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CORRIGENDA AND ADDENDA: *ÉTALE COHOMOLOGY, LEFSCHETZ THEOREMS AND NUMBER OF POINTS OF SINGULAR VARIETIES OVER FINITE FIELDS*

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Brian Conrad kindly pointed out to us that the proof of Proposition 9.8 in the article in question is incomplete.

We provide here the missing arguments together with a few other corrections and use the opportunity to indicate some new consequences of our results, and also mention some applications of the results in [S1]. In what follows, the supplementary references, including the original paper itself, are numbered as [S1], [S2], etc., while citations such as [1] refer to those in [S1]. Lemmas, propositions, etc., numbered such as 2.1, 8.4, &c., correspond to those in [S1]. A revised version of [S1] incorporating the corrections in this note is available as arXiv:0808.2169 [math.AG].

Base Field

Usually, at the beginning of each section of [S1], the assumptions on the base field k are specified. In addition, the following modifications are in order.

– In Statements 2.1, 2.4, 2.5 and 2.6, one should mention explicitely that X and Y are defined over k.

– In Remark 2.7, one has to assume that there is a proper linear section of codimension $s+1$ of X defined over k.

If k is algebraically closed, the Galois group $\mathbf{g}$ is trivial, and these conditions are fulfilled.

Further, the proof of part (i) of Prop. 8.7 uses Corollary 1.4 and it should be modified as follows:

– Take an extension k'/k in order to get a section Y defined over k'. This implies that the eigenvalues of the Frobenius of k' in $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$ are pure, and the same holds for the eigenvalues of the Frobenius of k, since they are roots of the preceding.

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Betti Numbers of Curves

The proof of Lemma 8.4 as given in [S1] is only valid on a finite field. This Lemma and its proof should be stated as follows.

8.4. Lemma. *Let K be an algebraically closed field, and X an irreducible projective curve in $\mathbb{P}_K^N$, with arithmetic genus $p_a(X)$. Let $\widetilde{X}$ be a nonsingular projective curve birationally equivalent to X, with geometric genus $g(\widetilde{X})$. Then we have the following.*

(i) *If d denotes the degree of X, then*

$$2g(\widetilde{X}) \leqslant b_1(X) \leqslant 2p_a(X) \leqslant (d-1)(d-2).$$

(ii) *If $K = \bar{k}$, where k is a finite field, and if X is defined over k, then*

$$b_1^+(X) = 2g(\widetilde{X}).$$

During the proof of the Lemma, we shall make use of the following standard construction, when X is a curve. This leads to an inequality between Hilbert polynomials.

8.5. *Remark* (Comparison of Hilbert polynomials). Let K be an algebraically closed field and X a closed subvariety in $\mathbb{P}_K^N$ distinct from the whole space, and r an integer such that $\dim X + 1 \leqslant r \leqslant N$. Let $\mathcal{C}_r(X)$ be the subvariety of $\mathbb{G}_{N-r,N}$ of linear varieties of codimension r meeting X. From the properties of the incidence correspondence Σ defined by

$$\Sigma = \left\{(x,\, E) \in \mathbb{P}^N \times \mathbb{G}_{N-r,N} : x \in E\right\},$$

it is easy to see that $\mathcal{C}_r(X) = \pi_2(\pi_1^{-1}(X))$ is irreducible and that the codimension of $\mathcal{C}_r(X)$ in $\mathbb{G}_{N-r,N}$ is equal to $r - \dim X$. Hence, the set of linear subvarieties of codimension r in $\mathbb{P}_K^N$ disjoint from X is a nonempty open subset $\mathcal{D}_r(X)$ of $\mathbb{G}_{N-r,N}$.

If E belongs to $\mathcal{D}_{n+2}(X)$, where $n = \dim X$, the projection π with center E gives rise to a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "\pi_X"'] & \mathbb{P}_K^N - E \arrow[d, "\pi"] \\ X' \arrow[r, "i'"] & \mathbb{P}_K^{n+1} \end{tikzcd}$$

such that X' is an irreducible hypersurface with $\deg X' = \deg X$, and where the restriction π_X is a finite birational morphism: denoting by $S(X)$ the homogeneous coordinate ring of X, we have an inclusion $S(X') \subset S(X)$, and $S(X)$ is a finitely generated module over $S(X')$. Hence, if $P_X(T) \in \mathbb{Q}[T]$ is the *Hilbert polynomial of* X [16, p. 52], we have

$$P_{X'}(t) \leqslant P_X(t) \quad \text{if } t \in \mathbb{N} \text{ and } t \to \infty.$$

Proof of Lemma 8.4. Let U be a regular open subscheme of X. Then, there is a commutative diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r] \arrow[d] & \widetilde{X} \arrow[d, "\pi"'] & \widetilde{S} \arrow[l] \arrow[d] \\ U \arrow[r] & X & S \arrow[l] \end{tikzcd}$$

where $\widetilde{X}$ is a nonsingular curve, where π is a proper morphism which is a birational isomorphism, and an isomorphism when restricted to $\widetilde{U}$, and

$$\operatorname{Sing} X \subset S = X \setminus U, \quad \widetilde{S} = \widetilde{X} \setminus \widetilde{U}.$$

The excision long exact sequence in compact cohomology [30, Rem. 1.30, p. 94] gives:

$$0 \longrightarrow H^0_c(X) \longrightarrow H^0_c(S) \longrightarrow H^1_c(U) \longrightarrow H^1_c(S) \longrightarrow 0$$

and there is a similar exact sequence if we replace X, U, S by $\widetilde{X}$, $\widetilde{U}$, $\widetilde{S}$. This implies

$$b_1(U) = b_1(X) - 1 + |S|, \qquad b_1(\widetilde{U}) = b_1(\widetilde{X}) - 1 + |\tilde{S}|,$$

and since U and $\widetilde{U}$ are isomorphic, we obtain

$$b_1(X) = b_1(\widetilde{X}) + d(X) = 2g(\widetilde{X}) + d(X), \quad \text{where} \quad d(X) = |\tilde{S}| - |S|,$$

since, as is well-known, $b_1(\widetilde{X}) = 2g(\widetilde{X})$. Let

$$\delta(X) = p_a(X) - g(\widetilde{X}).$$

Then $0 \leqslant d(X) \leqslant \delta(X)$ [S9, Prop. 1, p. 68]. Hence

$$b_1(X) = 2g(\widetilde{X}) + d(X) \leqslant 2g(\widetilde{X}) + 2\delta(X) = 2p_a(X).$$

This proves the first and second inequalities of (i). The Hilbert polynomial of X is [16, p. 54]:

$$P_X(T) = dT + 1 - p_a(X).$$

Apply now the construction of Remark 8.5 to X, and obtain a morphism $X \longrightarrow X'$, where X' is a plane curve of degree d. From the inequality $P_{X'}(t) \leqslant P_X(t)$ for t large, we get $p_a(X) \leqslant p_a(X')$. Now, by Example 4.3(ii),

$$p_a(X') = (d-1)(d-2)/2,$$

since X' is a plane curve of degree d, and so

$$p_a(X) \leqslant (d-1)(d-2)/2,$$

and this proves the third inequality of (i). Now, under the hypotheses of (ii), we have by [4, Thm. 2.1]:

$$P_1(X, T) = P_1(\widetilde{X}, T) \prod_{j=1}^{d(X)} (1 - \omega_j T),$$

where the numbers ω_j are roots of unity, and this implies the inequality in (ii). $\square$

The Penultimate Cohomology Group

Let k be a perfect field. Assume, as in Sec. 9 of [S1], that all projective varieties over k considered in this section have a k-rational nonsingular point. The proof given in [S1] of Prop. 9.8 could be completed as follows.

9.8. Proposition. *Let X be a normal projective variety of dimension $n \geqslant 2$ defined over k which is regular in codimension 2. Then there is a $\mathbf{g}$-equivariant isomorphism*

$$j_X : V_\ell(\mathrm{Alb}_w X) \xrightarrow{\sim} H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)).$$

If $(\mathbf{R}_{\mathrm{n,p}})$ holds, the same conclusion is true if one only assumes that X is regular in codimension 1.

Proof. Step 1. Assume that X is a subvariety in $\mathbb{P}^N_k$. Since X is regular in codimension 2, we deduce from Proposition 1.3 and Corollary 1.4 that $\mathcal{U}_{n-2}(X)$ contains a nonempty Zariski open set U_0 in the Grassmannian $\mathbb{G}_{N-n+2,N}$. On the other hand, any open set defined over $\bar{k}$ is defined over a finite extension k', and contains an open set defined over k (take the intersection of the transforms by the Galois group of k'/k). Let $U_1 \subset U_0$ be an open set defined over k. If $E \in U_1$, then $Y = X \cap E$ is a *typical surface* on X over k, i. e., a nonsingular proper linear section of dimension 2 in X. For such a typical surface Y, the closed immersion $\iota \colon Y \to X$ induces a homomorphism $\iota_* \colon \mathrm{Alb}_w Y \to \mathrm{Alb}_w X$. By Proposition 9.4(i), the set of linear varieties $E \in U_1$ such that ι_* is a purely inseparable isogeny contains as well a nonempty open subset $U \subset \mathbb{G}_{N-n+2,N}$ which is defined over k.

Step 2. Assume that $U(k)$ is nonempty. If $E \in U(k)$, we get a $\mathbf{g}$-equivariant isomorphism

$$V_\ell(\iota_*) \colon V_\ell(\mathrm{Alb}_w Y) \xrightarrow{\sim} V_\ell(\mathrm{Alb}_w X).$$

Since Y is nonsingular, we get from Poincaré Duality Theorem for nonsingular varieties [30, Cor. 11.2, p. 276] a $\mathbf{g}$-equivariant nondegenerate pairing

$$H^1(\bar{Y}, \mathbb{Q}_\ell) \times H^3(\bar{Y}, \mathbb{Q}_\ell(2)) \longrightarrow \mathbb{Q}_\ell,$$

from which we deduce a $\mathbf{g}$-equivariant isomorphism

$$\psi \colon \mathrm{Hom}(H^1(\bar{Y}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \longrightarrow H^3(\bar{Y}, \mathbb{Q}_\ell(2)).$$

Since (X, Y) is a semi-regular pair with Y nonsingular, from Corollary 2.1 we know that the Gysin map

$$\iota_* \colon H^3(\bar{Y}, \mathbb{Q}_\ell(2-n)) \longrightarrow H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$$

is an isomorphism. Now a $\mathbf{g}$-equivariant isomorphism of vector spaces over $\mathbb{Q}_\ell$:

$$j_X : V_\ell(\mathrm{Alb}_w X) \xrightarrow{\sim} H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$$

is defined as the isomorphism making the following diagram commutative:

$$\begin{tikzcd}
\mathrm{Hom}(V_\ell(\mathrm{Pic}_s Y)(-1), \mathbb{Q}_\ell) \arrow[r, "\varpi", "\sim"'] \arrow[d, "{}^t h_Y"', "\sim"] & V_\ell(\mathrm{Alb}_w Y) \arrow[r, "V_\ell(\iota_*)", "\sim"'] & V_\ell(\mathrm{Alb}_w X) \arrow[d, "j_X"'] \\
\mathrm{Hom}(H^1(\bar{Y}, \mathbb{Q}_\ell), \mathbb{Q}_\ell) \arrow[r, "\psi", "\sim"'] & H^3(\bar{Y}, \mathbb{Q}_\ell)(2) \arrow[r, "\iota_*", "\sim"'] & H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)).
\end{tikzcd}$$

Here ϖ is defined by the Weil pairing, and ${}^t h_Y$ is the transpose of the map h_Y defined in Proposition 9.6. Hence, the conclusion holds if $U(k) \neq \varnothing$.

STEP 3. Assume that k is an infinite field. One checks successively that if U is an open subset in an affine line, an affine space, or a Grassmannian, then $U(k) \neq \varnothing$ and the conclusion follows from Step 2.

STEP 4. Assume that k is a finite field. Then the following elementary result holds (as a consequence of Proposition 12.1, for instance).

Claim. *Let U be a nonempty Zariski open set in $\mathbb{G}_{r,N}$, defined over k, and $k_s = \mathbb{F}_{q^s}$ the extension of degree s of $k = \mathbb{F}_q$. Then there is an integer $s_0(U)$ such that $U(k_s) \neq \varnothing$ for every $s \geqslant s_0(U)$.*

Now take for U the open set in $\mathbb{G}_{N-n+2,N}$ introduced in Step 1. Choose any $s \geqslant s_0(U)$, and let $\mathbf{g}_s = \mathrm{Gal}(\bar{k}/k_s)$. Since $U(k_s) \neq \varnothing$, upon replacing k by k_s, we deduce from Step 2 a $\mathbf{g}_s$-equivariant isomorphism of $\mathbb{Q}_\ell$-vector spaces:

$$j_{X,s} \colon V_\ell(\mathrm{Alb}_w X) \xrightarrow{\sim} H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)).$$

This implies in particular that if $m = 2 \dim \mathrm{Alb}_w X$, then

$$\dim H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)) = \dim V_\ell(\mathrm{Alb}_w X) = m.$$

In each of these spaces, there is an action of $\mathbf{g} = \mathbf{g}_1$. By choosing bases, we identify both of them with $\mathbb{Q}_\ell^m$. Denote by $g_1 \in \mathrm{GL}_m(\mathbb{Q}_\ell)$ the matrix of the endomorphism $V_\ell(\varphi)$, where $\varphi \in \mathbf{g}$ is the geometric Frobenius, and by $g_2 \in \mathrm{GL}_m(\mathbb{Q}_\ell)$ the matrix of the Frobenius operator in $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$. The existence of the $\mathbf{g}_s$-equivariant isomorphism $j_{X,s}$ implies that g_1^s and g_2^s are conjugate. In order to finish the proof when k is finite, we must show that g_1 and g_2 are conjugate. This follows from the Conjugation Lemma below, since g_1 is semi-simple by [31, p. 203].

STEP 5. Assume now that $(\mathbf{R}_{\mathrm{n,p}})$ holds and that X is regular in codimension 1. Take $\widetilde{X}$ to be a nonsingular projective variety birationally equivalent to X over k. Then $\mathrm{Alb}_w \widetilde{X} = \mathrm{Alb}_w X$ since the Albanese-Weil variety is a birational invariant, and

$$H_+^{2n-1}(\widetilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)) = H_+^{2n-1}(X \otimes \bar{k}, \mathbb{Q}_\ell(n)),$$

by Proposition 8.1(ii). Now it is well known that $H^{2n-1}(\widetilde{X} \otimes \bar{k}, \mathbb{Q}_\ell)$ is pure, and the same holds for X, by Prop. 8.7(i). Hence,

$$H^{2n-1}(\widetilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)) = H^{2n-1}(X \otimes \bar{k}, \mathbb{Q}_\ell(n)).$$

Since the conclusion is true for a *nonsingular* variety, we obtain a $\mathbf{g}$-equivariant map

$$j_{\widetilde{X}} \colon V_\ell(\mathrm{Alb}_w \widetilde{X}) \xrightarrow{\sim} H^{2n-1}(\widetilde{X} \otimes \bar{k}, \mathbb{Q}_\ell(n)),$$

and this gives the required $\mathbf{g}$-equivariant isomorphism. $\square$

Conjugation Lemma. *Let K be a field of characteristic zero, and let g_1 and g_2 be two matrices in $\mathrm{GL}_n(K)$, with g_1 semi-simple. If g_2^s is conjugate to g_1^s for infinitely many prime numbers s, then g_2 is conjugate to g_1.*

Proof. Let $g_2 = su$ be the multiplicative Jordan decomposition of g_2 into its semi-simple and unipotent part. Take a and b prime with g_2^a conjugate to g_1^a. Then $s^a u^a$ is conjugate to g_1^a, and hence, $u^a = \mathbf{I}$, by the uniqueness of the Jordan decomposition. Similarly, we find $u^b = \mathbf{I}$. Hence $u = \mathbf{I}$ with the help of Bézout's equation, and g_2 is semisimple.

Take now two diagonal matrices d_1 and d_2 in $\mathrm{GL}_n(\bar{K})$ such that g_i is conjugate to d_i in $\mathrm{GL}_n(\bar{K})$. Two conjugate diagonal matrices are conjugate by an element of the group W of permutation matrices: if d_1^s and d_2^s are conjugate, then $d_2^s = (w_s d_1 w_s^{-1})^s$ with $w_s \in W$. Since W is finite, one of the sets

$$T(w) = \left\{ s \in \mathbb{N} \colon d_2^s = (w d_1 w^{-1})^s \right\}$$

contains infinitely many prime numbers. Take two prime numbers a and b in that set, then

$$d_2^a = h_1^a, \quad d_2^b = h_1^b, \quad h_1 = w d_1 w^{-1},$$

from which we deduce $d_2 = h_1$ by Bézout's equation. This implies that d_1 and d_2 are conjugate in $\mathrm{GL}_n(\bar{K})$, and the same holds for g_1 and g_2. But two elements of $\mathrm{GL}_n(K)$ which are conjugate in $\mathrm{GL}_n(\bar{K})$ are conjugate in $\mathrm{GL}_n(K)$. □

Addenda

One can improve some results in the paper, assuming that $(\mathbf{R}_{\mathrm{n,p}})$ holds. This may provide indications on the range of validity of the statements. For instance, the following proposition shows that the conclusion of Cor. 9.10 of [S1] is true without assuming that X is regular in codimension 2.

In what follows k is a perfect field of characteristic $p \geqslant 0$. Recall that a projective variety, regular in codimension one, which is a local complete intersection, is normal.

A1. Proposition. *Assume that $(\mathbf{R}_{\mathrm{n,p}})$ holds. Let X be a projective variety of dimension $n \geqslant 2$ defined over k.*

(i) *If X is normal, there is a $\mathbf{g}$-equivariant injective linear map*

$$H^1(\bar{X}, \mathbb{Q}_\ell) \longrightarrow \mathrm{Hom}(H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n)), \mathbb{Q}_\ell).$$

(ii) *If X is regular in codimension one and is a local complete intersection, this linear map is bijective.*

Proof. The proof of (i) follows the lines of the proof of [S1, Cor. 9.10], taking in account the last statement in Prop. 9.8 of the present note. In order to see that (ii) holds, remark that

$$\dim H^1(\bar{X}, \mathbb{Q}_\ell) = \dim H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$$

by Poincaré duality [S1, Rem. 2.7]. □

A2. Proposition. *Assume that* $(\mathbf{R}_{\mathrm{n,p}})$ *holds. Let* X *be a projective variety of dimension* $n \geqslant 2$ *defined over* k*, regular in codimension one, which is a local complete intersection. Then the canonical map*

$$\nu\colon \operatorname{Alb}_w X \longrightarrow \operatorname{Alb}_s X$$

is an isomorphism.

Proof. From Prop. A1(ii) and the proof of [S1, Cor. 9.10], we deduce that the homomorphism

$$V_\ell({}^t\nu)\colon V_\ell(\operatorname{Pic}_s X) \longrightarrow V_\ell(\operatorname{Pic}_w X)$$

is bijective, hence, ν is an isogeny by Tate's Theorem [31, Appendix I]. Since the kernel of ν is connected by Prop. 9.1(ii), it is trivial. □

Now Prop. A2 leads to an improvement of Prop. 10.10:

A3. Corollary. *Assume that* $(\mathbf{R}_{\mathrm{n,p}})$ *holds. Let* X *be a projective variety of dimension* n*, defined over the finite field* $k = \mathbb{F}_q$*, regular in codimension one, which is a local complete intersection. If* $g = \dim \operatorname{Alb}_w X$*, then*

$$q^{-g} P_1^\vee(X, q^n T) = P_{2n-1}(X, T). \qquad \square$$

Applications and complements

It may be interesting to note that some of the results of [S1] have found application in such diverse fields as group theory by T. Bandman & al. [S2], [S3], the study of Boolean functions by F. Rodier [S7], and Waring's problem in function fields by Y.-R. Liu and T. Wooley [S6] None of these applications are based on the results whose proof needed modifications or corrections, as outlined here. Improvements of some of the estimates in [S1] have also been obtained by A. Cafure and G. Matera [S4], [S5]. Finally, since Section 1 of [S1] includes a version of Bertini Theorem, it is worthwile to notice that deep results on Bertini Theorems over finite fields have been recently obtained by B. Poonen [S7].

Supplementary References

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