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*The American Mathematical Monthly*, Vol. 98, No. 3 (Mar., 1991), 235-237.

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Wed Jan 21 11:15:21 2004

# NOTES

EDITED BY DENNIS DETURCK, DAVID J. HALLENBECK, AND RODICA SIMON

## Algebraic Characterization of Compact Abelian Groups

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The well-known Banach-Stone theorem gives a nice algebraic characterization of compact Hausdorff topological spaces. One might wonder if the topological spaces are also endowed with a (compatible) group structure whether one can obtain a similar characterization for them to be the "same", i.e., to be topologically isomorphic. The purpose of this note is to give one such characterization which doesn't seem to have made its way into the literature yet. We begin by stating the result of Banach [**1**, p. 170] and Stone [**7**, p. 469] in a somewhat different form than the traditional version.

THEOREM 1. *Let $X$ and $Y$ be compact Hausdorff topological spaces. Then the following statements are equivalent.*

(1) *$X$ and $Y$ are homeomorphic.*
(2) *There exists an algebra isomorphism of $C(Y)$ onto $C(X)$.*

Here $C(X)$ (resp: $C(Y)$) denotes the space of complex valued continuous functions on $X$(resp: $Y$). These spaces clearly form an algebra (i.e., a vector space which is also a ring such that the scalar multiplication behaves nicely with the ring multiplication) over $\mathbb{C}$. In the traditional version the condition (2) requires a linear isometric isomorphism instead of an algebra isomorphism and proofs of this version could be found in [**2**, p. 442] or [**5**, p. 153]. Our proof of the result for topological groups would include a proof of Theorem 1 that is essentially as in [**4**] but does not use the language of Banach algebras and instead depends only on the following elementary fact, the proof of which is left as an exercise.

LEMMA A. *If $X$ is any compact topological space and $M$ is any maximal ideal of the ring $C(X)$, then there exists a unique $u \in X$ such that $M = \{f \in C(X): f(u) = 0\}$.*

We now recall the notion of a character on a topological group $G$ which by definition is a continuous homeomorphism from G into the "circle group" $\Gamma = \{z \in \mathbb{C}: |z| = 1\}$. The characters of $G$ form a subgroup of $C(G)$ which is usually denoted by $\hat{G}$. We will use the following basic

THEOREM 2. *Characters of a locally compact (Hausdorff) abelian group $G$ separate the points of $G$, i.e.,*

$$x \in G,\ y \in G,\ x \neq y \Rightarrow \text{there exists } \chi \in \hat{G} \text{ such that } \chi(x) \neq \chi(y).$$

For a proof of this theorem we refer to ([**4**], p. 120). Let us also remark that if $G$ is a compact group then $\hat{G}$ is discrete. Without further ado let us state and prove our main

THEOREM 3. *Let $G$ and $H$ be compact (Hausdorff) abelian groups. Then the following statements are equivalent.*

(1) *$G$ and $H$ are topologically isomorphic.*

(2) *There exists a character preserving algebra isomorphism of $C(H)$ onto $C(G)$ (i.e., $\hat{H}$ is mapped into $\hat{G}$).*

*Proof.* If $h: G \to H$ is a topological isomorphism (resp: homeomorphism) then it is easy to see that the map $T: C(H) \to C(G)$ defined by $T(g)(x) = g(h(x))$ for $g \in C(H)$, and $x \in G$ gives a character preserving algebra isomorphism (resp: algebra isomorphism). Conversely if $T: C(H) \to C(G)$ is any algebra isomorphism then we will show that $T$ is given by $g \mapsto g \circ h$ for some homeomorphism $h: G \to H$ which is a topological isomorphism if $T$ is character preserving. Given any $x \in G$ let $M_x$ denote the ideal $\{g \in C(H): T(g)(x) = 0\}$. Now $M_x$ is clearly a maximal ideal being the kernel of the homomorphism $g \mapsto T(g)(x)$ whose image is the field $\mathbb{C}$. Hence by Lemma A there exists a unique $y = h(x) \in H$ such that $M_x = \{g \in C(H): g(y) = 0\}$. In fact $T(g)(x) = g(y) = g(h(x))$ for every $g \in C(H)$ since $g - g(y)1_H$ is always in $M_x$. Similarly, given any $y \in H$ the ideal $\widehat{M_y} = \{f \in C(G): T^{-1}(f)(y) = 0\}$, is a maximal ideal and we can find a unique $x = \hat{h}(y) \in G$ such that $\widehat{M_y} = \{f \in C(G): f(x) = 0\}$. It is easy to verify that the maps $h$ and $\hat{h}$ defined this way are inverses of each other, i.e., $h \circ \hat{h} = \mathrm{id}_H$, and $\hat{h} \circ h = \mathrm{id}_G$. Thus $h$ is bijective. Now given any $x_0 \in G$, by Urysohn's lemma, there exists a neighbourhood $V$ of $h(x_0)$ and some $\varphi \in C(H)$ such that $\varphi(h(x_0)) = 1$ and $\varphi(u) = 0$ for all $u \in H \setminus V$ (note that $H$ is normal being compact and Hausdorff). Since the set $U = \{x \in G: T(\varphi)(x) \neq 0\}$ is an open subset of $G$ such that $x_0 \in U$ and $h(U) \subseteq V$, it follows that $h$ is continuous. Now both $G$ and $H$ are compact Hausdorff, and $h: G \to H$ is a continuous bijection, hence we conclude that $h$ is a homeomorphism. This proves Theorem 1. Now assume that $T$ preserves characters. Let $x_1 \in G, x_2 \in G$ be given. We have to show that $h(x_1 + x_2) = h(x_1) + h(x_2)$, but this is quite easy—thanks to Theorem 2. So let $\chi \in \hat{H}$ be any character. Then

$$\chi(h(x_1 + x_2)) = (\chi \circ h)(x_1 + x_2) = T(\chi)(x_1 + x_2)$$

and since $T(\chi)$ is a character, the expression on the left equals

$$T(\chi)(x_1)T(\chi)(x_2) = \chi(h(x_1))\chi(h(x_2)) = \chi(h(x_1) + h(x_2)).$$

This proves that $h$ is a homomorphism and hence a topological isomorphism. □

REMARK. Note that an argument similar to the one used above together with Theorem 8 on page 442 of [2] would show that if $G$ and $H$ are as in Theorem 3 and if $T$ is a linear isometry of $C(H)$ onto $C(G)$ which preserves characters then $G$ and $H$ are topologically isomorphic and in fact, there exists a topological isomorphism $h: G \to H$ and a character $\chi: G \to \Gamma$ such that $T(g) = \chi(g \circ h)$ for every $g \in C(H)$. Conversely any such $T$ is clearly a character preserving linear isometry of $C(H)$ onto $C(G)$.

*Example.* Let us consider one of the simplest examples of a compact abelian group, namely the circle group $\Gamma = \{z \in \mathbb{C}: |z| = 1\}$. So we are taking $G = H = \Gamma$. Now it is easy to see that the characters of $\Gamma$ are precisely the functions $z \mapsto z^n$ for $n \in \mathbb{Z}$ and the only topological isomorphisms among these are those functions for which $n = \pm 1$. Hence by the above theorem it follows that the only algebra automorphisms of $C(\Gamma)$ which preserve these countably many characters are the identity and the "conjugation" $f(z) \mapsto f(\bar{z})$. Also in view of the above remark we

see that the only character preserving linear isometries of $C(\Gamma)$ onto itself are $f(z) \mapsto z^n f(z)$ and $f(z) \mapsto z^n f(\bar{z})$ where $n$ varies over the set of all integers.

REMARK. If $G$ and $H$ are locally compact (Hausdorff) abelian groups then we can obtain a result similar to Theorem 3 with $C(G)$ and $C(H)$ replaced by $C_0(G)$ (viz., the algebra of continuous functions on $G$ which "vanish at infinity") and $C_0(H)$ respectively.

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## First-Class Functions

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In a recent issue of this MONTHLY, J. Fabrykowski [**1**] posed the problem of finding a sequence of functions continuous on [0, 1] whose pointwise limit is finite on the rationals and infinite on the irrationals. To analyze a proposed solution to the problem, he required several facts from number theory; in particular, facts about continued fractions and diophantine approximation. In fact, there are simple solutions to Fabrykowski's problem which use no number theory; we give one below. Having removed all number theory from the answer, we proceed to banish it from the question, replacing the rationals with an arbitrary $F_\sigma$ (that is, a countable union of closed sets). The main theorem is the following.

THEOREM 1. *There is a sequence of functions continuous on* [0, 1] *whose pointwise limit is finite on S and infinite on the complement of S if and only if S is an* $F_\sigma$.

I would like to thank my colleague, Béla Brindza, for insisting that we look for such a generalization, and for many fruitful discussions on the topic.

Let us first take care of the rationals. Gelbaum and Olmsted, in Example 4 of Chapter 7 of [**2**], give a sequence of continuous functions converging pointwise to the function $f$ given by

$$f(x) = 0 \text{ if } x \text{ is irrational,}$$

$$f(p/q) = 1/q \text{ if } p \text{ and } q \text{ are relatively prime integers,} \qquad q > 0.$$

Taking reciprocals yields a sequence of continuous functions solving Fabrykowski's problem. The construction in [**2**] makes no use of continued fractions, diophantine approximation, or other number-theoretical devices. The following construction is similar to that of [**2**], and arguably even a bit simpler: Enumerate the rationals in