Let $\hat{\lambda}^*$ denote the restriction of $\hat{\lambda}$ to the subset* $\mathrm{marad}(L)$ *of* $\mathrm{rad}(L)$. *Then $\hat{\lambda}^*$ is a bijection of* $\mathrm{marad}(L)$ *onto* $\mathrm{skel}(L^0)$ *and its inverse is given by $\hat{\mu}$.*

Proof. Follows from Lemmas 2.4, 2.5 and 2.6. $\square$

We now turn to the case of biladders and obtain an analogue of the above theorem. In the rest of this section, we shall denote $\Delta(S', S)$ simply by Δ. Thus,

$$\Delta = \Delta(S', S) = \partial S \cap \partial S' = \partial S \cap L(S').$$

First, we will need the following basic lemma.

Lemma 2.8. *For any* $R \in \text{rad}(\mathscr{L})$, *we have the following*:

(i) $R \cup \Delta \in \text{rad}(L)$.
(ii) $R \in \text{marad}(\mathscr{L}) \Leftrightarrow R \cup \Delta \in \text{marad}(L)$.

Proof. Let $R \in \text{rad}(\mathscr{L})$. Note that clearly $R \cup \Delta \subseteq L$ and $R \cap \Delta = \emptyset$; also note that by observation (1.7), $\Delta \in \text{rad}(L)$. Given any $(i,j) \in R$ and $(p,q) \in \Delta$, by observations (1.3) and (1.6), we have that

$$i < p,\ j < q \ \Rightarrow\ (i,j) \in L(S'), \quad \text{and} \quad i > p,\ j > q \ \Rightarrow\ (i,j) \notin L(S)$$

yielding a contradiction in both the cases. Coupled with the fact that both R and Δ are in rad(L), this shows that $R \cup \Delta \in \text{rad}(L)$ thus proving (i).

Now suppose $R \in \text{marad}(\mathscr{L})$ and there exists some $(i,j) \in L(S)$ such that $(i,j) \notin R \cup \Delta$ and $R \cup \Delta \cup \{(i,j)\} \in \text{rad}(L)$. Then $(i,j) \in L(S')$ lest the maximality of R is contradicted. Also since $(i,j) \notin \Delta$, it follows that $(i,j) \in L(S)^0 \cap L(S')$. Now let k be the least positive integer such that $(i+k, j+k) \notin L(S')$. Then $(i+k-1, j+k-1)$ is in $L(S') \subseteq L(S)$ and if it is not in $L(S)^0$, then $k > 1$ and $(i+k-1, j+k-1) \in \Delta$, which contradicts the fact that $\Delta \cup \{(i,j)\} \in \text{rad}(L)$. Hence $(i+k-1, j+k-1) \in L(S)^0$, and consequently $(i+k, j+k) \in L(S) \backslash L(S') = \mathscr{L}$. Moreover, for any $(p,q) \in \mathscr{L}$, in view of observation (1.3) we see that

$$p < i+k,\ q < i+k \Rightarrow p \leqslant i+k-1,\ q \leqslant i+k-1 \Rightarrow (p,q) \in L(S')$$

$$\Rightarrow\ (p,q) \notin \mathscr{L}$$

and

$$p > i+k,\ q > i+k \Rightarrow p > i,\ q > i \Rightarrow (p,q) \notin R,$$

where the last implication follows from the fact that $R \cup \{(i,j)\} \in \text{rad}(L)$. This shows that $R \cup \{(i+k, j+k)\} \in \text{rad}(\mathscr{L})$ contradicting the maximality of R in rad$(\mathscr{L})$.

Conversely, if for some $R \in \text{rad}(\mathscr{L})$ we have that $R \cup \Delta \in \text{marad}(L)$, then R is also maximal in rad$(\mathscr{L})$, owing to (i) and the fact that $\mathscr{L} \cap \Delta = \emptyset$. $\square$

Lemma 2.9. *Given any* $\mathscr{E} \in \text{skel}(\mathscr{L}^0)$, *we have* $\mu(\mathscr{E}) \in \text{marad}(\mathscr{L})$ *and* $\hat{\mu}(\mathscr{E}) = \mu(\mathscr{E}) \cup \Delta$. *Consequently*, *if we let* $M = m(1) + m(2) - 1$ *and* $\delta_0 = |\Delta(S', S)|$, *then we have* $|\mu(\mathscr{E})| = M - \delta_0$.

Proof. Let $\mathscr{E} \in \text{skel}(\mathscr{L}^{\text{o}})$. Since $\mu(\mathscr{E}) = \hat{\mu}(\mathscr{E}) \cap \mathscr{L}$ it follows from Lemma 2.4 that $\mu(\mathscr{E}) \in \text{rad}(\mathscr{L})$. We shall now show that $\hat{\mu}(\mathscr{E}) \subseteq \mu(\mathscr{E}) \cup \Delta$. To this effect, let there be given any $(i,j) \in \hat{\mu}(\mathscr{E})$. If $(i,j) \notin \mathscr{L}$ and $(i,j) \notin \Delta$, then $(i,j) \in L(S') \cap L^0$; suppose this is the case. Let $t, (\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_{t+1}, \beta_{t+1})$ be the unique elements corresponding to $E = \mathscr{E}$ as in Definition 2.3. Now there exists a unique $r \in [1, t+1]$ such that either (I) $i = \alpha_r$ and $\beta_r \leqslant j < \beta_{r-1}$ or (II) $\alpha_{r-1} < i \leqslant \alpha_r$ and $j = \beta_{r-1}$. In the first case, since $(i,j) \in L^0$, it follows that $i < m(1)$ so that $r \neq t+1$ and $(\alpha_r, \beta_r) \in \tilde{E} \cap L^0 = \mathscr{E} \subseteq \mathscr{L}$;

whereas, since $(i,j)\in L(S')$ and $r\neq t+1$, it follows that $(\alpha_r,\beta_r)\in L(S')$ yielding a contradiction. In the second case, we can similarly conclude that $r\neq 1$ and $(\alpha_{r-1},\beta_{r-1})\in\mathscr{L}$ and $(\alpha_{r-1},\beta_{r-1})\in L(S')$ yielding a contradiction once again. Thus, $\hat{\mu}(\mathscr{E})\subseteq\mu(\mathscr{E})\cup\Delta$. Now, by (i) of Lemma 2.8, $\mu(\mathscr{E})\cup\Delta\in\mathrm{rad}(\mathscr{L})$ and hence, by the maximality of $\hat{\mu}(\mathscr{E})$ in $\mathrm{rad}(L)$, we must have $\hat{\mu}(\mathscr{E})=\mu(\mathscr{E})\cup\Delta$. The remaining assertions follow readily from Lemmas 2.4 and 2.8. □

Thus, we obtain the map $\mu\colon \mathrm{skel}(\mathscr{L}^{\mathrm{o}})\to\mathrm{marad}(\mathscr{L})$. As in the case of ladders, we now find out more about μ as well as λ. The results are described in the following two lemmas.

Lemma 2.10. *For any* $R\in\mathrm{rad}(\mathscr{L})$, *we have* $R\subseteq\mu(\lambda(R))$, *and moreover*,

$$R\in\mathrm{marad}(\mathscr{L})\ \Leftrightarrow\ R=\mu(\lambda(R))\ \Leftrightarrow\ |R|=m(1)+m(2)-1-|\Delta|.$$

Proof. Let $R\in\mathrm{rad}(\mathscr{L})$. Then $R\in\mathrm{rad}(L)$ and by Lemma 2.5 and (vi) of Lemma 2.2 it follows that $R\subseteq\hat{\mu}(\hat{\lambda}(R))\cap\mathscr{L}=\hat{\mu}(\lambda(R))\cap\mathscr{L}=\mu(\lambda(R))$. Moreover, in view of the fact that $\Delta\cap\mathscr{L}=\emptyset$, we see that

$R\in\mathrm{marad}(\mathscr{L})$

$\Leftrightarrow R\cup\Delta\in\mathrm{marad}\ (L)$	by Lemma 2.8
$\Leftrightarrow R\cup\Delta=\hat{\mu}(\hat{\lambda}(R\cup\Delta))$	by Lemmas 2.5 and 2.8
$\Leftrightarrow R=(R\cup\Delta)\cap\mathscr{L}=\hat{\mu}(\hat{\lambda}(R\cup\Delta))\cap\mathscr{L}$	by Lemmas 2.5 and 2.8
$\Leftrightarrow R=\hat{\mu}(\lambda(R))\cap\mathscr{L}$	by Lemma 2.8 and (v), (vi) of Lemma 2.2.
$\Leftrightarrow R=\mu(\lambda(R))$.	

Finally, since $R\cap\Delta=\emptyset$, it follows from Lemmas 2.5 and 2.8 that

$$R\in\mathrm{marad}(\mathscr{L})\ \Leftrightarrow\ R\cup\Delta\in\mathrm{marad}(L)\ \Leftrightarrow\ |R|=m(1)+m(2)-1-|\Delta|.\qquad\square$$

Lemma 2.11. *Given any* $\mathscr{E}\in\mathrm{skel}(\mathscr{L}^0)$ *and* $R\subseteq\mathscr{L}$ *we have that*

$$R\in\mathrm{rad}(L)\ \textit{and}\ \lambda(R)=\mathscr{E}\ \Leftrightarrow\ \mathscr{E}\subseteq R\subseteq\mu(\mathscr{E}).$$

In particular, $\lambda(\mu(\mathscr{E}))=\mathscr{E}$ *and* $\mu(\mathscr{E})$ *is the unique maximal element of the set* $\lambda^{-1}(\mathscr{E})=\{R'\in\mathrm{rad}(\mathscr{L}):\lambda(R')=\mathscr{E}\}$.

Proof. If $R\in\mathrm{rad}(\mathscr{L})$ and $\lambda(R)=\mathscr{E}$ then, by Lemma 2.9 and (iii) of Lemma 2.2, we see that $\mathscr{E}=\lambda(R)\subseteq R\subseteq\mu(\lambda(R))=\mu(\mathscr{E})$. Conversely, if $\mathscr{E}\subseteq R\subseteq\mu(\mathscr{E})$ then, by Lemma 2.9, $R\in\mathrm{rad}(\mathscr{L})$, and, in view of Lemma 2.5 and (vi) of Lemma 2.2, it follows that $\lambda(R)=\hat{\lambda}(R)=\mathscr{E}$. This proves the desired equivalence. The remaining assertions are now evident. □

We are now ready to prove the main result of this section.

Theorem 2.12. *Let* λ^* *denote the restriction of* λ *to the subset* $\mathrm{marad}(\mathscr{L})$ *of* $\mathrm{rad}(\mathscr{L})$. *Then* λ^* *is a bijection of* $\mathrm{marad}(L)$ *onto* $\mathrm{skel}(L^0)$ *and its inverse is given by* μ.

Proof. Follows from Lemmas 2.9, 2.10 and 2.11. □

Finally, in this section, we obtain an enumerative consequence of the correspondence described in the above theorem. Recall that for any $l \in \mathbb{N}$, $\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$ denotes the set of all skeletons $\mathscr{E}$ in $\mathscr{L}^{\mathrm{o}}$ of 'size' l, i.e., $|\mathscr{E}| = l$, and that for any $d \in \mathbb{N}$, $\mathrm{rad}(\mathscr{L}, d)$ denotes the set of all radicals R in $\mathscr{L}$ of 'size' d, i.e., $|R| = d$.

Theorem 2.13. *Let* $M = m(1) + m(2) - 1$ *and* $\delta_0 = |\Delta(S', S)|$. *Then for any* $d \in \mathbb{N}$, *we have*

$$|\mathrm{rad}(\mathscr{L}, d)| = \sum_{l \in \mathbb{N}} |\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)| \binom{M - \delta_0 - l}{d - l},$$

where the summation on the right is essentially finite.

Proof. Let $d \in \mathbb{N}$. Since $\lambda : \mathrm{rad}(\mathscr{L}) \to \mathrm{skel}(\mathscr{L}^{\mathrm{o}})$ is onto, we have

$$\mathrm{rad}(\mathscr{L}, d) = \coprod_{\mathscr{E} \in \mathrm{skel}(\mathscr{L}^{\mathrm{o}})} \lambda^{-1}(\mathscr{E}) \cap \mathrm{rad}(\mathscr{L}, d) = \coprod_{l \in \mathbb{N}} \coprod_{\mathscr{E} \in \mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)} \lambda^{-1}(\mathscr{E}) \cap \mathrm{rad}(\mathscr{L}, d).$$

By Lemma 2.11, we see that $\lambda^{-1}(\mathscr{E}) = \{R : \mathscr{E} \subseteq R \subseteq \mu(\mathscr{E})\}$, and consequently,

$$\lambda^{-1}(\mathscr{E}) \cap \mathrm{rad}(\mathscr{L}, d) = \{R : \mathscr{E} \subseteq R \subseteq \mu(\mathscr{E}) \text{ and } |R| = d\}.$$

The desired formula now follows from the above equalities by applying Lemma 1.3 and Lemma 2.9. The essential finiteness follows, for example, by noting that the binomial coefficient in the above summation vanishes if $l > d$. □

3. Enumeration of skeletons in a biladder

Let S' and S be in $\mathrm{lad}(m)$ with $S' \leqslant S$. Let $h = \mathrm{len}(S)$ and $h' = \mathrm{len}(S')$. Let $\mathscr{L}$ and $\mathscr{L}^{\mathrm{o}}$ denote the biladder $\mathscr{L}(S', S)$ and its interior $\mathscr{L}(S', S)^0$, respectively. Let $l \in \mathbb{N}$. We will continue to assume that $h = \mathrm{len}(S) \neq 0$. Further, we shall assume that $h' = \mathrm{len}(S') \neq 0$. These assumptions will be tacitly used in some of the results proved in this section.

Our aim in this section is to count the set $\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$ of all skeletons in $\mathscr{L}^{\mathrm{o}}$ of size l. Now an element of $\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$ can be represented by a pair of sequences of length l, one of which is (strictly) increasing and the other one is (strictly) decreasing. Thus if $\mathscr{L}^{\mathrm{o}}$ were the entire rectangle $[1, m(1)] \times [1, m(2)]$, then $|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)|$ would simply equal $\binom{m(1)}{l}\binom{m(2)}{l}$. However, the problem of finding $|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)|$ appears increasingly difficult even if $\mathscr{L}^{\mathrm{o}}$ were the set obtained by cutting off a corner each from the top left-hand side and the bottom right-hand side of the above rectangle. Naturally, in order to find $|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)|$, the condition for a pair of 'increasing–decreasing' sequences to correspond to a skeleton within $\mathscr{L}^{\mathrm{o}}$ needs to be expressed in a more suitable manner. We do this by decomposing the set $\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$ into smaller and smaller disjoint subsets. Finding the cardinalities of the 'components' along with some manipulation, then results into a concrete (but complicated!) formula for $|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)|$.

We begin by introducing some notation which will be used in the rest of this paper. Given any $h^* \in \mathbb{N}$, we let $\mathbb{N}(h^*)$ denote the set of all sequences of length h^* whose terms are nonnegative integers, i.e.,

$$\mathbb{N}(h^*) = \{u = (u(1), u(2), \ldots, u(h^*)):\ u(k) \in \mathbb{N} \text{ for all } k \in [1, h^*]\}.$$

Given any $u \in \mathbb{N}(h^*)$ and $k \in [1, h^*]$, we may sometimes write u_k to mean $u(k)$. Next, given any $m^* \in \mathbb{N}^+$ and $l^* \in \mathbb{N}$, we let

$$\mathrm{inc}\,(m^*, l^*) = \{a \in \mathbb{N}(l^*):\ 0 < a(1) < a(2) < \cdots < a(l^*) < m^*\}$$

and

$$\mathrm{dec}\,(m^*, l^*) = \{b \in \mathbb{N}(l^*):\ m^* > b(1) > b(2) > \cdots > b(l^*) > 0\}.$$

Finally, a note concerning notation. Given any two sets X and Y and a map $f : X \to Y$, by $\mathrm{im}(f)$ we denote the image of f, i.e., $\mathrm{im}(f)$ is the subset $\{f(x):\ x \in X\}$ of Y; for any $y \in Y$, by $f^{-1}(y)$ or $f^{-1}\{y\}$ we denote the set $\{x \in X:\ f(x) = y\}$. Observe that if Z is any set with $\mathrm{im}(f) \subseteq Z \subseteq Y$, then we have

$$X = \coprod_{z \in Z} f^{-1}(z) \quad \text{where } \coprod \text{ denotes disjoint union as usual.}$$

The desired decomposition of $\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$ will be achieved by using, among other things, the maps defined below.

Definition 3.1. (i) Given any $\mathscr{E} \in \mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$, define $\mathscr{E}^1$ and $\mathscr{E}^2$ to be the unique elements in $\mathrm{inc}(m(1), l)$ and $\mathrm{dec}(m(2), l)$, respectively such that $\mathscr{E} = \{(\mathscr{E}^1(i), \mathscr{E}^2(i)):\ i \in [1, l]\}$.

(ii) Define the 'projection map' π: $\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l) \to \mathrm{inc}(m(1), l)$ by $\pi(\mathscr{E}) = \mathscr{E}^1$, for all $\mathscr{E} \in \mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)$.

(iii) Define the map ρ: $\mathrm{inc}(m(1), l) \to \mathbb{N}(h) \times \mathbb{N}(h')$ by putting

$$\rho(a) = (e, e') \quad \text{for } a \in \mathrm{inc}(m(1), l),$$

where we have temporarily let e and e' denote the unique elements of $\mathbb{N}(h)$ and $\mathbb{N}(h')$, respectively, given by

$$e(k) = |\{i \in [1, l]:\ a(i) < S(1, k)\}| \quad \text{for all } k \in [1, h]$$

and

$$e'(k') = |\{i \in [1, l]:\ a(i) < S'(1, k')\}| \quad \text{for all } k' \in [1, h'].$$

Lemma 3.2. *Let* $(e,e')\in\mathbb{N}(h)\times\mathbb{N}(h')$. *Then we have the following*:

(i) $\rho^{-1}\{(e,e')\}$ *is equal to*

$$\{a\in\operatorname{inc}(m(1),l)\colon \{i\in[1,l]\colon a(i)<S(1,k)\}=[1,e(k)]\ \forall\ k\in[1,h],$$
$$\text{and } \{i\in[1,l]\colon a(i)<S'(1,k')\}=[1,e'(k')]\ \forall\ k\in[1,h']\}.$$

(ii) *If* $(e,e')\in\operatorname{im}(\rho)$ *then we have*

$$0\leqslant e(1)\leqslant\cdots\leqslant e(h)=l \quad \textit{and} \quad 0\leqslant e'(1)\leqslant\cdots\leqslant e'(h')=l. \tag{6}$$

Proof. Let $a\in\rho^{-1}\{(e,e')\}$. For $k\in[1,h]$, if we let

$$i_k=\max\{i\in[1,l]\colon a(i)<S(1,k)\}$$

then, in view of the fact that $a\in\operatorname{inc}(m(1),l)$, it follows that

$$\{i\in[1,l]\colon a(i)<S(1,k)\}=[1,i_k],$$

therefore $i_k=e(k)$. It is now evident that $\rho^{-1}\{(e,e')\}\subseteq$ [RHS of (i)]. The other inclusion being obvious, this proves (i).

Given any $(e,e')\in\operatorname{im}(\rho)$, we can find some $a\in\operatorname{inc}(m(1),l)$ such that $\rho(a)=(e,e')$. Now for any $k\in[1,h-1]$ and $k'\in[1,h'-1]$, since $S(1,k)<S(1,k+1)$ and $S'(1,k')<S'(1,k'+1)$, we have $e(k)\leqslant e(k+1)$ and $e'(k')\leqslant e'(k'+1)$. Moreover, since $S(1,h)=S'(1,h')=m(1)$ and $a(i)<m(1)$ for all $i\in[1,l]$, we have $e(h)=e'(h')=l$. This proves (ii). □

Lemma 3.3. *For any* $a\in\mathbb{N}(l)$, *let* $\bar{a}$ *denote the set* $\{a(1),a(2),\ldots,a(l)\}$. *For any* $(e,e')\in\mathbb{N}(h)\times\mathbb{N}(h')$, *let* $e(0)=0=e'(0)$, *and let* $A(e,e')$ *denote the set*

$$\{a\in\operatorname{inc}(m(1),l)\colon |\bar{a}\cap[S(1,k-1),S(1,k))|=e(k)-e(k-1)\ \forall\ k\in[1,h] \text{ and}$$
$$|\bar{a}\cap[S'(1,k'-1),S'(1,k'))|=e'(k')-e'(k'-1)\,\forall\ k'\in[1,h']\}.$$

Then we have the following:

(i) $\rho^{-1}\{(e,e')\}=A(e,e')$ *for all* $(e,e')\in\mathbb{N}(h)\times\mathbb{N}(h')$.
(ii) *The set* $\operatorname{skel}(\mathscr{L}^{\mathrm{o}},l)$ *decomposes as*

$$\operatorname{skel}(\mathscr{L}^{\mathrm{o}},l)=\coprod_{a\in\operatorname{inc}(m(1),l)}\pi^{-1}(a)=\coprod_{(e,e')}\ \coprod_{a\in A(e,e')}\pi^{-1}(a),$$

where (e,e') *range over all elements of* $\mathbb{N}(h)\times\mathbb{N}(h')$ *satisfying* (6).

Proof. Follows from Lemma 3.2. □

Now let us fix (until Lemma 3.8) some $(e,e')\in\mathbb{N}(h)\times\mathbb{N}(h')$ satisfying (6), and also let us fix some $a\in\rho^{-1}\{(e,e')\}$. We now proceed to find $|\pi^{-1}(a)|$. To this effect we first decompose the set $\pi^{-1}(a)$ by using the map defined below.

Definition 3.4. Define the map $\theta:\pi^{-1}(a) \to \mathbb{N}(h)\times\mathbb{N}(h')$ by putting, for any $\mathscr{E}\in\pi^{-1}(a), \theta(\mathscr{E})=(f,f')$, where we have temporarily let f and f' to be the unique elements of $\mathbb{N}(h)$ and $\mathbb{N}(h')$, respectively, such that

$$f(k)=|\{j\in[1,l]:\mathscr{E}^2(j)\geqslant S(2,k)\}| \quad \text{for all } k\in[1,h]$$

and

$$f'(k')=|\{j\in[1,l]:\mathscr{E}^2(j)> S'(2,k'-1)\}| \quad \text{for all } k'\in[1,h'].$$

Lemma 3.5. *Let* $(f,f')\in\mathbb{N}(h)\times\mathbb{N}(h')$. *Then we have the following*:

(i) $\theta^{-1}\{(f,f')\}$ *equals*

$$\begin{aligned}\{\mathscr{E}\in \operatorname{skel}(\mathscr{L}^{\mathrm{o}},l)\colon\ & \mathscr{E}^1=a,\ \{j\in[1,l]:\mathscr{E}^2(j)\geqslant S(2,k)\}=[1,f(k)],\\ & \{j\in[1,l]\colon \mathscr{E}^2(j)>S'(2,k'-1)\}=[1,f'(k')]\\ & \text{for all } k\in[1,h] \text{ and } k'\in[1,h']\}.\end{aligned}$$

(ii) *If* $(f,f')\in \operatorname{im}(\theta)$, *then we have*

$$\begin{aligned} &0\leqslant f(1)\leqslant\cdots\leqslant f(h)=l, \qquad 0\leqslant f'(1)\leqslant\cdots\leqslant f'(h')=l,\\ &f(k)\leqslant e(k) \quad \text{for all } k\in[1,h], \qquad f'(k')\geqslant e'(k') \quad \text{for all } k'\in[1,h'].\end{aligned} \tag{7}$$

Proof. If $\mathscr{E}\in\theta^{-1}(f,f')$ then, in view of the fact that $\mathscr{E}^2\in \operatorname{dec}(m(2),l)$, it follows that

$$\{j\in[1,l]\colon \mathscr{E}^2(j)\geqslant S(2,k)\}=[1,f(k)] \quad \text{for all } k\in[1,h]$$

and

$$\{j\in[1,l]\colon \mathscr{E}^2(j)> S'(2,k'-1)\}=[1,f'(k')] \quad \text{for all } k'\in[1,h'].$$

This shows that $\theta^{-1}(f,f')$ is contained in the set on the right-hand side of the equality asserted in (i). The other inclusion being obvious, this proves (i).

Now let us assume that $(f,f')\in \operatorname{im}(\theta)$. Fix some $\mathscr{E}\in \operatorname{skel}(\mathscr{L}^{\mathrm{o}},l)$ such that $\theta(\mathscr{E})=(f,f')$. For all $k\in[1,h-1]$ and $k'\in[1,h'-1]$, we have $S(2,k)\geqslant S(2,k+1)$ and $S'(2,k'-1)\geqslant S'(2,k')$, and therefore it follows that $f(k)\leqslant f(k+1)$ and $f'(k')\leqslant f'(k'+1)$. Moreover, since $S(2,h)=1$, we see that $f(h)=l$, and since $\mathscr{E}\cap L(S')=\emptyset$ and $[1,m(1)]\times[1,S'(2,h'-1)]\subseteq L(S')$, we also see that $f'(h')=l$.

Now, by Lemma 3.2, we have that

$$\{i\in[1,l]\colon \mathscr{E}^1(i)< S(1,k)\}=[1,e(k)] \quad \text{for all } k\in[1,h]$$

and

$$\{i\in[1,l]\colon \mathscr{E}^1(i)< S'(1,k')\}=[1,e'(k')] \quad \text{for all } k'\in[1,h'].$$

Consequently, if for some $k \in [1,h]$, we have $f(k) > e(k)$, then $1 \leqslant f(k) \leqslant l$ and $\mathscr{E}^1(f(k)) \geqslant S(1,k)$. Also by (i) we have $\mathscr{E}^2(f(k)) \geqslant S(2,k)$. Thus

$$\mathscr{E}^1(f(k)) \in [S(1,k), m(1)) \quad \text{and} \quad \mathscr{E}^2(f(k)) \notin [1, S(2,k)).$$

This is a contradiction since $([S(1,k), m(1)) \times [S(2,k), m(2)]) \cap L(S)^0 = \emptyset$. Similarly, if for some $k' \in [1,h']$ we have $f'(k') < e'(k')$, then it can be seen that we have $1 \leqslant e'(k') \leqslant l$ and

$$\mathscr{E}^2(e'(k')) \in [1, S'(2,k'-1)] \quad \text{and} \quad \mathscr{E}^1(e'(k')) \in [1, S'(1,k')),$$

which is contradiction since $[1, S'(1,k')) \times [1, S'(2,k'-1)] \subseteq L(S')$. This proves (ii). $\square$

Lemma 3.6. *Let $(f,f') \in \mathbb{N}(h) \times \mathbb{N}(h')$ be such that* (7) *holds. Let $B(f,f')$ denote the set*

$$\{b \in \operatorname{dec}(m(2),l)\colon |\{j \in [1,l]\colon b(j) \geqslant S(2,k)\}| = f(k), \text{ for all } k \in [1,h] \text{ and}$$
$$|\{j \in [1,l]\colon b(j) > S'(2,k'-1)\}| = f'(k') \text{ for all } k' \in [1,h']\}.$$

Then we have the following:

(i) *There exists a one-to-one correspondence between $\theta^{-1}\{(f,f')\}$ and $B(f,f')$. In particular, $|\theta^{-1}\{(f,f')\}| = |B(f,f')|$.*

(ii) *Given any $b \in \mathbb{N}(l)$, let $\bar{b} = \{b(1), b(2), \ldots, b(l)\}$. Also, let $f(0) = 0$ and $f'(h'+1) = l$. Then $B(f,f')$ equals*

$$\{b \in \operatorname{dec}\,(m(2),l)\colon |\bar{b} \cap [S(2,k), S(2,k-1))| = f(k) - f(k-1)\ \forall k \in [1,h],$$
$$\text{and } |\bar{b} \cap (S'(2,k'), S'(2,k'-1)]| = f'(k'+1) - f'(k')\ \forall k' \in [1,h']\}.$$

Proof. Consider the map $\mathscr{E} \mapsto \mathscr{E}^2$ of $\theta^{-1}\{(f,f')\} \to B(f,f')$. By Definition 3.4 we see that this map is well defined; moreover it is clearly injective. To show that it is surjective as well, let there be given any $b \in B(f,f')$. Let $\mathscr{E} \in \operatorname{skel}([1,m(1)] \times [1,m(2)], l)$ be the unique element such that $\mathscr{E}^1 = a$ and $\mathscr{E}^2 = b$. It suffices to show that $\mathscr{E} \subseteq \mathscr{L}^{\mathrm{o}} = L(S)^0 \backslash L(S')$. Given any $i \in [1,l]$, we have $a(i) < m(1)$, and hence there exists a unique $k \in [1,h]$ such that $S(1,k-1) \leqslant a(i) < S(1,k)$; consequently, $i > e(k-1)$ (because $e(0) = 0$ and for $k > 1$, $\{i' \in [1,l] : a(i') < S(1,k-1)\} = [1, e(k-1)]$), and, therefore, $i > f(k-1)$, so that $b(i) < S(2,k-1)$. Also $S(2,0) = m(2) > b(i)$ for all $i \in [1,l]$. Thus, we have

$$(a(i), b(i)) \in [S(1,k-1), S(1,k)) \times [1, S(2,k-1)) \subseteq L(S) \quad \text{for all } i \in [1,l].$$

Similarly, given any $j \in [1,l]$, there exists a unique $k' \in [1,h']$ with the property that $S'(1,k'-1) \leqslant a(j) < S'(1,k')$; consequently, $j \leqslant e'(k') \leqslant f'(k')$, and therefore $b(j) > S'(2,k'-1)$ (because $\{j \in [1,l]\colon b(j) > S'(2,k'-1)\} = [1, f'(k')]$). Thus, for any $j \in [1,l]$, we have

$$(a(j), b(j)) \notin \coprod_{k' \in [1,h']} [S'(1,k'-1), S'(1,k']) \times [1, S'(2,k'-1)] = L(S').$$

This proves the first assertion. The second assertion is evident in view of the fact that $f'(1)=0=f(0)$. $\square$

Having characterized $\theta^{-1}\{(f,f')\}$ as above, it is now a relatively easy matter to find its cardinality. The cardinality of $\rho^{-1}\{(e,e')\}$ can also be found in an analogous manner in view of Lemma 3.3. We separate the essential enumerative argument in the general proposition below.

Proposition 3.7. *Let C be a finite set. Suppose $\{C_k\colon 1\leqslant k\leqslant h\}$ and $\{C'_{k'}\colon 1\leqslant k'\leqslant h'\}$ are two families of subsets of C, and $\{w_k\colon 1\leqslant k\leqslant h\}$ and $\{w'_{k'}: 1\leqslant k'\leqslant h'\}$ are two families of nonnegative integers such that*

$$C=\coprod_{k=1}^{h} C_k=\coprod_{k'=1}^{h'} C'_{k'} \quad \textit{and} \quad l=\sum_{k=1}^{h} w_k=\sum_{k'=1}^{h'} w'_{k'}.$$

Let $\mathscr{P}(C,l)$ denote the set of all subsets of C of *cardinality l, and let*

$$\mathscr{S}=\{E\in\mathscr{P}(C,l)\colon |E\cap C_k|=w_k\ \forall\ k\in[1,h] \textit{ and } |E\cap C'_{k'}|=w'_{k'}\ \forall\ k'\in[1,h']\}.$$

Then, upon letting $C_{k,k'}=C_k\cap C'_{k'}$ for all $k\in[1,h]$ and $k'\in[1,h']$, we have

$$|\mathscr{S}|=\sum_{\alpha=(\alpha_{kk'})}\ \prod_{\substack{1\leqslant k\leqslant h\\ 1\leqslant k'\leqslant h'}} \binom{|C_{k,k'}|}{\alpha_{kk'}},$$

where the sum is taken over all $h\times h'$ matrices $\alpha=(\alpha_{kk'})\in M_{h\,h'}(\mathbb{N},l)$ such that

$$\sum_{k=1}^{h}\alpha_{kk'}=w'_{k'}\ \ \forall\ k'\in[1,h'] \quad \textit{and} \quad \sum_{k'=1}^{h'}\alpha_{kk'}=w_k\ \ \forall\ k\in[1,h].$$

Proof. Follows by noting that $C=\coprod C_{k,k'}$, where the disjoint union is taken over all $(k,k')\in[1,h]\times[1,h']$, and that the map $E\mapsto(E\cap C_{k,k'})_{(k,k')\in[1,h]\times[1,h']}$ sets up a one-to-one correspondence:

$$\mathscr{S}\ \leftrightarrow\ \coprod_{\alpha}\prod_{(k,k')}\{E^*\colon E^*\subseteq C_{k,k'} \text{ and } |E^*|=\alpha_{kk'}\},$$

where the direct product of sets is taken over all $(k,k')\in[1,h]\times[1,h']$, and the disjoint union is taken over all $\alpha=(\alpha_{kk'})\in M_{h\,h'}(\mathbb{N},l)$ such that $\sum_{k=1}^{h}\alpha_{kk'}=w'_{k'}\ \forall\ k'\in[1,h']$ and $\sum_{k'=1}^{h'}\alpha_{kk'}=w_k\ \forall\ k\in[1,h]$. $\square$

Notation. Given a ladder $\mathscr{L}=\mathscr{L}(S',S)$, where $S'\in\mathrm{lad}[m',h']$ and $S\in\mathrm{lad}[m,h]$ are such that $S'\leqslant S$, and $(k,k')\in[1,h]\times[1,h']$ we let

$$\mu_{\mathscr{L}}(k,k')=|[S(1,k-1),S(1,k))\cap[S'(1,k'-1),S'(1,k'))|$$

and

$$v_{\mathscr{L}}(k,k')=|[S(2,k),S(2,k-1))\cap(S'(2,k'),S'(2,k'-1)]|.$$

Lemma 3.8. *Given any* $(e,e') \in \mathbb{N}(h) \times \mathbb{N}(h')$ *satisfying* (6) [*in particular, any* $(e,e') \in \mathrm{im}\ (\rho)$], *we have that*

$$|\rho^{-1}\{(e,e')\}| = \sum_{\alpha=(\alpha_{kk'})} \prod_{\substack{1\leqslant k\leqslant h\\ 1\leqslant k'\leqslant h'}} \binom{\mu_{\mathscr{L}}(k,k')}{\alpha_{kk'}},$$

where the sum is taken over all $h \times h'$ *matrices* $\alpha = (\alpha_{kk'}) \in M_{h\,h'}(\mathbb{N}, l)$ *such that*

$$\sum_{k=1}^{h} \alpha_{kk'} = e'(k') - e'(k'-1)\ \forall k' \quad \text{and} \quad \sum_{k'=1}^{h'} \alpha_{kk'} = e(k) - e(k-1)\ \forall k,$$

where by convention, $e(0) = e'(0) = 0$. (8)

Proof. Let $(e,e') \in \mathbb{N}(h) \times \mathbb{N}(h')$ be such that (6) holds. Further, let $C = [1, m(1))$ and for $1 \leqslant k \leqslant h$ and $1 \leqslant k' \leqslant h'$, let

$$C_k = [S(1,k-1), S(1,k)) \quad \text{and} \quad C'_{k'} = [S'(1,k'-1), S'(1,k')),$$

$$w_k = e(k) - e(k-1) \quad \text{and} \quad w'_{k'} = e'(k') - e'(k'-1).$$

Then in view of Lemma 3.2 we clearly see that

$$C = \coprod_{k=1}^{h} C_k = \coprod_{k'=1}^{h'} C'_{k'} \quad \text{and} \quad l = \sum_{k=1}^{h} w_k = \sum_{k'=1}^{h'} w'_{k'}.$$

Now the assertion readily follows from Lemmas 3.2 and 3.3, and Proposition 3.7 by noting that the map $a \mapsto \bar{a}$ sets up a natural one-to-one correspondence between $\mathrm{inc}(m(1), l)$ and the set consisting of all subsets E of $[1, m(1))$ with $|E| = l$. $\square$

Using arguments similar to those used in the proof of the above lemma, we can easily obtain a formula for $|\theta^{-1}\{(f,f')\}|$ as given below. This time we leave the details to the reader.

Lemma 3.9. *Given any* $(f,f') \in \mathbb{N}(h) \times \mathbb{N}(h')$ *satisfying* (7) [*in particular, any* $(f,f') \in \mathrm{im}(\theta)$], *we have that*

$$|\theta^{-1}\{(e,e')\}| = \sum_{\beta=(\beta_{kk'})} \prod_{\substack{1\leqslant k\leqslant h\\ 1\leqslant k'\leqslant h'}} \binom{v_{\mathscr{L}}(k,k')}{\beta_{kk'}},$$

where the sum is taken over all $h \times h'$ *matrices* $\beta = (\beta_{kk'}) \in M_{h\,h'}(\mathbb{N}, l)$ *such that*

$$\sum_{k=1}^{h} \beta_{kk'} = f'(k') - f'(k'-1) \quad \forall k', \qquad \sum_{k'=1}^{h'} \beta_{kk'} = f(k) - f(k-1) \quad \forall k,$$

where by convention, $f(0) = 0$ *and* $f'(h'+1) = l$. (9)

Proof. Follows from Lemmas 3.5, 3.6 and Proposition 3.7. □

Let us define

$$\mathscr{S}(\mathscr{L}^{\mathrm{o}}, l) = \sum_{\substack{\alpha,\beta \in M_{h\,h'}(\mathbb{N},l) \\ \sigma(\beta)\leqslant\sigma(\alpha),\ \tau(\beta)\leqslant\tau(\alpha)}} \prod_{\substack{1\leqslant k\leqslant h \\ 1\leqslant k'\leqslant h'}} \binom{\mu_{\mathscr{L}}(k,k')}{\alpha_{kk'}} \binom{v_{\mathscr{L}}(k,k')}{\beta_{kk'}}.$$

With this notation, some of the results obtained in this section can be combined into the following theorem.

Theorem 3.10. *Given any* $l \in \mathbb{N}$, *we have*

$$|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)| = \mathscr{S}(\mathscr{L}^{\mathrm{o}}, l).$$

Proof. By Lemma 3.3, we have

$$\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l) = \coprod_{(e,e')} \coprod_{a \in \rho^{-1}\{(e,e')\}} \pi^{-1}(a),$$

where the first disjoint union is taken over all $(e,e') \in \mathbb{N}(h) \times \mathbb{N}(h')$ satisfying (6). Now for any $a \in \rho^{-1}\{(e,e')\}$,

$$\pi^{-1}(a) = \coprod_{(f,f')} \theta^{-1}\{(f,f')\},$$

where the disjoint union is taken over all $(f,f') \in \mathbb{N}(h) \times \mathbb{N}(h')$ satisfying (7); hence by Lemma 3.9 we see that $|\pi^{-1}(a)|$ depends only on the pair (e,e') and not on the choice of $a \in \rho^{-1}\{(e,e')\}$. Consequently,

$$|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)| = \sum_{(e,e')} |\rho^{-1}\{(e,e')\}| \sum_{(f,f')} |\theta^{-1}\{(f,f')\}|,$$

where (e,e') and (f,f') range over all elements of $\mathbb{N}(h) \times \mathbb{N}(h')$ satisfying (6) and (7) respectively. Now by applying Lemmas 3.8 and 3.9 we find that $|\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, l)|$ is given by

$$\sum_{(e,e')} \sum_{\alpha=(\alpha_{kk'})} \prod_{\substack{1\leqslant k\leqslant h \\ 1\leqslant k'\leqslant h'}} \binom{\mu_{\mathscr{L}}(k,k')}{\alpha_{kk'}} \sum_{(f,f')} \sum_{\beta=(\beta_{kk'})} \prod_{\substack{1\leqslant k\leqslant h \\ 1\leqslant k'\leqslant h'}} \binom{v_{\mathscr{L}}(k,k')}{\beta_{kk'}},$$

where $\alpha = (\alpha_{kk'})$ and $\beta = (\beta_{kk'})$ range over all elements of $M_{h\,h'}(\mathbb{N}, l)$ satisfying (8) and (9), respectively.

Fix $(e,e') \in \mathbb{N}(h) \times \mathbb{N}(h')$ satisfying (6) and $\alpha = (\alpha_{kk'}) \in M_{h\,h'}(\mathbb{N}, l)$ satisfying (8). Then the union

$$\bigcup_{(f,f')} \left\{ \beta = (\beta_{kk'}) \in M_{h\,h'}(\mathbb{N}, l) \colon \sum_{k=1}^{h} \beta_{kk'} = f'(k'+1) - f'(k')\ \forall k' \in [1,h'] \text{ and } \sum_{k'=1}^{h'} \beta_{kk'} = f(k+1) - f(k)\ \forall k \in [1,h] \right\}$$

taken over all $(f,f')\in\mathbb{N}(h)\times\mathbb{N}(h')$ satisfying (7), can be easily seen to be disjoint and to equal the set

$$\{\beta=(\beta_{kk'})\in M_{h\,h'}(\mathbb{N},l)\colon\ \sigma_p(\beta)\leqslant e(p)=\sigma_p(\alpha)\ \forall p\in[1,h] \text{ and}$$
$$\tau_q(\beta)\leqslant l-e(q)=\tau_q(\alpha)\ \forall q\in[1,h']\},$$

where it may be recalled that for $\beta=(\beta_{kk'})\in M_{h\,h'}(\mathbb{N},l)$, $p\in[1,h]$ and $q\in[1,h']$ we have put

$$\sigma_p(\beta)=\sum_{k=1}^{p}\sum_{k'=1}^{h'}\beta_{kk'} \quad\text{and}\quad \tau_q(\beta)=\sum_{k'=q}^{h'}\sum_{k=1}^{h}\beta_{kk'}.$$

Finally, we note that the union

$$\bigcup_{(e,e')}\left\{\alpha=(\alpha_{kk'})\in M_{h\,h'}(\mathbb{N},l)\colon\ \sum_{k=1}^{h}\alpha_{kk'}=e'(k')-e'(k-1')\ \forall k'\in[1,h'] \text{ and } \sum_{k'=1}^{h'}\alpha_{kk'}=e(k)-e(k-1)\ \forall k\in[1,h]\right\}$$

taken over all $(e,e')\in\mathbb{N}(h)\times\mathbb{N}(h')$ satisfying (6), is also easily seen to be disjoint and to equal the set $M_{h\,h'}(N,l)$. It follows that the above formula for $|\mathrm{skel}(\mathscr{L}^{\circ},l)|$ can be written as

$$\sum_{\alpha\in M_{h\,h'}(\mathbb{N},l)}\ \sum_{\substack{\beta\in M_{h\,h'}(\mathbb{N},l)\\ \sigma(\beta)\leqslant\sigma(\alpha),\ \tau(\beta)\leqslant\tau(\alpha)}}\ \prod_{\substack{1\leqslant k\leqslant h\\ 1\leqslant k'\leqslant h'}}\binom{\mu_{\mathscr{L}}(k,k')}{\alpha_{kk'}}\binom{v_{\mathscr{L}}(k,k')}{\beta_{kk'}}.$$

This proves the theorem. $\square$

Remark 3.11. Since we have assumed in the beginning of this section that $h'\neq 0$, the above result seems to apply only to 'proper' biladders $\mathscr{L}=L(S)\backslash L(S')$ with $L(S')\neq\emptyset$. To cover the important special case when $\mathscr{L}=L=L(S)$ is a ladder (so that $h'=0$), we note that Kulkarni [20] (see also, [21, Theorem 4]; [19, Proposition 4]) has already proved that $|\mathrm{skel}(L^{\circ},l)|=\mathscr{S}(L^{\circ},l)$, where

$$\mathscr{S}(L^{\circ},l)=\sum_{\mathbf{e}}\sum_{\mathbf{f}}\binom{\mu_L(k)}{e_k}\binom{v_L(k)}{f_k},$$

where

$$\mu_L(k)=|[S(1,k-1),S(1,k))|=S(1,k)-S(1,k-1)\quad\text{for } 1\leqslant k\leqslant h$$

and

$$v_L(k)=|[S(2,k),S(2,k-1))|=S(2,k-1)-S(2,k)\quad\text{for } 1\leqslant k\leqslant h$$

and where the first summation is over all $\mathbf{e}=(e_1,\dots,e_h)\in\mathbb{N}(h)$ such that

$$0\leqslant e_i\leqslant l\quad\text{for } 1\leqslant i\leqslant h \text{ and } e_h=l$$

while the second summation is over all $\mathbf{f}=(f_1,\dots,f_h)\in\mathbb{N}(h)$ such that

$$0\leqslant f_i\leqslant e_i \quad \text{for } 1\leqslant i\leqslant h \text{ and } f_h=l.$$

It may be noted that this formula follows from our Theorem 3.10 by an obvious extension of $L(S)$ to a larger rectangle $[1,m(1)+1]\times[1,m(2)+1]$ and taking $L(S')$ to be the top hook of this enlarged rectangle, i.e., letting S' be the unique ladder generating bisequence of length 2 such that $(S'(1,1),S'(2,1))=(1,1)$. At any rate, we have a formula $|\mathrm{skel}(\mathscr{L}^{\mathrm{o}},l)|$ also in the case $h'=0$ and in this way $\mathscr{S}(\mathscr{L}^{\mathrm{o}},l)$ is defined and Theorem 3.10 holds for any biladder $\mathscr{L}$.

4. Enumeration of indexed monomials

Let $S',S\in\mathrm{lad}(m)$ be such that $\mathrm{len}(S)\neq 0$ and $S'\leqslant S$. Let $\mathscr{L}$ and $\mathscr{L}^{\mathrm{o}}$ denote the biladder $\mathscr{L}(S',S)$ and its interior $\mathscr{L}(S',S)^0$, respectively.

In this section, we shall use the results obtained so far to find a formula for the Hilbert function of $K[\mathscr{L}]/I_{p+1}(\mathscr{L})$ for the given biladder $\mathscr{L}$ and any $p\in\mathbb{N}^+$. As remarked in Section 0, by Theorem 1.4, it suffices to find a formula for the cardinality of the set $\mathrm{mon}(\mathscr{L},V,p)$ of monomials in $\mathscr{L}$ of degree V and index $\leqslant p$. To this end, we first notice that the latter problem can be easily reduced to the problem of finding a formula for the number of the so-called p-fold radicals in $\mathscr{L}$ of a given length. Then comes one of the main steps, namely that a p-fold radical is split into a radical and a $(p-1)$-fold radical in a smaller biladder. Such splitting is unique in some sense and it allows us to obtain the desired formula recursively. It may be remarked that the above splitting can be viewed as a refinement of the superskeleton decomposition used in [3].

Lemma 4.1. *Given any* $V\in\mathbb{N}$ *and* $p\in\mathbb{N}^+$, *we have*

$$|\mathrm{mon}(\mathscr{L},V,p)|=\sum_{d=0}^{V}\binom{V-1}{V-d}|\mathrm{rad}^p(\mathscr{L},d)|.$$

Proof. It is clear that if $\theta\in\mathrm{mon}(\mathscr{L},p,V)$, then $|\mathrm{supp}(\theta)|\leqslant V$. Now consider the map

$$\Phi:\mathrm{mon}(\mathscr{L},p,V)\to\coprod_{d=0}^{V}\mathrm{rad}^p(\mathscr{L},d)$$

defined by $\theta\overset{\Phi}{\mapsto}\mathrm{supp}(\theta)$. Given any $d\in[0,V]$ and $R\in\mathrm{rad}^p(\mathscr{L},d)$, by Lemma 1.2 we have

$$|\Phi^{-1}(R)|=\begin{bmatrix}d-1\\V-d\end{bmatrix}=\binom{V-1}{V-d}.$$

Since the RHS above is independent of the choice of R within $\mathrm{rad}^p(\mathscr{L},d)$, the desired equality is immediate. $\square$

As a corollary of this lemma and the theorem proved in the previous section we can obtain a formula for the number of monomials of degree V and index $\leqslant 1$ as follows. It may be noted that this generalizes the result of Kulkarni [20] which gives such a formula in the case of a ladder.

Theorem 4.2. *Let $M = m(1) + m(2) - 1$ and $\delta_0 = |\Delta(S', S)|$. Then for any $d \in \mathbb{N}$ and any $V \in \mathbb{N}$, we have*

$$|\mathrm{mon}(\mathscr{L}, V, 1)| = \sum_{\ell=0}^{V} \begin{bmatrix} V-\ell \\ M-\delta_0-1 \end{bmatrix} \mathscr{S}(\mathscr{L}^{\mathrm{o}}, \ell).$$

Consequently, the Hilbert function as well as the Hilbert polynomial of $K[\mathscr{L}]/I_2(\mathscr{L})$ is given by the formula above. Moreover, if $\mathscr{L}$ is nonempty, then this formula is a polynomial in the parameter V with coefficients in $\mathbb{Q}$, of degree $M - \delta_0 - 1$ and leading coefficient $(1/(M-\delta_0-1)!)\sum_{\ell\in\mathbb{N}} \mathscr{S}(\mathscr{L}^{\mathrm{o}}, \ell)$.

Proof. Note that if $\mathscr{L} = \emptyset$, then $\delta_0 = M$, and thus the desired result clearly holds. Now assume that $\mathscr{L} \neq \emptyset$. By Lemma 4.1 and Theorem 2.13 we see that

$$|\mathrm{mon}(\mathscr{L}, V, 1)| = \sum_{d=0}^{V} \binom{V-1}{V-d} \sum_{\ell=0}^{d} \binom{M-\delta_0-\ell}{d-\ell} |\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, \ell)|.$$

Interchanging the summations, we can write the RHS as

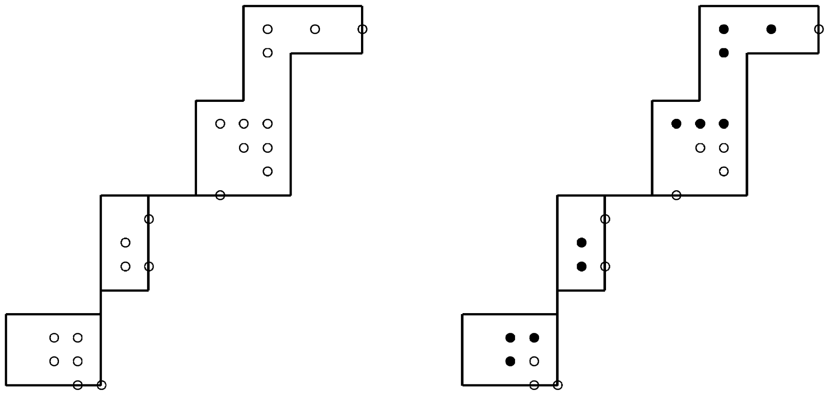
$$\begin{aligned} &\sum_{\ell=0}^{V} \left\{ \sum_{d=\ell}^{V} \binom{V-1}{V-d} \binom{M-\delta_0-\ell}{d-\ell} \right\} |\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, \ell)| \\ &\quad = \sum_{\ell\in\mathbb{N}} \begin{bmatrix} M-\delta_0-1 \\ V-\ell \end{bmatrix} |\mathrm{skel}(\mathscr{L}^{\mathrm{o}}, \ell)|, \end{aligned}$$

where the last step follows from (iv) of Lemma 1.1. Finally, we note that since $\mathscr{L} \neq \emptyset$, we have $M - \delta_0 - 1 \geqslant 1$, and hence in view of (v) and (vi) of Lemma 1.11 as well as Theorem 3.10 and Remark 3.11, we get the formula for $|\mathrm{mon}(\mathscr{L}, V, 1)|$ as asserted. From Theorem 1.4 it follows that this formula gives the Hilbert function of $K[\mathscr{L}]/I_2(\mathscr{L})$. It is also the Hilbert polynomial since the binomial coefficients appearing in this formula are evidently polynomials in V. It is clear that the degree in V of this polynomial is $M - \delta_0 - 1$ and the coefficient of $V^{M-\delta_0-1}$ is equal to $(1/(M-\delta_0-1)!)\sum_{\ell\in\mathbb{N}} \mathscr{S}(\mathscr{L}^{\mathrm{o}}, \ell)$, which is nonzero since $\mathscr{L}$ is nonempty. $\square$

We shall now attempt to generalize the above result by first finding $|\mathrm{rad}^p(\mathscr{L}, d)|$ for any nonnegative integer d. The splitting discussed in the beginning of this section is achieved by means of the map Γ obtained from the following definition.

Definition 4.3. Given any $p \in \mathbb{N}^+$ and $R \in \mathrm{rad}^p(\mathscr{L})$, we define

$$\gamma(R) = \mu(\lambda(R)) \cap R \quad \text{and} \quad \Gamma(R) = (\gamma(R), R \setminus \gamma(R)).$$



(a) (b)

Fig. 5.

The construction of $\gamma(R)$ (and hence of $\Gamma(R)$) can be illustrated by Figs. 5(a) and (b). Here, we consider a 3-fold radical R, which is indicated by the hollow circles in Fig. 5(a), and the corresponding radical $\gamma(R)$ is given by the bullets or thick circles in Fig. 5(b).

Lemma 4.4. *For any* $p \in \mathbb{N}^+$ *and* $R \in \mathrm{rad}^p(\mathscr{L})$, *we have*

$$\mathrm{ind}(\gamma(R)) \leqslant 1 \quad \textit{and} \quad \mathrm{ind}(R \backslash \gamma(R)) \leqslant p-1.$$

Proof. By Lemma 2.9, we see that $\mathrm{ind}(\gamma(R)) \leqslant 1$ since $\gamma(R) \subseteq \mu(\lambda(R))$. Now if $R = \emptyset$, then both the assertions are trivially true and thus we assume that $R \neq \emptyset$. Hence $\lambda(R) \neq \emptyset$, and we can find $l \in \mathbb{N}^+$, and integers $i_1 < \cdots < i_l$ and $j_1 < \cdots < j_l$ such that $\lambda(R) = \{(i_1, j_1), \ldots, (i_l, j_l)\}$. Recall that we take $i_0 = 0$, $i_{l+1} = m(1)$, $j_0 = m(2)$, $j_{l+1} = 1$, and for $s \in [1, l]$,

$$i_s = \min\{i \in (i_{s-1}, m(1)]: (i,j) \in R \cap \mathscr{L}^{\mathrm{o}} \quad \text{for some } j \in [1, j_{s-1})\} \tag{10}$$

and

$$j_s = \min\{j \in [1, j_{s-1}): (i_s, j) \in R \cap \mathscr{L}^{\mathrm{o}}\}. \tag{11}$$

If $p = 1$ then $R \in \mathrm{rad}(\mathscr{L})$, and hence by Lemma 2.10, $R \subseteq \mu(\lambda(R))$ so that $\gamma(R) = R$ and $\mathrm{ind}(R \backslash \gamma(R)) = 0$. Thus, we shall now assume that $p > 1$. Suppose, if possible, $\mathrm{ind}(R \backslash \gamma(R)) \geqslant p$. Then we can find $(a_1, b_1), (a_2, b_2), \ldots, (a_p, b_p)$ in $R \backslash \gamma(R)$ such that $a_1 < a_2 < \cdots < a_p$ and $b_1 < b_2 < \cdots < b_p$. Since $p > 1$, it follows from observation (1.5) that $(a_1, b_1) \in R \cap \mathscr{L}^{\mathrm{o}}$, and so by (10) we see that $i_1 \leqslant a_1$. Further, if $i_1 = a_1$, then by (11), we have $j_1 \leqslant b_1$, and thus $(a_1, b_1) \in \{i_1\} \times [j_1, j_0) \subseteq \mu(\lambda(R))$, which is a contradiction. Hence $i_1 < a_1$, and so there exists a unique $k \in [1, l]$ such that $i_k < a_1 \leqslant i_{k+1}$. Now if $b_1 < j_k$ then by (10), $a_1 = i_{k+1}$, and hence by (11), $b_1 \geqslant j_{k+1}$. Consequently, $k + 1 \leqslant l$ and $(a_1, b_1) \in \{i_{k+1}\} \times [j_{k+1}, j_k) \subseteq \mu(\lambda(R))$, which is a contradiction. Also if $b_1 = j_k$, then $(a_1, b_1) \in (i_k, i_{k+1}] \times \{j_k\} \subseteq \mu(\lambda(R))$, again yielding a contradiction. Thus, we must have $j_k < b_1$, which gives us the sequence $(i_k, j_k), (a_1, b_1), \ldots, (a_p, b_p)$ of elements of R such that $i_k < a_1 < \cdots < a_p$ and $j_k < b_1 < \cdots < b_p$, contrary to the assumption that $\mathrm{ind}(R) \leqslant p$. $\square$

Thus, Γ defines a map of $\mathrm{rad}^p(\mathscr{L})$ into $\mathrm{rad}(\mathscr{L}) \times \mathrm{rad}^{p-1}(\mathscr{L})$. This map is obviously injective. A characterization of the image of Γ is given by the lemma below.

Lemma 4.5. *Let $p \in \mathbb{N}^+$ and $R \in \mathrm{rad}^p(\mathscr{L})$. Then there exists a unique $S_R \in \mathrm{lad}(m)$ with*

$$\mathrm{len}(S_R) \neq 0, \quad S' \leqslant S_R \leqslant S \quad \textit{and} \quad \Delta(S', S_R) = \Delta(S', S), \tag{12}$$

such that

$$\gamma(R) \subseteq \partial S_R, \quad R \backslash \gamma(R) \subseteq \mathscr{L}(S_R, S) \quad \textit{and} \quad \widetilde{\lambda(R)} = \lambda(\widetilde{\gamma(R)}) = \mathscr{N}(S_R).$$

Conversely, given any $S^* \in \mathrm{lad}(m)$ with

$$\mathrm{len}(S^*) \neq 0, \quad S' \leqslant S^* \leqslant S \quad \textit{and} \quad \Delta(S', S^*) = \Delta(S', S) \tag{13}$$

and any $(U, U^*) \in \mathrm{rad}(\mathscr{L}) \times \mathrm{rad}^{p-1}(\mathscr{L})$ such that

$$U \subseteq \partial S^*, \quad U^* \subseteq \mathscr{L}(S^*, S) \quad \textit{and} \quad \widetilde{\lambda(U)} = \mathscr{N}(S^*) \tag{14}$$

there exists a unique $S_R \in \mathrm{lad}(m)$ such that $\Gamma(R) = (U, U^*)$ and $S_R = S^*$.

Proof. Let $R \in \mathrm{rad}^p(\mathscr{L})$. Let $t \in \mathbb{N}$ and $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_{t+1}, \beta_{t+1})$ be the unique elements such that $\widetilde{\lambda(R)} = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}$ with

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_t \leqslant \alpha_{t+1} = m(1)$$

and

$$m(2) = \beta_0 \geqslant \beta_1 > \beta_2 > \cdots > \beta_t \geqslant \beta_{t+1} = 1.$$

Define $S_R \in \mathrm{lad}[m, t+1]$ by $S_R(1,0) = 1$, $S_R(2,0) = m(2)$, $S_R(1,k) = \alpha_k$ and $S_R(2,k) = \beta_k$ for all $k \in [1, t+1]$. It is clear that $\mathrm{len}(S_R) \neq 0$ and S_R is uniquely determined by R. Further, by observation (1.2) and Lemma 2.9, we have

$$\partial S_R = \hat{\mu}(\lambda(R)) = \mu(\lambda(R)) \cup \Delta(S', S) \subseteq L(S).$$

Hence, in view of observations (1.4) and (1.8), it follows that $L(S_R) \subseteq L(S)$, i.e., $S_R \leqslant S$. To see that $S' \leqslant S_R$, let there be given any $(i,j) \in L(S')$. Then we can find a unique $r \in [1, t+1]$ such that $\alpha_{r-1} < i \leqslant \alpha_r$. Suppose if possible $j > \beta_{r-1}$. Then we must have $r > 1$, and by observation (1.5), we have $(\alpha_{r-1}, \beta_{r-1}) \in L(S')^{\mathrm{o}}$. Moreover, since $r-1 \in [1,t]$, it follows that either $(\alpha_{r-1}, \beta_{r-1}) \in \lambda(R) \subseteq \mathscr{L}$ or $(\alpha_{r-1}, \beta_{r-1}) \in \partial S$. In any case, this is a contradiction because in view of observations (1.4) and (1.8) we have $L(S')^{\mathrm{o}} \subseteq L(S)^{\mathrm{o}}$. This shows that $(i,j) \in (\alpha_{r-1}, \alpha_r] \times [1, \beta_{r-1}] \subseteq L(S_R)$. Thus, $S' \leqslant S_R$. Also, since $\mu(\lambda(R)) \subseteq \mathscr{L}$, we have

$$\Delta(S', S_R) = \partial S_R \cap L(S') = \Delta(S', S) \cap L(S') = \Delta(S', S).$$

This proves (11). Further, since $\partial S_R = \hat{\mu}(\lambda(R))$, we have

$$\gamma(R) = \mu(\lambda(R)) \cap R = \hat{\mu}(\lambda(R)) = \partial S_R.$$

We now proceed to show that $R \setminus \gamma(R) \subseteq \mathscr{L}(S_R, S) = L(S) \setminus L(S_R)$. To this end, let $(i,j) \in R \setminus \gamma(R)$. Since $R \subseteq \mathscr{L}$, we have $\gamma(R) = \hat{\mu}(\lambda(R)) \cap R$, and hence $(i,j) \notin \hat{\mu}(\lambda(R)) = \partial S_R$. So, it suffices to show that $(i,j) \notin L(S_R)^{\circ}$. Now write $\lambda(R) = \{(i_1, j_1), \ldots, (i_l, j_l)\}$ where $(i_1, j_1), \ldots, (i_l, j_l)$ are as in Definition 2.1. Now if $(i,j) \in L(S_R)^{\circ}$, then $(i,j) \in R \cap \mathscr{L}$, and therefore $i_1 \leqslant i < m(1)$. Hence we can find a unique $s \in [1, l]$ such that $i_s \leqslant i < i_{s+1}$, where by convention, $i_{l+1} = m(1)$. Now if $i = i_s$, then by Definition 2.1, we have $j \geqslant j_s$. Also, if $i > i_s$, and if we had $j < j_s$, then $s + 1 \leqslant l$ and $i_{s+1} \leqslant i$, which is a contradiction. Thus in any case $j \geqslant j_s$. Since $\lambda(R) \subseteq \hat{\lambda}(R) = \mathscr{N}(S_R)$, we can find $k^* \in [1, t]$ such that $(i_s, j_s) = (S_R(1, k^*), S_R(2, k^*))$. Moreover, since $(i,j) \in L(S_R)^{\circ}$, there exists a unique $k \in [1, t+1]$ such that

$$S_R(1, k-1) \leqslant i < S_R(1, k) \quad \text{and} \quad 1 \leqslant j < S_R(2, k-1).$$

In particular, $S_R(1, k^*) = i_s \leqslant i < S_R(1, k)$, and hence $k^* \leqslant k - 1$. Consequently, $j_s = S_R(2, k^*) \geqslant S_R(2, k-1) > j$, which is a contradiction. This proves that $R \setminus \gamma(R) \subseteq \mathscr{L}(S_R, S)$. The equality $\widetilde{\lambda(R)} = \mathscr{N}(S_R)$ is obvious whereas the equality $\widetilde{\lambda(R)} = \lambda(\widetilde{\gamma(R)})$ follows from Lemma 2.11 in view of the fact that $\lambda(R) \subseteq \gamma(R) \subseteq \mu(\lambda(R))$.

To prove the converse, let there be given any $S^* \in \operatorname{lad}(m)$ satisfying (13) and $(U, U^*) \in \operatorname{rad}(\mathscr{L}) \times \operatorname{rad}^{p-1}(\mathscr{L})$ satisfying (14). Put $R = U \cup U^*$. Since U and U^* are disjoint, it follows that $\operatorname{ind}(R) \leqslant p$. Thus $R \in \operatorname{rad}^p(\mathscr{L})$. All the remaining assertions will readily follow if we show that $U = \gamma(R)$. To this end, we first show that $\lambda(R) = \lambda(U)$. Write $\lambda(R) = \{(i_1, j_1), \ldots, (i_l, j_l)\}$ where $(i_1, j_1), \ldots, (i_l, j_l)$ and (i_0, j_0) are as in Definition 2.1. Suppose $\lambda(R) \nsubseteq \lambda(U)$. Then we can find s which is the least integer in $[1, l]$ with the property that $(i_s, j_s) \notin \lambda(U)$. Now $(i_s, j_s) \in L(S)^{\circ}$, and so in view of observation (1.9), we can find some $k \in [1, \operatorname{len}(S^*)]$ such that $S^*(1, k-1) \leqslant i_s < S^*(1, k)$. Further, since $(i_s, j_s) \in U \cup U^*$, it follows that $(i_s, j_s) \notin L(S^*)^{\circ}$, and hence $j_s \geqslant S^*(2, k-1)$. In particular, $k > 1$. Also, by observation (1.4), $(S^*(1, k-1), S^*(2, k-1))$ is in $L(S)^{\circ}$; in fact it is in $\mathscr{L}^{\circ}$ because if it were in $L(S')$, then it would be in $\Delta(S', S^*) = \Delta(S', S)$, and hence in ∂S, which is a contradiction. Further, if $i_{s-1} \geqslant S^*(1, k-1)$, then $s > 1$ and since $j_{s-1} > j_s \geqslant S^*(2, k-1)$, it follows that $(i_{s-1}, j_{s-1}) \notin L(S^*)$. This contradicts the minimality of s because $\lambda(U) \subseteq \widetilde{\lambda(U)} \subseteq L(S^*)$. Therefore, $i_{s-1} < S^*(1, k-1) \leqslant i_s$. It follows that

$$(S^*(1, k-1), S^*(2, k-1)) \in \widetilde{\lambda(U)} \cap \mathscr{L}^{\circ} \subseteq \lambda(U) \cap \mathscr{L}^{\circ} \subseteq R \cap \mathscr{L}^{\circ}.$$

Now, in view of Definition 2.1, we obtain that $S^*(1, k-1) = i_s$ and $S^*(2, k-1) = j_s$. Consequently, $(i_s, j_s) \in \widetilde{\lambda(U)} \cap \mathscr{L}^{\circ} \subseteq \lambda(U)$, which is contrary to our assumption. Thus, we have shown that $\lambda(R) \subseteq \lambda(U)$. To prove the reverse inclusion, we note that $\lambda(U) \subseteq \mathscr{N}(S^*)$, and let $k \in [1, \operatorname{len}(S^*) - 1]$ be such that $(S^*(1, k), S^*(2, k)) \in \lambda(U)$. Let $(i_0, j_0), (i_1, j_1), \ldots, (i_l, j_l)$ be as above so that $\lambda(R) = \{(i_1, j_1), \ldots, (i_l, j_l)\}$. Since $\lambda(U) \subseteq U \cap \mathscr{L}^{\circ} \subseteq R \cap \mathscr{L}^{\circ}$, we can find $s \in [1, l]$ such that $i_{s-1} < S^*(1, k) \leqslant i_s$. We claim that $S^*(2, k-1) < j_{s-1}$. This claim is obvious if $s = 1$. Moreover, if $s > 1$, then using the inclusion $\lambda(R) \subseteq \lambda(U)$, we get $(i_{s-1}, j_{s-1}) = (S^*(1, k'), S^*(2, k'))$ for some

$k' \in [1, \mathrm{len}(S^*) - 1]$. Now clearly, $k' < k$, and hence $S^*(2,k) < S^*(2,k') = j_{s-1}$. This proves our claim. Next, in view of Definition 2.1, we obtain that $i_s = S^*(1,k)$. Using the inclusion $\lambda(R) \subseteq \lambda(U)$ once again, we can deduce that $(S^*(1,k), S^*(2,k)) = (i_s, j_s)$. Thus $\lambda(R) = \lambda(U)$. Hence,

$$\widetilde{\lambda(R)} = \widetilde{\lambda(U)} = \mathcal{N}(S^*)$$

and, therefore, $\hat{\mu}(\lambda(R)) = \partial S^*$. Consequently,

$$U \subseteq \partial S^* \cap R \subseteq \hat{\mu}(\lambda(R)) \cap R = \mu(\lambda(R)) \cap R = \gamma(R).$$

Moreover, $\gamma(R) = \hat{\mu}(\lambda(R)) \cap R \subseteq \partial S^*$, and hence $\gamma(R) \cap U^* = \emptyset$, as desired. The remaining assertions are now evident. $\square$

Lemma 4.6. *Let $M = m(1) + m(2) - 1$ and $\delta_0 = |\Delta(S', S)|$. Then for any $d \in \mathbb{N}$ and $p \in \mathbb{N}^+$, we have*

$$|\mathrm{rad}^p(\mathcal{L}, d)| = \sum_{S^*} \sum_{d_1 + d_2 = d} \binom{M - \delta_0 - v^*}{d_1 - v^*} |\mathrm{rad}^{p-1}(\mathcal{L}^*, d_2)|,$$

where the first sum is over all $S^* \in \mathrm{lad}(m)$ satisfying (13), *and the second sum is over all $(d_1, d_2) \in \mathbb{N} \times \mathbb{N}$ such that $d_1 + d_2 = d$, and where $v^* = |\mathcal{N}(S^*) \setminus [\mathcal{N}(S) \cap \mathcal{N}(S^*)]|$ and $\mathcal{L}^*$ denotes the biladder $\mathcal{L}(S^*, S)$.*

Proof. In view of Lemma 4.5, we see that $\mathrm{rad}^p(\mathcal{L}, d)$ is in one-to-one correspondence with

$$\coprod_{S^*} \coprod_{d_1 + d_2 = d} \left\{ \begin{array}{c} (U, U^*)\colon\ U \subseteq \partial S^* \cap \mathrm{rad}(\mathcal{L}, d_1),\ \widetilde{\lambda(U)} = \mathcal{N}(S^*), \\ \text{and } U^* \in \mathrm{rad}^{p-1}(\mathcal{L}^*, d_2), \end{array} \right\},$$

where the first disjoint union is taken over all $S^* \in \mathrm{lad}(m)$ satisfying (13). Note that in each of the components in the decomposition above, U and U^* vary independently. Moreover, we claim that the condition on U is equivalent to asserting that

$$U \subseteq \partial S^* \cap \mathcal{L}, \quad |U| = d_1 \quad \text{and} \qquad U \supseteq \lambda(U) = \mathcal{N}(S^*) \setminus [\mathcal{N}(S) \cap \mathcal{N}(S^*)]. \tag{15}$$

To verify this claim, suppose $U \subseteq \partial S^* \cap \mathrm{rad}(\mathcal{L}, d_1)$ and $\widetilde{\lambda(U)} = \mathcal{N}(S^*)$. Then clearly $U \subseteq \partial S^* \cap \mathcal{L}$, and

$$\lambda(U) \subseteq \widetilde{\lambda(U)} \cap L(S)^{\mathrm{o}} = \mathcal{N}(S^*) \cap L(S)^{\mathrm{o}} \subseteq \mathcal{N}(S^*) \setminus [\mathcal{N}(S) \cap \mathcal{N}(S^*)].$$

Moreover, since $\mathcal{N}(S^*) \setminus \lambda(U) = \widetilde{\lambda(U)} \setminus \lambda(U) \subseteq \mathcal{N}(S)$, we have $\mathcal{N}(S^*) \setminus \mathcal{N}(S) \subseteq \lambda(U)$. This proves (15). Conversely, if (15) holds, then in view of observation (1.7), we see that $U \subseteq \partial S^* \cap \mathrm{rad}(\mathcal{L}, d_1)$. Further, if $(\alpha, \beta) \in \widetilde{\lambda(U)} \setminus \lambda(U)$, then $(\alpha, \beta) = (S(1,k), S(2,k))$ for some $k \in [1, \mathrm{len}(S) - 1]$. Now let $k^* \in [1, \mathrm{len}(S^*) - 1]$ be the unique integer such that $S^*(1,k^*) \leqslant \alpha < S^*(1, k^*+1)$. We must have $S^*(2,k^*) \leqslant S(2,k)$ because otherwise $S(2,k) \in [1, S^*(2,k^*))$ so that $(S(1,k), S(2,k)) \in L(S^*)^{\mathrm{o}} \subseteq L(S)^{\mathrm{o}}$, which is a contradiction. Thus, $(S^*(1,k^*), S^*(2,k^*))$ is in $[1,\alpha] \times [1,\beta]$, and since the intersection of the latter with $\lambda(U)$ is empty, it follows that

$$(S^*(1,k^*), S^*(2,k^*)) \notin \lambda(U) = \mathcal{N}(S^*) \setminus [\mathcal{N}(S) \cap \mathcal{N}(S^*)].$$

Thus $(S^*(1,k^*), S^*(2,k^*)) \in \mathcal{N}(S)$, and this forces that $(S^*(1,k^*), S^*(2,k^*)) = (S(1,k), S(2,k))$. Hence $(\alpha,\beta) \in \mathcal{N}(S^*)$. Thus, $\widetilde{\lambda(U)} \subseteq \mathcal{N}(S^*)$. Further, if $(\alpha,\beta) \in \mathcal{N}(S) \cap \mathcal{N}(S^*)$, then $([1,\alpha] \times [1,\beta]) \cap \mathcal{N}(S^*) = \{(\alpha,\beta)\}$, and therefore $(\alpha,\beta) \in \widetilde{\lambda(U)}$. This proves our claim. Lastly, observe that if U satisfies (15), then $\partial S^* \cap \mathcal{L} = \mu(\lambda(U))$, and thus in view of Lemma 2.9, we have $|\partial S^* \cap \mathcal{L}| = M - \delta_0$. Now applying Lemma 1.3, we see that

$$|\{U\colon |U| = d_1 \text{ and } \mathcal{N}(S^*)\backslash[\mathcal{N}(S) \cap \mathcal{N}(S^*)] \subseteq U \subseteq \partial S^* \cap \mathcal{L}\}|$$
$$= \binom{M - \delta_0 - v^*}{d_1 - v^*}.$$

This yields the desired formula. $\square$

We are now ready to state and prove the main results of this paper. First, we need some notation, which will be useful in the sequel.

Notation. Given any $p \in \mathbb{N}^+$, we let $\mathcal{D}_p(\mathcal{L})$ denote the set of all $(p-1)$-tuples $\mathbf{S} = (S_1,\ldots,S_{p-1})$ of elements of $\mathrm{lad}(m)$ such that

$$\mathrm{len}(S_i) \neq 0, \quad \varDelta(S_{i-1},S_i) = \varDelta(S_{i-1},S) \qquad \text{for } 1 \leqslant i \leqslant p-1$$

and

$$S' \leqslant S_1 \leqslant \cdots \leqslant S_{p-1} \leqslant S.$$

Given any $\mathbf{S} = (S_1,\ldots,S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$, we let $S_0 = S'$ and

$$\delta(\mathbf{S}) = \sum_{i=0}^{p-1} \delta_i \quad \text{where } \delta_i = |\partial S_{i-1} \cap \partial S| \text{ for } 0 \leqslant i \leqslant p-1;$$

further, we let $\mathcal{L}_{p-1} = \mathcal{L}(S_{p-1},S)$, and

$$v(\mathbf{S}) = \sum_{i=1}^{p-1} v_i \quad \text{where } v_i = |\mathcal{N}(S_i)\backslash(\mathcal{N}(S_i) \cap \mathcal{N}(S))| \text{ for } 1 \leqslant i \leqslant p-1.$$

Given $\mathbf{S} = (S_1,\ldots,S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$ and $u \in \mathbb{N}$, we let

$$F_u(\mathbf{S}) = \sum_{\ell \geqslant 0} \binom{v(\mathbf{S}) + \ell}{u} \mathcal{S}(\mathcal{L}^{\circ}_{p-1}, \ell).$$

Note that since $\mathcal{S}(\mathcal{L}^{\circ}_{p-1}, \ell)$ vanishes for all large enough ℓ, the summation above is essentially finite. Finally, we let, as before, $M = m(1) + m(2) - 1$.

Observe that $\mathcal{D}_p(\mathcal{L})$ is nonempty since it always contains the $(p-1)$-tuple $(S,\ldots,S)$. If $p = 1$, $\mathcal{D}_p(\mathcal{L})$ is the nonempty set containing the empty tuple $\mathbf{S}$ (say), and in this case $\delta(\mathbf{S}) = \delta_0$, $v(\mathbf{S}) = 0$ and $\mathcal{L}_{p-1} = \mathcal{L}$.

Lemma 4.7. *If $\mathcal{L}$ is nonempty, then for any $p \in \mathbb{N}^+$ and $\mathbf{S} = (S_1,\ldots,S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$, we have*

$$pM - \delta(\mathbf{S}) - v(\mathbf{S}) - 1 \geqslant 0.$$

Proof. Let $i \in [1, p-1]$. Then $S_i \leqslant S$. We claim that

$$\mathcal{N}(S_i) \cap \partial S \subseteq \mathcal{N}(S).$$

To see this, suppose a node $P = (S_i(1,l), S_i(2,l))$ of S_i is in ∂S. Then by observation (1.2), we have either $P = (\alpha, S(2,k-1))$ with $S(1,k-1) < \alpha \leqslant S(1,k)$ or $P = (S(1,k), \beta)$ with $S(2,k) \leqslant \beta < S(2,k-1)$, for some $k \in [1,h]$. Now $S(2,l-1) > S(2,l)$ and thus in the first case, we have

$$(\alpha, S(2,k-1)+1) = (S_i(1,l), S_i(2,l)+1) \in L(S_i) \subseteq L(S)$$

which is a contradiction since $\alpha > S(1,k-1)$. Similarly, in the second case

$$(S(1,k)+1, \beta) = (S_i(1,l)+1, S_i(2,l)) \in L(S_i) \subseteq L(S)$$

which is a contradiction unless $\beta = S(2,k)$. Thus, $P = (S(1,k), S(2,k))$. Moreover, $k \neq h$ since $P \in \mathcal{N}(S_i)$. This proves the claim. As a consequence, $\partial S \cap \partial S_i$ and $\mathcal{N}(S_i) \setminus (\mathcal{N}(S_i) \cap \mathcal{N}(S))$ are disjoint and their union is contained in ∂S_i. Therefore,

$$\delta_i + v_i \leqslant |\partial S_i| = M \quad \text{for } 1 \leqslant i \leqslant p-1.$$

Further, since $\mathcal{L} \neq \emptyset$, we have $M - \delta_0 - 1 \geqslant 0$, and hence

$$pM - \delta(\mathbf{S}) - v(\mathbf{S}) - 1 = (M - \delta_0 - 1) + \sum_{i=1}^{p-1} (M - (\delta_i + v_i)) \geqslant 0. \qquad \square$$

Theorem 4.8. *Given any $p \in \mathbb{N}^+$ and $d \in \mathbb{N}$, we have*

$$|\mathrm{rad}^p(\mathcal{L}, d)| = \sum_{\mathbf{S} \in \mathcal{D}_p(\mathcal{L})} \sum_{\ell \geqslant 0} \binom{pM - \delta(\mathbf{S}) - v(\mathbf{S}) - \ell}{d - v(\mathbf{S}) - \ell} \mathcal{S}(\mathcal{L}^0_{p-1}, \ell),$$

where the summation on the right is essentially finite.

Proof. If $p = 1$, then the result follows from Theorem 2.13. If $p > 1$, we see from Lemma 4.6 using induction on p that $|\mathrm{rad}^p(\mathcal{L}, d)|$ is equal to

$$\sum_{\mathbf{S} \in \mathcal{D}_p(\mathcal{L})} \sum_{d_1 + \cdots + d_p = d} \prod_{i=1}^{p-1} \binom{M - \delta_i - v_i}{d_i - v_i} |\mathrm{rad}(\mathcal{L}_{p-1}, d_p)|.$$

Now the desired result follows from Theorem 2.13 using (iv) and (vii) of Lemma 1.1. The essential finiteness is evident since $\mathcal{S}(\mathcal{L}^{\circ}_{p-1}, \ell)$ vanishes for all large enough ℓ. $\square$

Theorem 4.9. *Given any $p \in \mathbb{N}^+$ and $V \in \mathbb{N}$, we have*

$$|\mathrm{mon}(\mathcal{L}, V, p)| = F(V),$$

where

$$F(V) = \sum_{u \geqslant 0} \sum_{\mathbf{S} \in \mathcal{D}_p(\mathcal{L})} (-1)^u F_u(\mathbf{S}) \binom{V + pM - \delta(\mathbf{S}) - 1 - u}{pM - \delta(\mathbf{S}) - 1 - u}.$$

Consequently, the Hilbert function as well as the Hilbert polynomial of the ladder determinantal ring $K[\mathscr{L}]/I_{p+1}(\mathscr{L})$ *is given by* $F(V)$.

Proof. Consider first the trivial case when $\mathscr{L}=\emptyset$; here we have $|\mathrm{mon}(\mathscr{L},V,p)|=0$, for any $V\in\mathbb{N}$. Now $\mathscr{L}=\emptyset$ corresponds to taking $S'=S$. Hence, in this case

$$\mathbf{S}=(S_1,\dots,S_{p-1})\in\mathscr{D}_p(\mathscr{L}) \;\Leftrightarrow\; S_i=S \;\forall i \Rightarrow v(\mathbf{S})=0 \quad\text{and}\quad \delta(\mathbf{S})=pM.$$

Further, $\mathscr{L}_{p-1}=\emptyset$, and so $\mathscr{S}(\mathscr{L}^0_{p-1},\ell)=1$ if $\ell=0$, and 0 otherwise. Hence,

$$F_u(\mathbf{S})=\sum_{\ell\geqslant 0}\binom{l}{u}\mathscr{S}(\mathscr{L}^0_{p-1},\ell)=\begin{cases}1 & \text{if } u=0,\\ 0 & \text{if } u>0.\end{cases}$$

Consequently, for any $V\in\mathbb{N}$, we have

$$F(V)=\binom{V+pM-\delta(\mathbf{S})-1}{pM-\delta(\mathbf{S})-1}=\binom{V-1}{-1}=0.$$

Thus, we shall now assume that $\mathscr{L}$ is nonempty. Fix some $V\in\mathbb{N}$. By Lemma 4.1 and Theorem 4.8, we can write $|\mathrm{mon}(\mathscr{L},V,p)|$ as

$$\sum_{d=0}^{V}\sum_{\mathbf{S}\in\mathscr{D}_p(\mathscr{L})}\sum_{\ell\geqslant 0}\binom{V-1}{V-d}\binom{pM-\delta(\mathbf{S})-v(\mathbf{S})-\ell}{d-v(\mathbf{S})-\ell}\mathscr{S}(\mathscr{L}^0_{p-1},\ell).$$

Interchanging summations and using (iv) of Lemma 1.1 together with Lemma 4.7 and (vi) of Lemma 1.1, we can write the above expression as

$$\sum_{\mathbf{S}\in\mathscr{D}_p(\mathscr{L})}\sum_{\ell\geqslant 0}\left[\begin{matrix}V-v(\mathbf{S})-\ell\\ pM-\delta(\mathbf{S})-1\end{matrix}\right]\mathscr{S}(\mathscr{L}^0_{p-1},\ell). \tag{16}$$

Now, in view of (iv) and (vii) of Lemma 1.1, we have

$$\left[\begin{matrix}V-v(\mathbf{S})-\ell\\ pM-\delta(\mathbf{S})-1\end{matrix}\right]=\sum_{u\geqslant 0}(-1)^u\left[\begin{matrix}V\\ pM-\delta(\mathbf{S})-1-u\end{matrix}\right]\binom{v(\mathbf{S})+\ell}{u}.$$

Using this, we obtain the desired expression for $|\mathrm{mon}(\mathscr{L},V,p)|$ by interchanging summations once again. The remaining assertions follow from Theorem 1.4 and the fact that the binomial coefficients appearing in $F(V)$ are polynomials in V while the coefficients $(-1)^uF_u(\mathbf{S})$ are independent of V. □

Corollary 4.10. *For any* $p\in\mathbb{N}^+$, $I_{p+1}(\mathscr{L})$ *is a Hilbertian ideal.*

Proof. Follows from Theorem 4.9. □

Remark 4.11. 1. Lemmas 4.5 and 4.6 may motivate the use of biladders although one may only be interested in (one-sided) ladders. Indeed, even if $\mathscr{L}$ were a ladder to begin with, the $\mathscr{L}^*$ that one obtains in Lemma 4.6 is necessarily a biladder. Thus it makes sense to have the results of Sections 2 and 3 in the general case of biladders.

2. The formulae in Theorems 4.8 and 4.9 are no doubt complicated and perhaps they may seem unworthy of being called ‘explicit’, in view of the rather unwieldy summation over $\mathcal{D}_p(\mathcal{L})$. Indeed, from this viewpoint, only the formula in the case of $p=1$ is truly explicit. Nevertheless, these formulae are much more explicit than any of the known formulae (e.g., those given by [3, Section 4; 17, Corollary 4.3]) in the general case of biladder, and to illustrate this we will show that our formulae can be used to deduce some interesting information about the variety associated to $I_{p+1}(\mathcal{L})$. For example, it is shown in the next section that one can derive fairly simple estimates for the degree of the Hilbert polynomial. It may also be observed that, as Krattenthaler and Prohaska [19, Section 7] seem to suggest, it appears unlikely that an elegant and simple formula for the Hilbert function of $I_{p+1}(\mathcal{L})$ can be found.

5. Applications

It is well known that if $\mathcal{H}(V)$ is the Hilbert polynomial of a projective variety, and if

$$\mathcal{H}(V)=\frac{e}{d!}V^d+c_1V^{d-1}+\cdots+c_d \quad \text{with } e,c_1,\ldots,c_d\in\mathbb{Z} \text{ and } e\neq 0$$

then the degree d of $\mathcal{H}(V)$ is the dimension of that projective variety and the ‘normalized leading coefficient’ e is its order or the multiplicity. Note that the dimension d can also be read off from the Hilbert series; indeed, we have

$$\sum_{V=0}^{\infty}\mathcal{H}(V)t^V=\frac{\mathcal{P}(t)}{(1-t)^{d+1}} \quad \text{where } \mathcal{P}(t)\in\mathbb{Z}[t] \text{ with } \mathcal{P}(1)\neq 0.$$

Moreover, $d+1$ is the (Krull) dimension of the corresponding homogeneous co-ordinate ring. The polynomial $\mathcal{P}(t)$ is sometimes called the h-polynomial (and its coefficient vector is referred to as an h-vector) of the vanishing ideal of that variety.

With this in view, we now attempt to extract some information about the degree as well as the leading coefficient of the Hilbert polynomial $F(V)$ of the ladder determinantal variety $\mathcal{V}_{p+1}(\mathcal{L})$, and the corresponding Hilbert series. First, we need some notation.

Fix some $p\in\mathbb{N}^+$ and a biladder $\mathcal{L}=\mathcal{L}(S',S)$. Let M, $\mathcal{D}_p(\mathcal{L})$, $\delta(\mathbf{S})$, $\mathcal{L}_{p-1}$, $F_u(\mathbf{S})$ and $F(V)$ be as defined in Section 4. Further, we let

$$\delta^*(\mathcal{L})=\min\{\delta(\mathbf{S})\colon \mathbf{S}\in\mathcal{D}_p(\mathcal{L})\}$$

and

$$\mathcal{D}_p^*(\mathcal{L})=\{\mathbf{S}\in\mathcal{D}_p(\mathcal{L})\colon \delta(\mathbf{S})=\delta^*(\mathcal{L})\}.$$

Proposition 5.1. *The degree of the Hilbert polynomial $F(V)$ of $\mathcal{V}_{p+1}(\mathcal{L})$ equals*

$$pM-\delta^*(\mathcal{L})-1=p(m(1)+m(2)-1)-\delta^*(\mathcal{L})-1$$

and the normalized leading coefficient equals

$$\sum_{\mathbf{S}\in\mathscr{D}_p^*(\mathscr{L})} |\mathrm{skel}(\mathscr{L}^{\mathrm{o}}_{p-1})| = \sum_{\mathbf{S}\in\mathscr{D}_p^*(\mathscr{L})} \sum_{\ell\geqslant 0} \mathscr{S}(\mathscr{L}^{\mathrm{o}}_{p-1}, \ell).$$

Proof. From Theorem 4.9, we see that $F(V)$ is a sum of terms of the form

$$(-1)^u F_u(\mathbf{S}) \binom{V + pM - \delta(\mathbf{S}) - 1 - u}{pM - \delta(\mathbf{S}) - 1 - u},$$

where the coefficients $(-1)^u F_u(\mathbf{S})$ are independent of V. Clearly, the binomial coefficient above is a polynomial in V of degree $pM - \delta(\mathbf{S}) - 1 - u$, and this degree is maximum when $u = 0$ and $\delta(\mathbf{S}) = \delta^*(\mathscr{L})$. The corresponding leading coefficient

$$\frac{1}{(pM - \delta^*(\mathscr{L}) - 1)!} \sum_{\mathbf{S}\in\mathscr{D}_p^*(\mathscr{L})} F_0(\mathbf{S}) = \frac{1}{(pM - \delta^*(\mathscr{L}) - 1)!} \sum_{\mathbf{S}\in\mathscr{D}_p^*(\mathscr{L})} \sum_{\ell\geqslant 0} \mathscr{S}(\mathscr{L}^{\mathrm{o}}_{p-1}, \ell)$$

is clearly positive since $\mathscr{D}_p(\mathscr{L})$ is nonempty. $\square$

Theorem 5.2. *Let t be an indeterminate over $\mathbb{Q}$ and let*

$$\mathscr{P}_{\mathscr{L}}(t) = \sum_{\mathbf{S}\in\mathscr{D}_p^*(\mathscr{L})} \hbar_{\mathbf{S}}(t) t^{v(\mathbf{S})} (1-t)^{\delta(\mathbf{S}) - \delta^*(\mathscr{L})} \quad \text{where } \hbar_{\mathbf{S}}(t) = \sum_{\ell\geqslant 0} \mathscr{S}(\mathscr{L}^{\mathrm{o}}_{p-1}, \ell) t^{\ell}.$$

Then the Hilbert series of $I_{p+1}(\mathscr{L})$ is given by

$$\frac{\mathscr{P}_{\mathscr{L}}(t)}{(1-t)^{pM - \delta^*(\mathscr{L})}}.$$

In particular, $\mathscr{P}_{\mathscr{L}}(t)$ gives the h-polynomial of $I_{p+1}(\mathscr{L})$.

Proof. From expression (16) in the proof of Theorem 4.9, we see that the Hilbert series is given by

$$\sum_{V\geqslant 0} \sum_{\mathbf{S}\in\mathscr{D}_p(\mathscr{L})} \sum_{\ell\geqslant 0} \begin{bmatrix} V - v(\mathbf{S}) - \ell \\ pM - \delta(\mathbf{S}) - 1 \end{bmatrix} \mathscr{S}(\mathscr{L}^0_{p-1}, \ell) t^V. \tag{17}$$

Given any $\mathbf{S} \in \mathscr{D}_p(\mathscr{L})$ and $\ell \geqslant 0$, by Lemma 4.7 and (vi) and (vii) of Lemma 1.1, we have

$$\begin{bmatrix} V - v(\mathbf{S}) - \ell \\ pM - \delta(\mathbf{S}) - 1 \end{bmatrix} = (-1)^{V - v(\mathbf{S}) - \ell} \binom{\delta(\mathbf{S}) - pM}{V - v(\mathbf{S}) - \ell}.$$

Thus, by interchanging summations and using (v) of Lemma 1.1, we can write (17) as

$$\sum_{\mathbf{S}\in\mathscr{D}_p(\mathscr{L})} \sum_{\ell\geqslant 0} \mathscr{S}(\mathscr{L}^0_{p-1}, \ell)\, t^{v(\mathbf{S})+\ell} \sum_{V = v(\mathbf{S})+\ell}^{\infty} \binom{\delta(\mathbf{S}) - pM}{V - v(\mathbf{S}) - \ell} (-t)^{V - v(\mathbf{S}) - \ell}.$$

Therefore, by Binomial Theorem, we see that the Hilbert series is given by

$$\sum_{\mathbf{S}\in\mathscr{D}_p(\mathscr{L})} \frac{\hbar_{\mathbf{S}}(t)t^{\nu(\mathbf{S})}}{(1-t)^{pM-\delta(\mathbf{S})}}$$

and in view of Proposition 5.1, this implies the desired result. □

In general, it does not appear very easy to compute more explicitly the coefficients of the h-polynomial described above or for that matter, even the degree of the denominator. However, it is not difficult to get a simple upper bound for this degree. We shall make use of the following elementary observations concerning the boundary of a ladder.

Proposition 5.3. *If* $S^* \in \mathrm{lad}(m)$ *with* $\mathrm{len}(S^*) \neq 0$, *then*

$$|\partial S^*| = M = m(1) + m(2) - 1 \tag{18}$$

and moreover, we can write $\partial S^* = \{P_1, P_2, \ldots, P_M\}$, *where*

$$P_1 = (1, m(2)), \quad P_M = (m(1), 1), \quad P_j - P_{j-1} = (1,0) \text{ or } (0,1) \quad \text{for } 1 < j \leqslant M. \tag{19}$$

Proof. Follows from observation (1.2). □

Lemma 5.4. *Given any* $\mathbf{S} = (S_1, \ldots, S_{p-1}) \in \mathscr{D}_p(\mathscr{L})$, *we have*

$$\delta_i = |\partial S_i \cap \partial S| \geqslant \min\{\delta_0 + 2i, M\} \quad \text{for } 1 \leqslant i \leqslant p-1 \tag{20}$$

and consequently, if $t = \max\{i \in [1,p]\colon \delta_0 + 2(i-1) \leqslant M\}$, *then*

$$\delta(\mathbf{S}) = \sum_{i=0}^{p-1} \delta_i \geqslant t\delta_0 + t(t-1) + (p-t)M. \tag{21}$$

Further, if we assume that $p < (M - \delta_0 + 1)/2$, *then we have*

$$\begin{aligned} &\delta^*(\mathscr{L}) \geqslant p\delta_0 + p(p-1) \quad \text{and} \\ &\dim \mathscr{V}_{p+1}(\mathscr{L}) \leqslant p(m(1) + m(2) - p - \delta_0) - 1. \end{aligned} \tag{22}$$

Proof. We prove (20) by induction on i. The case of $i = 0$ being trivial, assume that $i \geqslant 1$ and that $\delta_{i-1} \geqslant \min\{\delta_0 + 2i - 2, M\}$. Now since $\partial S_i \cap \partial S_{i-1} = \partial S_{i-1} \cap \partial S$, we have $\partial S_i \cap \partial S \supseteq \partial S_{i-1} \cap \partial S$, and so $\delta_i \geqslant \delta_{i-1}$. Thus in case $\delta_{i-1} \geqslant M$, we have $\delta_i \geqslant M \geqslant \min\{\delta_0 + 2i, M\}$. Suppose $\delta_{i-1} \leqslant M - 1$. Then, by (18), $|\partial S_i \cap \partial S_{i-1}| \leqslant |\partial S_{i-1}| - 1$, and so we can write $\partial S_i = \{P_1, P_2, \ldots, P_M\}$, where P_j's are as in (19), and further, we can find $r \in [1, M]$ such that

$$P_r \notin \partial S_{i-1} \quad \text{but} \quad P_j \in \partial S_{i-1} \qquad \text{for } 1 \leqslant j < r. \tag{23}$$

Also, let us write

$$\partial S_{i-1} = \{Q_1, Q_2, \ldots, Q_M\} \quad \text{and} \quad \partial S = \{R_1, R_2, \ldots, R_M\},$$

where Q_j and R_j satisfy the conditions in (19). Using (19) and (23), we see that $P_j = Q_j$ for $1 \leqslant j < r$. Moreover, since $\partial S_i \cap \partial S_{i-1} = \partial S_{i-1} \cap \partial S$, we have $P_j = R_j$ for $1 \leqslant j < r$. Now $P_r \neq Q_r$ and $P_{r-1} = R_{r-1}$, and thus in view of (19), we have that $P_r \notin \partial S_{i-1}$ and further, $R_r = P_r$ or $R_r = Q_r$. But if $R_r = Q_r$, then $Q_r \in \partial S_{i-1} \cap \partial S \subseteq \partial S_i$, and this forces that $Q_r = P_r$, which is a contradiction. Thus, $R_r = P_r$ and so

$$\delta_i = |\partial S_i \cap \partial S| \geqslant |\partial S_{i-1} \cap \partial S| + |\{P_r\}| = \delta_{i-1} + 1.$$

In particular, if $\delta_{i-1} = M - 1$, then $\delta_i \geqslant M$, and hence (20) holds. Now suppose $\delta_{i-1} < M - 1$. Then we can also find some $s \in [1, M]$ such that $r < s < M$ and

$$P_s \notin \partial S_{i-1} \quad \text{but } P_j \in \partial S_{i-1} \quad \text{for } s < j \leqslant M.$$

Arguing as in the case of P_r, we obtain that $P_s = R_s$, and thus

$$\delta_i = |\partial S_i \cap \partial S| \geqslant |\partial S_{i-1} \cap \partial S| + |\{P_r, P_s\}| = \delta_{i-1} + 2.$$

Hence, using the induction hypothesis, we obtain $\delta_i \geqslant \delta_0 + 2i$. This proves (20).

Next, if t is as defined in the lemma, then by (20), we have

$$\delta(\mathbf{S}) = \sum_{i=0}^{p-1} \delta_i \geqslant \sum_{i=0}^{t-1} (\delta_0 + 2i) + \sum_{i=t}^{p-1} M = t\delta_0 + t(t-1) + (p-t)M.$$

Thus (21) is proved. To prove (22) consider the quadratic function

$$q(t) = t\delta_0 + t(t-1) + (p-t)M.$$

Its derivative with respect to t equals $2(t - t_0)$, where $t_0 = (M - \delta_0 + 1)/2$. Hence $q(t)$ is strictly decreasing for $t < t_0$, and thus if $p < t_0$, then we have $q(t) \geqslant q(p)$ for all $t \in [1, p]$. This yields (22). $\square$

Remark 5.5. It may be noted that the condition $p < (M - \delta_0 + 1)/2$ in Lemma 5.4 is not restrictive. Indeed, the intersection $\partial S' \cap \partial S$ of the boundaries of $L(S)$ and $L(S')$ can be split into 'horizontal overlaps' and 'vertical overlaps' (see Fig. 2(b)); if δ_h is the cardinality of the former and δ_v is the cardinality of the latter, then we have $\delta_0 = |\partial S' \cap \partial S| \leqslant \delta_h + \delta_v$. Moreover, in order that a $(p+1) \times (p+1)$ minor of X has its entries in $\mathscr{L} = L(S) \backslash L(S')$, the columns should avoid the horizontal overlaps and the rows should avoid the vertical overlaps. Thus, $p + 1 \leqslant m(2) - \delta_h$ and $p + 1 \leqslant m(1) - \delta_v$. This implies that $2p < m(1) + m(2) - (\delta_h + \delta_v) \leqslant m(1) + m(2) - \delta_0$, and so $p < (M - \delta_0 + 1)/2$.

Corollary 5.6. *Let $L = L(S)$ be the ladder corresponding to S. Assume that $p + 1 \leqslant \min\{m(1), m(2)\}$. Then* $\dim \mathscr{V}_{p+1}(L) \leqslant p(m(1) + m(2) - p) - 1$.

Proof. Apply Lemma 5.4 with S' as the empty ladder generating bisequence, and note that in this case $\delta_0 = 0$ and $t = p$. $\square$

Remark 5.7. The upper bound for $\dim \mathscr{V}_{p+1}(L)$ in the corollary above is, in fact, the dimension of the classical determinantal variety corresponding to the ideal $I_{p+1}(X)$ generated by the $(p+1)\times(p+1)$ minors of X (see, e.g., [2, Theorem 20.15]). In this particular case of (one-sided) ladders, the same bound also follows from the expression for the Hilbert series in [19, Theorem 2]. However, this bound need not be attained, in general. For instance, in the example considered in [19, p. 1022], the (projective) dimension is seen to be 103 whereas the value predicted by the above bound as well as by [19, Theorem 2] is 124. Notice that this shows that the numerator of the Hilbert series as it appears in [19, Theorem 2] does not give the true h-polynomial of $I_{p+1}(L)$.

Finally, in this section, we illustrate how in some cases the actual value of $\dim \mathscr{V}_{p+1}(\mathscr{L})$ can be determined.

Example 5.8. (1) Suppose $\mathscr{L}$ is the full rectangle $[1,m(1)]\times[1,m(2)]$, which corresponds to the case when $L(S')$ is empty and S is the unique element of $\mathrm{lad}(m)$ of length 1; also suppose $p<\min\{m(1),m(2)\}$. More generally, let $\mathscr{L}$ be a ladder $L(S)$ (so that S' is the empty ladder generating bisequence) where $S\in\mathrm{lad}(m)$ is of length $h>0$ with $S(1,1)\geqslant p$ and $S(2,h-1)\geqslant p$. In this case, by (21), we have

$$\delta^*(\mathscr{L})=\min\{\delta(\mathbf{S})\colon \mathbf{S}\in\mathscr{D}_p(\mathscr{L})\}\geqslant p(p-1).$$

Moreover, if we take $\mathbf{S}=(S_1,\ldots,S_{p-1})$, where S_i's are the 'hook-like' ladder generating bisequences determined by the conditions

$$\mathrm{len}(S_i)=2,\quad (S_i(1,1),S_i(2,1))=(i,i)\qquad \text{for } 1\leqslant i\leqslant p-1$$

then it is easy to see that for $1\leqslant i\leqslant p-1$, we have

$$\partial S_i\cap\partial S_{i-1}=\{(1,m(2)),\ldots,(i-1,m(2)),(m(1),1),\ldots,(m(1),i-1)\}=\partial S_{i-1}\cap\partial S,$$

and therefore $\mathbf{S}\in\mathscr{D}_p(\mathscr{L})$ and $\delta_i=|\partial S_i\cap\partial S|=2i$ for $0\leqslant i\leqslant p-1$; hence $\delta(\mathbf{S})=p(p-1)$. It follows that $\delta^*(\mathscr{L})=p(p-1)$, and therefore,

$$\dim \mathscr{V}_{p+1}(\mathscr{L})=p(m(1)+m(2)-p)-1.$$

(2) Let $\mathscr{L}$ be a ladder with one corner missing, i.e, $L(S')$ is the empty ladder and $\mathscr{L}=L(S)$, where $S\in\mathrm{lad}[m]$ is of length 2. Assume that $p<\min\{m(1),m(2)\}$. Let $(S(1,1),S(2,1))=(a,b)$. Then we have

$$\delta^*(\mathscr{L})=\begin{cases} p(p-1) & \text{if } a\geqslant p \text{ and } b\geqslant p,\\ p(p-1)+(p-a)(m(2)-b) & \text{if } a<p \text{ and } b\geqslant p,\\ p(p-1)+(p-b)(m(1)-a) & \text{if } a\geqslant p \text{ and } b<p,\\ m(1)(p-b)+m(2)(p-a)+ab-p & \text{if } a<p \text{ and } b<p \end{cases}\qquad(24)$$

and consequently,

$$\dim \mathscr{V}_{p+1}(\mathscr{L}) = \begin{cases} p(m(1)+m(2)-p)-1 & \text{if } a \geqslant p \text{ and } b \geqslant p, \\ p(m(1)+b-p)+a(m(2)-b)-1 & \text{if } a < p \text{ and } b \geqslant p, \\ p(a+m(2)-p)+b(m(1)-a)-1 & \text{if } a \geqslant p \text{ and } b < p, \\ bm(1)+am(2)-ab-1 & \text{if } a < p \text{ and } b < p. \end{cases}$$

To see this, note that when $a \geqslant p$ and $b \geqslant p$, the result follows from the preceding example. In the remaining cases, we can argue as in the proof of Lemma 5.4 and the preceding example. Thus, for instance, if $a < p$ and $b \geqslant p$, then for any $\mathbf{S} \in \mathscr{D}_p(\mathscr{L})$, we must have

$$\delta_i \geqslant 2i \quad \text{for } 0 \leqslant i < a \qquad \delta_i \geqslant 2i + (m(2)-b) \quad \text{for } a \leqslant i \leqslant p-1.$$

Moreover, there exists a configuration $\mathbf{S} \in \mathscr{D}_p(\mathscr{L})$ for which the above inequalities are equalities. This proves (24). Alternatively, we can directly prove the formula for $\dim \mathscr{V}_{p+1}(\mathscr{L})$ in the last three cases from the following simple observations. If $a < p$ and $b \geqslant p$, then $\mathscr{V}_{p+1}(\mathscr{L})$ is a cylinder over the determinantal variety $\mathscr{V}_{p+1}(Y)$, where Y is the $m(1) \times b$ rectangular submatrix of X obtained by taking the first b columns of X. The case when $a \geqslant p$ and $b < p$ is similar. Lastly, when $a < p$ and $b < p$, we have $I_{p+1}(\mathscr{L}) = (0)$.

Remark 5.8. 1. The observation about the dimension of $\dim \mathscr{V}_{p+1}(L)$ in the first example above may perhaps explain why in Kulkarni's formula for the Hilbert function of $\mathscr{V}_2(L)$, the degree is the same as that in the case of $\mathscr{V}_2(X)$. Indeed, when $p=1$, the conditions $S(1,1) \geqslant p$ and $S(2,h-1) \geqslant p$ always hold.

2. In the case $\mathscr{L}$ is the full rectangle $[1,m(1)] \times [1,m(2)]$, Proposition 5.1 gives a curious formula for the multiplicity of the classical determinantal ideal $I_{p+1}(X)$. It may be interesting to compare this with the more elegant formulae described in [17, p. 17].

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