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Subclose Families, Threshold Graphs, and the Weight Hierarchy of Grassmann and Schubert Codes

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Dedicated to Gilles Lachaud on his sixtieth birthday

Abstract. We discuss the problem of determining the complete weight hierarchy of linear error correcting codes associated to Grassmann varieties and, more generally, to Schubert varieties in Grassmannians. In geometric terms, this corresponds to the determination of the maximum number of $\mathbb{F}_q$-rational points on sections of Schubert varieties (with nondegenerate Plücker embedding) by linear subvarieties of a fixed (co)dimension. The problem is partially solved in the case of Grassmann codes, and one of the solutions uses the combinatorial notion of a close family. We propose a generalization of this to what is called a subclose family. A number of properties of subclose families are proved, and its connection with the notion of threshold graphs and graphs with maximum sum of squares of vertex degrees is outlined.

1. Introduction

It has been almost a decade since the first named author and Gilles Lachaud wrote [**5**] where alternative proofs of Nogin's results on higher weights of Grassmann codes [**14**] were given and Schubert codes were introduced. Originally, much of [**5**] was conceived as a side remark in [**6**]. But in retrospect, it appears to have been a good idea to write [**5**] as an independent article and use the opportunity to propose therein a conjecture concerning the minimum distance of Schubert codes. This conjecture has been of some interest, and after being proved, in the affirmative, in a number of special cases (cf. [**1, 17, 7, 9**]), the general case appears to have been settled very recently by Xiang [**19**]. The time sees ripe, therefore, to up the ante and think about more general questions. It is with this in view, that we discuss in this paper the problem of determining the complete weight hierarchy of Schubert codes and, in particular, the Grassmann codes. In fact, the case of Grassmann codes and the determination of higher weights in the cases not covered by the result in [**14**] and [**5**] has already been considered in some recent work (cf. [**10, 11, 8**]). What is proposed here is basically a plausible approach to tackle the general case.

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This paper is organized as follows. In Section 2 below we recall the combinatorial notion of a close family, and introduce a more general notion of a subclose family. A number of elementary properties of subclose families are proved here, including a nice duality that prevails among these. Basic notions concerning linear error correcting codes, such as the minimum distance and more generally, the higher weights are reviewed in Section 3. Further, we state here a general conjecture that relates the higher weights of Grassmann codes and Schubert codes with subclose families. Finally, in Section 4, we recall threshold graphs and optimal graphs and then show that, in a special case, subclose families are closely related to these well-studied notions in graph theory. As an application we obtain explicit bounds on the sum of squares of degrees of a simple graph in terms of the number of vertices and edges, which seem to ameliorate and complement some of the known results on this topic that has been of some interest in graph theory (cf. [**3, 2, 15**]).

2. Close Families and Subclose Families

Fix integers ℓ, m such that $1 \leq \ell \leq m$. Set

$$k := \binom{m}{\ell} \quad \text{and} \quad \mu := \max\{\ell, m-\ell\} + 1.$$

Let $[m]$ denote the set $\{1, \ldots, m\}$ of first m positive integers. Given any nonnegative integer j, let $I_j[m]$ denote the set of all subsets of $[m]$ of cardinality j.

Let $\Lambda \subseteq I_\ell[m]$. Following [**6**], we call Λ a *close family* if $|A \cap B| = \ell - 1$ for all $A, B \in \Lambda$. Suppose $|\Lambda| = r$. Then Λ is said to be of *Type I* if there exists $S \in I_{\ell-1}[m]$ and $T \subseteq [m] \setminus S$ with $|T| = r$ such that

$$\Lambda = \{S \cup \{t\} : t \in T\},$$

whereas Λ is said to be of *Type II* if there exists $S \in I_{\ell-r+1}[m]$ and $T \subseteq [m] \setminus S$ with $|T| = r$ such that

$$\Lambda = \{S \cup T \setminus \{t\} : t \in T\}.$$

Basic results about close families are as follows.

Proposition 2.1 (Structure Theorem for Close Families). *Let $\Lambda \subseteq I_\ell[m]$. Then Λ is close if and only if Λ is either of Type I or of Type II.*

This is proved in [**6**, Thm. 4.2]. An immediate consequence is the following.

Corollary 2.2. *Let r be a nonnegative integer. A close family in $I_\ell[m]$ of cardinality r exists if and only if $r \leq \mu$. In greater details, a close family of Type I in $I_\ell[m]$ exists if and only if $r \leq m - \ell + 1$, whereas a close family of Type II in $I_\ell[m]$ exists if and only if $r \leq \ell + 1$.*

We use this opportunity to state the following elementary result which complements Proposition 2.1. This is not stated explicitly in [**5, 6**], but a related result is proved in [**8**] where we obtain an algebraic counterpart of Proposition 2.1 in the setting of exterior algebras and the Hodge star operator.

Proposition 2.3 (Duality). *Given $\Lambda \subseteq I_\ell[m]$, let $\Lambda^* := \{[m] \setminus A : A \in \Lambda\} \subseteq I_{m-\ell}[m]$. Then Λ is close in $I_\ell[m]$ of type I if and only if Λ^* is close in $I_{m-\ell}[m]$ of type II.*

PROOF. Given $S \in I_{\ell-1}[m]$ and $T \subseteq [m] \setminus S$ with $|T| = r$, observe that

$$[m] \setminus (S \cup \{t\}) = ([m] \setminus (S \cup T)) \cup T \setminus \{t\}$$

for every $t \in T$. □

As explained in [**5**], Corollary 2.2 essentially accounts for the barrier on r for which the higher weights d_r of Grassmann codes $C(\ell, m)$ are hitherto known (see, e.g., [**14, 5**]). Recently some attempts have been made to break this barrier (cf. [**10, 11, 8**]) but the complete weight hierarchy $\{d_r : 1 \leq r \leq k\}$ is still not known. We will comment more on this in Section 3. For the time being, we introduce a combinatorial generalization of close families which may play some role in the determination of higher weights.

Given a subset $\Lambda = \{A_1, \cdots, A_r\}$ of $I_\ell[m]$, we define

$$K_\Lambda = \sum_{i<j} |A_i \cap A_j|.$$

Further, given any nonnegative integer $r \leq k$, we define

$$K_r(\ell, m) := \max\{K_\Lambda : \Lambda \subseteq I_\ell[m] \text{ and } |\Lambda| = r\}.$$

We call Λ a *subclose family* if $K_\Lambda = K_r(\ell, m)$ where $r = |\Lambda|$. It is clear that for each nonnegative integer $r \leq k$, there exists a subclose family of cardinality r.

PROPOSITION 2.4. *Given $\Lambda \subseteq I_\ell[m]$, we have*

$$K_\Lambda \leq (\ell - 1)\binom{|\Lambda|}{2}.$$

Moreover, equality holds if and only if Λ is a close family. Consequently, for any nonnegative integer $r \leq k$, we have

$$K_r(\ell, m) \leq (\ell - 1)\binom{r}{2}.$$

Moreover, equality holds if and only if $r \leq \mu$.

PROOF. The last assertion follows from Corollary 2.2. The remaining assertions are obvious. □

It is an interesting question to determine $K_r(\ell, m)$ for any r. The first few values are given by the above result. We shall now determine some more. To this end, let us first observe the following analogue of Proposition 2.3.

PROPOSITION 2.5 (First Duality Theorem). *Given $\Lambda \subseteq I_\ell[m]$, consider the family of complements of sets in Λ, viz., $\Lambda^* := \{[m] \setminus A : A \in \Lambda\} \subseteq I_{m-\ell}[m]$. Then Λ is a subclose family in $I_\ell[m]$ if and only if Λ^* is a subclose family in $I_{m-\ell}[m]$. Moreover,*

$$K_r(\ell, m) = \binom{r}{2}(2\ell - m) + K_r(m - \ell, m) \quad \textit{for } 0 \leq r \leq k.$$

Proof: Write A^c for $[m] \setminus A$ for $A \in I_\ell[m]$. Then for any $A, B \in I_\ell[m]$, we have

$$|A \cap B| = m - |A^c \cup B^c| = m - (m - \ell) - (m - \ell) + |A^c \cap B^c| = (2\ell - m) + |A^c \cap B^c|.$$

Thus if $r := |\Lambda|$ and we write $\Lambda = \{A_1, \ldots, A_r\}$, then

$$K_\Lambda = \sum_{1 \leq i < j \leq r} (2\ell - m) + |A_i^c \cap A_j^c| = \binom{r}{2}(2\ell - m) + K_{\Lambda^*}.$$

Now as Λ varies over families in $I_\ell[m]$ of cardinality r, the dual Λ^* varies over families in $I_{m-\ell}[m]$ of cardinality r. It follows that Λ is a subclose family in $I_\ell[m]$ if and only if Λ^* is a subclose family in $I_{m-\ell}[m]$. Moreover,

$$K_r(\ell,m) = \binom{r}{2}(2\ell - m) + K_r(m-\ell,m) \quad \text{for } 0 \le r \le k. \qquad \square$$

Recall that given any $a, b \in \mathbb{Z}$, the binomial coefficient $\binom{a}{b}$ is defined by

$$\binom{a}{b} := \begin{cases} \dfrac{a(a-1)\cdots(a-b+1)}{b!} & \text{if } b \ge 0, \\ 0 & \text{if } b < 0. \end{cases}$$

With this in view, we may permit a and b to take negative values. We record some elementary properties of binomial coefficients, which will be useful in the sequel.

Lemma 2.6. *Given any integers a, b, c, d, e, we have the following.*

(i) $\binom{a}{b} = \binom{a}{a-b}$ *if and only if either $a \ge 0$ or $a < b < 0$*

(ii) $\binom{a}{b} = 0$ *if and only if either $b < 0$ or $b > a \ge 0$*

(iii) $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$.

(iv) $\binom{a+b}{c-e} = \sum_{j=e}^{c} \binom{a+d}{c-j}\binom{b-d}{j-e}$.

(v) *If $a \ge 0$, then* $b\binom{a}{b} = a\binom{a-1}{b-1} = a\binom{a-1}{a-b}$.

Proof. Both (i) and (ii) are straightforward. Proofs of (iii) and (iv) are also elementary; see, for example, Lemma 3.2 and Corollary 3.4 of [**4**]. Finally, (v) is readily verified when $a \ge 1$ and $b \ge 1$; the case when $a = 0$ or $b \le 0$ follows from (i) and (ii). $\square$

The value of $K_r(\ell,m)$ for the maximum permissible parameter r is determined below.

Proposition 2.7.

$$K_k(\ell,m) = m\binom{\nu}{2}, \qquad \textit{where} \qquad \nu := \binom{m-1}{\ell-1}.$$

Proof. Observe that

$$\frac{m}{\ell}\nu = \binom{m}{\ell} = k, \quad \text{that is,} \quad \nu = \frac{\ell k}{m}.$$

Write $I_\ell[m] = \{A_1, \ldots, A_k\}$. Then

$$K_k(\ell,m) = \sum_{1 \le i < j \le k} |A_i \cap A_j| = \frac{1}{2}\left(\sum_{i,j} |A_i \cap A_j| - \sum_{1 \le i = j \le k} \ell\right),$$

and consequently,

$$K_k(\ell,m) = \frac{1}{2}\left[U - k\ell\right], \quad \text{where} \quad U := \sum_{A,B \in I_\ell[m]} |A \cap B|. \tag{2.1}$$

Thus it suffices to determine U, which is more symmetric than $K_k(\ell, m)$. To find U, note that for any $A, B \in I_\ell[m]$, the intersection $A \cap B$ is a subset E, say, of $[m]$ of cardinality $i \leq \ell$. Now,

$$\begin{aligned}
U &= \sum_{\substack{E\subseteq[m]\\ |E|\leq\ell}} \sum_{\substack{A,B\in I_\ell[m]\\ A\cap B=E}} |E| \\
&= \sum_{i=0}^{\ell} \sum_{\substack{E\subseteq[m]\\ |E|=i}} i|\{(A,B)\in I_\ell[m]\times I_\ell[m] : A\cap B = E\}| \\
&= \sum_{i=0}^{\ell} \binom{m}{i}\left[i\binom{m-i}{\ell-i}\binom{m-\ell}{\ell-i}\right] \\
&= \sum_{i=0}^{\ell} i\binom{m}{i}\left[\binom{m-i}{m-\ell}\binom{m-\ell}{\ell-i}\right] \qquad \text{[by Lemma 2.6 (i)]} \\
&= \sum_{i=1}^{\ell} m\left[\binom{m-1}{m-i}\binom{m-i}{m-\ell}\right]\binom{m-\ell}{\ell-i} \qquad \text{[by Lemma 2.6 (v)]} \\
&= m\binom{m-1}{m-\ell}\sum_{i=1}^{\ell}\binom{\ell-1}{\ell-i}\binom{m-\ell}{\ell-i} \qquad \text{[by Lemma 2.6 (iii)]} \\
&= m\binom{m-1}{\ell-1}\sum_{i=1}^{\ell}\binom{\ell-1}{i-1}\binom{m-\ell}{\ell-i} \qquad \text{[by Lemma 2.6 (i)]} \\
&= m\binom{m-1}{\ell-1}\binom{m-1}{\ell-1} \qquad \text{[by Lemma 2.6 (iv)]} \\
&= m\nu^2.
\end{aligned}$$

Therefore, equation (2.1) becomes

$$K_k(\ell, m) = \frac{1}{2}\left[U - k\ell\right] = \frac{1}{2}\left[m\nu^2 - m\nu\right] = m\binom{\nu}{2},$$

as desired. □

The above result will be helpful to establish yet another version of duality among subclose families. But first we need a preliminary result whose proof is similar in spirit to the proof above.

Lemma 2.8. *Given any $A \in I_\ell[m]$, we have*

$$\sum_{\substack{B\in I_\ell[m]\\ B\neq A}} |A\cap B| = \ell(\nu-1), \qquad \textit{where} \qquad \nu := \binom{m-1}{\ell-1}.$$

Proof. As in the proof of Proposition 2.7, we have

$$\begin{aligned}
\sum_{\substack{B\in I_\ell[m]\\ B\neq A}} |A\cap B| &= \sum_{i=0}^{\ell-1}\sum_{\substack{E\subseteq A\\ |E|=i}}\sum_{\substack{B\in I_\ell[m]\\ B\cap A=E}} |E| \\
&= \sum_{i=0}^{\ell-1}\binom{\ell}{i} i \binom{m-\ell}{\ell-i} \\
&= \ell\sum_{i=1}^{\ell-1}\binom{\ell-1}{i-1}\binom{m-\ell}{\ell-i} \\
&= \ell\left[-1+\sum_{i=1}^{\ell}\binom{\ell-1}{i-1}\binom{m-\ell}{\ell-i}\right] \\
&= \ell\left[\binom{m-1}{\ell-1}-1\right],
\end{aligned}$$

where the last equality follows from part (iv) of Lemma 2.6. □

Proposition 2.9 (Second Duality Theorem). *Given $\Lambda \subseteq I_\ell[m]$, consider the complement $\Lambda^c := I_\ell[m]\setminus\Lambda$. Then Λ is a subclose family in $I_\ell[m]$ if and only if Λ^c is a subclose family in $I_\ell[m]$. Moreover, if we let $r := |\Lambda|$ and $\nu := \binom{m-1}{\ell-1}$, then*

$$K_{\Lambda^c} = m\binom{\nu}{2} - r\ell(\nu-1) + K_\Lambda. \tag{2.2}$$

Consequently,

$$K_{k-r}(\ell,m) = m\binom{\nu}{2} - r\ell(\nu-1) + K_r(\ell,m) \quad \text{for } 0\le r\le k. \tag{2.3}$$

Proof. Let $\Lambda \subseteq I_\ell[m]$. Write $\Lambda = \{A_1,\dots,A_r\}$ and $\Lambda^c = \{B_1,\dots,B_{k-r}\}$. Then $I_\ell[m] = \{A_1,\dots,A_r,B_1,\dots,B_{k-r}\}$, and we clearly have

$$K_k(\ell,m) = K_{I_\ell[m]} = \sum_{1\le i<j\le r}|A_i\cap A_j| + \sum_{1\le i<j\le k-r}|B_i\cap B_j| + \sum_{\substack{1\le i\le r\\ 1\le j\le k-r}}|A_i\cap B_j|.$$

Thus, in view of Proposition 2.7, we see that

$$m\binom{\nu}{2} = K_k(\ell,m) = K_\Lambda + K_{\Lambda^c} + \sum_{A\in\Lambda}\sum_{B\in\Lambda^c}|A\cap B|.$$

Further, in view of Lemma 2.8, we can write

$$\begin{aligned}
\sum_{A\in\Lambda}\sum_{B\in\Lambda^c}|A\cap B| &= \sum_{A\in\Lambda}\left(\sum_{\substack{B\in I_\ell[m]\\ B\neq A}}|A\cap B| - \sum_{\substack{B\in\Lambda\\ B\neq A}}|A\cap B|\right) \\
&= \sum_{A\in\Lambda}\ell(\nu-1) - \sum_{A\in\Lambda}\sum_{\substack{B\in\Lambda\\ B\neq A}}|A\cap B| \\
&= r\ell(\nu-1) - 2K_\Lambda.
\end{aligned}$$

It follows that

$$K_{\Lambda^c} = m\binom{\nu}{2} - r\ell(\nu-1) + K_\Lambda.$$

This implies that $K_{\Lambda^c} \le m\binom{\nu}{2} - r\ell(\nu-1) + K_r(\ell,m)$, and the equality holds if and only if Λ is subclose. Consequently,

$$K_{k-r}(\ell,m) = m\binom{\nu}{2} - r\ell(\nu-1) + K_r(\ell,m),$$

and moreover, Λ is subclose if and only if Λ^c is subclose. □

COROLLARY 2.10. *If $s \in \mathbb{Z}$ is such that $k-\mu \le s \le k$, then*

$$K_s(\ell,m) = m\binom{\nu}{2} - \ell(\nu-1)(k-s) + (\ell-1)\binom{k-s}{2}, \quad \textit{where} \quad \nu := \binom{m-1}{\ell-1}.$$

In particular, if $m \ge 4$, then

$$K_s(2,m) = m\binom{m-1}{2} - 2\,(k-s)\,(m-2) + \binom{k-s}{2}, \quad \textit{for} \ \binom{m-1}{2} \le s \le \binom{m}{2}.$$

PROOF. Given $s \in \mathbb{Z}$ with $k-\mu \le s \le k$, observe that $r := k-s$ satisfies $0 \le r \le \mu$. Now use (2.3) together with Proposition 2.4 to obtain the first equality. The second equality follows from the first by noting that if $\ell = 2$ and $m \ge 4$, then $\mu = m-1 = \nu$ and $k-\mu = \binom{m-1}{2}$. □

EXAMPLE 2.11. Using the above results, one can readily compile a table of values of $K_r(\ell,m)$ for $\ell = 2$ and for small values of m. For example, we have

r	1	2	3	4	5	6	7	8	9	10
$K_r(2,5)$	0	1	3	6	8	12	15	19	24	30

and

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$K_r(2,6)$	0	1	3	6	10	12	15	19	24	30	34	39	45	52	60

where it may be noted that the barrier μ on the values of r is 4 in the first table and 5 in the second table. That is where the pattern begins to change.

3. Higher Weights of Grassmann Codes and Schubert Codes

We have made it amply clear in the Introduction that the combinatorial considerations in the preceding section were motivated by problems in Coding Theory, more specifically, the determination of the higher weights of linear codes associated to Grassmann and Schubert varieties. In this section, we begin by describing some relevant background, set up some notation, and then state a precise conjecture that relates these higher weights to subclose families.

Fix integers k, n with $1 \le k \le n$ and a prime power q. Let C be a linear $[n,k]_q$-code, i.e., let C be a k-dimensional subspace of the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$ with q elements. Given any $x = (x_1,\ldots,x_n)$ in $\mathbb{F}_q^n$, let

$$\operatorname{supp}(x) := \{i : x_i \ne 0\} \quad \text{and} \quad \|x\| := |\operatorname{supp}(x)|$$

denote the *support* and the *(Hamming) norm* of x. More generally, for $D \subseteq \mathbb{F}_q^n$, let

$$\operatorname{supp}(D) := \{i : x_i \ne 0 \text{ for some } x = (x_1,\ldots,x_n) \in D\} \quad \text{and} \quad \|D\| := |\operatorname{supp}(D)|$$

denote the *support* and the *(Hamming) norm* of D. The *minimum distance* or the *Hamming weight* of C is defined by $d(C) := \min\{\|x\| : x \in C \text{ with } x \neq 0\}$. More generally, following [**18**], for any positive integer r, the rth *higher weight* or the rth *generalized Hamming weight* $d_r = d_r(C)$ of the code C is defined by

$$d_r(C) := \min\{\|D\| : D \text{ is a subspace of } C \text{ with } \dim D = r\}.$$

Note that $d_1(C) = d(C)$. If C is *nondegenerate*, i.e., if C is not contained in a coordinate hyperplane of $\mathbb{F}_q^n$, then it is easy to see that

$$0 < d_1(C) < d_2(C) < \cdots < d_k(C) = n.$$

See, for example, [**16**] for a proof as well as a great deal of basic information about higher weights of codes. The set $\{d_r(C) : 1 \leq r \leq k\}$ is often referred to as the *(complete) weight hierarchy* of the code C. It is usually interesting, and difficult, to determine the complete weight hierarchy of a given code.

An equivalent way of describing codes is via the language of projective systems, explained, for example in [**16, 14, 5**]. A $[n,k]_q$-projective system X is a (multi)set of n points in the projective space $\mathbb{P}^{k-1}$ over $\mathbb{F}_q$. We say X is *nondegenerate* if it is not contained in a hyperplane of $\mathbb{P}^{k-1}$. An $[n,k]_q$-nondegenerate projective system gives rise to a unique (up to isomorphism) nondegenerate $[n,k]_q$-linear code C_X. The minimum distance of C_X corresponds to maximizing the number of points of hyperplane sections of X, while the rth higher weight corresponds to maximizing the number of points of sections of X by codimension r projective linear subspaces. More precisely, for $0 \leq r \leq k$, we have

$$d_r(C_X) = n - \max\left\{|X \cap \Pi| : \Pi \text{ is a projective subspace of codimension } r \text{ in } \mathbb{P}^{k-1}\right\}.$$

Linear codes associated to projective systems given by the $\mathbb{F}_q$-rational points of higher dimensional projective algebraic varieties defined over $\mathbb{F}_q$ have been of much interest lately, and we refer to the recent survey by Little [**12**] for more on this. We are particularly interested in the case of Grassmann variety $G_{\ell,m}$ and its Schubert subvarieties $\Omega_\alpha = \Omega_\alpha(\ell,m)$ with its nondegenerate Plücker embedding in $\mathbb{P}^{k-1}$ and $\mathbb{P}^{k_\alpha-1}$, respectively. Here, as in Section 2, ℓ, m are fixed positive integers with $\ell \leq m$ and $k := \binom{m}{\ell}$, while α varies over the natural indexing set for points of $\mathbb{P}^{k-1}$, namely,

$$I(\ell,m) := \{\beta = (\beta_1,\ldots,\beta_\ell) \in \mathbb{Z}^\ell : 1 \leq \beta_1 < \cdots < \beta_\ell \leq m\},$$

and for any $\alpha \in I(\ell,m)$,

$$k_\alpha := |I_\alpha(\ell,m)| \quad \text{where} \quad I_\alpha(\ell,m) := \{\beta \in I(\ell,m) : \beta_i \leq \alpha_i \text{ for all } i = 1,\ldots,\ell\}.$$

We identify $\mathbb{P}^{k_\alpha-1}$ with $\left\{p = (p_\beta) \in \mathbb{P}^{k-1} : p_\beta = 0 \text{ for all } \beta \in I(\ell,m) \setminus I_\alpha(\ell,m)\right\}$ so that $\Omega_\alpha(\ell,m) = G_{\ell,m} \cap \mathbb{P}^{k_\alpha-1}$. For precise definitions of $G_{\ell,m}$ and Ω_α, and their Plücker embeddings, we refer to [**5**] and [**7**] or the references therein. The linear codes corresponding to $G_{\ell,m}$ and $\Omega_\alpha(\ell,m)$ are denoted by $C(\ell,m)$ and $C_\alpha(\ell,m)$ respectively. The length n of $C(\ell,m)$ and n_α of $C_\alpha(\ell,m)$ are respectively given by

$$n = |G_{\ell,m}(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m-1)(q^m-q)\cdots(q^m-q^{\ell-1})}{(q^\ell-1)(q^\ell-q)\cdots(q^\ell-q^{\ell-1})} \quad \text{and} \quad n_\alpha = |\Omega_\alpha(\mathbb{F}_q)|.$$

The dimension k of $C(\ell,m)$ and k_α of $C_\alpha(\ell,m)$ are respectively given by

$$k = |I(\ell,m)| = \binom{m}{\ell} \quad \text{and} \quad k_\alpha := |I_\alpha(\ell,m)|.$$

A number of explicit formulas for n_α and k_α are given in [**7**].

Given any $\Lambda \subseteq I(\ell, m)$, we let Π_Λ denote the intersection of the corresponding Plücker coordinate hyperplanes; more precisely,

$$\Pi_\Lambda := \left\{ p = (p_\beta) \in \mathbb{P}^{k-1} : p_\beta = 0 \text{ for all } \beta \in \Lambda \right\}.$$

Note that Π_Λ is a projective linear subspace of codimension $|\Lambda|$ in $\mathbb{P}^{k-1}$, and also that if $\Lambda \subseteq I_\alpha(\ell, m)$, then $\Pi_\Lambda \cap \mathbb{P}^{k_\alpha - 1}$ is a projective linear subspace of codimension $|\Lambda|$ in $\mathbb{P}^{k_\alpha - 1}$.

There is a natural one-to-one correspondence between the indexing set $I(\ell, m)$ and the set $I_\ell[m]$ defined in the previous section, given simply by

$$\beta = (\beta_1, \ldots, \beta_\ell) \longmapsto \bar{\beta} = \{\beta_1, \ldots, \beta_\ell\}.$$

With this in view, we shall identify $I(\ell, m)$ with $I_\ell[m]$, and apply the notions and results of Section 2 for $I_\ell[m]$ and its subfamilies to $I(\ell, m)$ and its subfamilies. In particular, we can talk about subclose families in $I(\ell, m)$. We are now ready to propose a plausible fact about the higher weights of $C(\ell, m)$ and $C_\alpha(\ell, m)$.

Conjecture 3.1. *Let r be a positive integer. If $r \leq k$, then the rth higher weight of the Grassmann code $C(\ell, m)$ is given by*

$$d_r(C(\ell, m)) = \begin{bmatrix} m \\ \ell \end{bmatrix}_q - \max\left\{ |G_{\ell,m}(\mathbb{F}_q) \cap \Pi_\Lambda| : \Lambda \subseteq I(\ell, m) \text{ is subclose and } |\Lambda| = r \right\}.$$

More generally, given any $\alpha \in I(\ell, m)$, if $r \leq k_\alpha$, then the rth higher weight of the Schubert code $C_\alpha(\ell, m)$ is given by

$$d_r(C_\alpha(\ell, m)) = n_\alpha - \max\left\{ |\Omega_\alpha(\mathbb{F}_q) \cap \Pi_\Lambda| : \Lambda \subseteq I_\alpha(\ell, m) \text{ is subclose and } |\Lambda| = r \right\}.$$

The evidence we have in favor of this conjecture is as follows.

(1) The conjecture is true in the case of $C(\ell, m)$ for $1 \leq r \leq \max\{\ell, m-\ell\}+1$. (See [**5**].)
(2) The conjecture is true in the case of $C(2, m)$ for $r = \max\{2, m-2\} + 2$. (See [**8**].)
(3) The conjecture is true in the case of $C_\alpha(\ell, m)$ for $r = 1$. (See [**19**].)
(4) The conjecture is true in the case of $C_\alpha(\ell, m)$ where α is a submaximal element of $I_\alpha(\ell, m)$ [so that the corresponding Schubert variety Ω_α is of codimension 1 in $G_{\ell,m}$] for $1 \leq r \leq \max\{\ell, m-\ell\}$. (See [**7**].)

It may be remarked that the notion of a subclose family and the above conjecture is similar to, yet distinct from, the notion of a Schubert union introduced in [**10**] and the corresponding conjecture of Hansen, Johnsen and Ranestad [**10, 11**] that the higher weights are attained by Schubert unions. It may also be noted that for a given r, there may be more than one subclose family of cardinality r. Thus the above conjecture does not pinpoint to a single such family but simply narrows down the search for such a family.

4. Threshold Graphs, Optimal Graphs and Subclose Families

When $\ell = 2$, the elements of the family $I_\ell[m]$ of ℓ-subsets of $[m] := \{1, \ldots, m\}$ can be viewed as the edges of a graph. It is, therefore, natural to investigate if the combinatorial notions and results in Section 2 have analogues and extensions in the rich and diverse field of graph theory. We will attempt to address these concerns in this section.

Given any $\Lambda \subseteq I_2[m]$, we denote by G_Λ the graph whose vertex set is $[m]$ and the edge set is Λ. Note that this is a simple (undirected) graph. Conversely, any simple graph on $[m]$ is of the form G_Λ for a unique $\Lambda \subseteq I_2[m]$. To say that Λ is close corresponds to saying that any two edges of G_Λ are incident. Thus, Proposition 2.1 corresponds to the following elementary result in graph theory.

PROPOSITION 4.1. *A simple graph in which any two edges are incident is either a star or a triangle.*

The analogue of subclose family is more interesting. Before explaining this, let us recall some notions from graph theory.[1] Let G be a (m, r)-graph, i.e., a graph with m vertices (assumed to be elements of the set $[m]$) and r edges. We denote by $g_i = g_i(G)$ the *degree* of the vertex i, viz., the number of edges emanating from it. The sequence $(g_1, \ldots, g_m)$ is called the *degree sequence* of G. It is well-known and easy to see that

$$\sum_{i=1}^{m} g_i = 2r. \tag{4.1}$$

A simple graph G is said to be a *threshold graph* if G can be constructed from a one-vertex graph by repeatedly adding an isolated vertex or a universal one (i.e., a vertex adjacent to every other vertex). A simple (m, r)-graph G is said to be (m, r)-*optimal*, or simply, *optimal* if

$$\Sigma(G) := \sum_{i=1}^{m} g_i(G)^2$$

is maximum among all simple (m, r)-graphs. Threshold graphs are a topic of considerable interest in graph theory, and we refer to [**13**] for more on this. It is easy to see that an optimal graph is a threshold graph (cf. [**15**, Fact 3]). In [**15**] it is shown that an optimal graph is one among certain six explicit classes of graphs. However, as the authors of [**15**] say, the complete characterization of optimal graphs remains an open question. The following explicit bound for $\Sigma(G)$ for a (m, r)-graph G is given by de Caen [**3**].

$$\Sigma(G) \le C(r, m) \quad \text{for } m \ge 2, \quad \text{where} \quad C(r, m) := r\left(\frac{2r}{m-1} + m - 2\right). \tag{4.2}$$

A somewhat more general bound has been obtained by Das [**2**]; however, this bound is not a pure function of m and r, but involves the maximum and the minimum among the vertex degrees $g_1, \ldots, g_m$.

The relation between optimal graphs and subclose families is given below.

PROPOSITION 4.2. *Let Λ be a subset of $I_2[m]$ with $|\Lambda| = r$. Then*

$$K_\Lambda = \frac{1}{2}\Sigma(G_\Lambda) - r \quad \textit{or equivalently,} \quad \Sigma(G_\Lambda) = 2K_\Lambda + 2r.$$

Consequently, Λ is a subclose family if and only if G_Λ is an optimal graph. Moreover,

$$\max\{\Sigma(G) : G \textit{ is a simple } (m, r)\textit{-graph}\} = 2K_r(2, m) + 2r.$$

[1] We are using here a notation that is consistent with the notation of Section 2. Inconvenience caused, if any, to graph theorists, who may be more used to letting n be the number of vertices, e the number of edges, and d_i the degree of the vertex i, is regretted.

PROOF. Write $\Lambda = \{A_1, \dots, A_r\}$. Note that $A_i \cap A_j$ is either empty or singleton for $1 \le i < j \le r$. Thus,

$$K_\Lambda = \sum_{i<j} |A_i \cap A_j| = \sum_{i<j} \sum_{v \in A_i \cap A_j} 1 = \sum_{v \in [m]} \sum_{\substack{i<j \\ v \in A_i \cap A_j}} 1 = \sum_{v \in [m]} \binom{g_v}{2}.$$

Further, in view of (4.1), we have

$$K_\Lambda = \frac{1}{2} \sum_{v \in [m]} g_v^2 - r = \frac{1}{2}\Sigma(G_\Lambda) - r.$$

Since G_Λ varies over all simple (m,r)-graphs as Λ varies over subsets of $I_2[m]$ of cardinality r, it follows that Λ is subclose if and only if G_Λ is optimal, and moreover,

$$\max\{\Sigma(G) : G \text{ is a simple } (m,r)\text{-graph}\} = 2K_r(2,m) + 2r,$$

as desired. □

COROLLARY 4.3. *Assume that $m \ge 4$. Then for any simple (m,r)-graph G, we have*

$$\Sigma(G) \le r(r+1) \quad \textit{for } r \le m-1, \tag{4.3}$$

and the equality holds if and only if G is a star with $r+1$ vertices (and $m-r-1$ isolated vertices).

PROOF. Since $m \ge 4$, we have $\mu := \max\{2, m-2\} + 1 = m-1$. Thus, thanks to Proposition 2.4, we have

$$K_r(2,m) \le \binom{r}{2} \quad \text{for} \quad r \le m-1.$$

Now apply Proposition 4.2 to obtain (4.3). The assertion about the equality follows from Proposition 4.1. □

Already, the trivial bound given by the Corollary above is superior to de Caen's bound (4.2) in several cases. Indeed, an easy calculation shows that

$$C(r,m) - r(r+1) = \frac{r(m-3)(m-1-r)}{m-1} > 0 \quad \text{for } r < m-1 \text{ and } m \ge 4.$$

The above result may also be compared with one of the cases where equality holds in a bound for $\Sigma(G)$ obtained by Das [**2**].

It is well-known that dual graphs $\overline{G}$ of optimal graphs G [defined in such a way that the non-edges of G are the edges of $\overline{G}$] are optimal (cf. [**15**, Fact 1]). This corresponds precisely to the special case $\ell = 2$ of the Second Duality Theorem (Proposition 2.9). Further, it is easy to see that if $\ell = 2$, then (2.2) corresponds precisely to the following elementary relation for simple (m,r)-graphs G:

$$\Sigma(\overline{G}) = m(m-1)^2 - 4r(m-1) + \Sigma(G).$$

The following bound, dual to the trivial bound given by (4.3), appears to be new.

COROLLARY 4.4. *If G is a simple (m,r)-graph such that $\binom{m-1}{2} \le r \le \binom{m}{2}$, then*

$$\Sigma(G) \le m(m-1)(m-2) + (k-r)(k-r-1) - 4(k-r)(m-2) + 2r, \quad \textit{where} \quad k := \binom{m}{2}.$$

PROOF. Follows from Corollary 2.10 and Proposition 4.2. □

Finally, we remark that the First Duality Theorem (Proposition 2.5) has no analogue in the setting of optimal graphs for the simple reason that it relates optimal graphs to objects that are not graphs, but hypergraphs. Indeed, as far as we know, not much seems to be known about threshold hypergraphs and optimal hypergraphs. Perhaps the notion of a subclose family and the results of Section 2 may be of some help in this direction.

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