ON THE ENUMERATION OF INDEXED MONOMIALS AND THE COMPUTATION OF HILBERT FUNCTIONS OF LADDER DETERMINANTAL VARIETIES

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Abstract. We outline the computation of an explicit formula for the Hilbert function of the ladder determinantal varieties defined by the vanishing of all minors of a fixed size of a rectangular matrix with indeterminate entries such that the indeterminates in these minors are restricted to lie in some ladder shaped region of the rectangular array. Finding such a formula is equivalent to enumerating the set of monomials of a fixed degree such that the support of these monomials is a subset of a 'ladder' and satisfies a certain "index condition". We also describe applications of this formula for estimating the dimension of ladder determinantal varieties.

1. Introduction

Typically, a ladder is a subset of a rectangle which looks as follows.

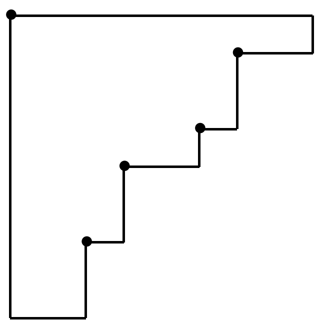

Figure 1

Given a rectangular matrix with indeterminate entries, the algebraic variety defined by all minors of a fixed size within a ladder shaped subset of the rectangle, is usually called a *ladder determinantal variety.* These varieties are intimately related to Schubert varieties in flag manifolds (cf. [21], [22]). Motivated partly by this connection, the ladder determinantal varieties (and, in fact, more general varieties of this kind) were introduced by Abhyankar [1]. It is known that such varieties are irreducible (cf. [1], [23]), Cohen-Macaulay (cf. [12]) and normal (cf. [5], [19]). Some of these properties can also be deduced using the connection with Schubert varieties from the work of Ramanathan [25] (see also [3], [17], [18], [24] [26]). Using [25] or otherwise (cf. [7], [11]) one knows also that most of these varieties have rational

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singularities. Other properties such as Gorensteinness as well as the determination of the divisor class group have also been investigated (cf. [5], [6], [20]).

We consider in this paper the problem of finding an explicit formula for the Hilbert function of ladder determinantal varieties. More precisely, we consider an $m(1) \times m(2)$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over a field K, and a subset $\mathcal{L}$ of the rectangle $\{(i,j) : 1 \le i \le m(1),\ 1 \le j \le m(2)\}$ such that $\mathcal{L}$ is like a (one-sided) ladder as in Fig. 1 or, more generally, a 'biladder' or a two-sided ladder (See Section 2 for precise definitions). Let $K[\mathcal{L}]$ denote the ring of polynomials in the indeterminates $\{X_{ij} : (i,j) \in \mathcal{L}\}$ with coefficients in K. Let $I_p(\mathcal{L})$ denote the ideal of $K[\mathcal{L}]$ generated by all $p \times p$ minors of X in $K[\mathcal{L}]$.

From a combinatorial viewpoint, the problem of finding the Hilbert function of $I_p(\mathcal{L})$ is equivalent to enumerating a set of monomials in $K[\mathcal{L}]$ satisfying a certain 'index condition' and of a fixed degree. Indeed, from the work of Abhyankar [1], we know that such "indexed monomials" form a K-basis for the graded components of the residue class ring $K[\mathcal{L}]/I_p(\mathcal{L})$. In fact, this equivalence is a basic starting point for the arguments in [2], [15], [16] and in this paper as well. The connection with indexed monomials is explained in details in Section 2.

The problem of finding explicitly the Hilbert function of the residue class ring $K[\mathcal{L}]/I_p(\mathcal{L})$ or of the corresponding projective variety $\mathcal{V}_p(\mathcal{L})$ was first studied by Kulkarni in his 1985 thesis [15] (see also [16]). There he obtained a nice formula in the first nontrivial case of $p = 2$. It may be noted that in the degenerate case when $\mathcal{L}$ is the entire rectangle $[1, m(1)] \times [1, m(2)]$, the ideal $I_p(\mathcal{L})$ reduces to the classical determinantal ideal $I_p(X)$ that arises frequently in Algebraic geometry and Invariant Theory. In the case of $I_p(X)$, the Hilbert function is explicitly known from the work of Abhyankar [1]. In particular, the Hilbert function of $K[X]/I_p(X)$ coincides with the Hilbert polynomial of $I_p(X)$ for all nonnegative integers; ideals with this property are called *hilbertian*. For a survey of Abhyankar's work, see [8] and for a short proof of a formula for the Hilbert function of $K[X]/I_p(X)$, see [4] or [9]. Returning to ladders, it was shown in 1989 by Abhyankar and Kulkarni [2] that the ideals $I_p(\mathcal{L})$ are also hilbertian for any $p > 1$ and any biladder $\mathcal{L}$; in fact, this result is applicable to sets more general than biladders, called generalized ladders or saturated sets (see Section 2 for details). Ladder determinantal ideals such as $I_p(\mathcal{L})$ were considered from the viewpoint of Gröbner bases and lattice paths by Herzog and Trung [12]. They showed that one can describe the Hilbert function of $K[\mathcal{L}]/I_p(\mathcal{L})$ in terms of the f-vector of the associated simplicial complex. While this would also prove the Hilbertianness of $I_p(\mathcal{L})$, there still remains the problem of finding explicitly the Hilbert function of $K[\mathcal{L}]/I_p(\mathcal{L})$. To this end, Conca and Herzog [4] conjectured a 'remarkable formula' for the Hilbert series (that is, the generating function for the sequence of values of the Hilbert function) in the case of one-sided ladders. Recently, Krattenthaler and Prohaska [14] have established this Conjecture using the so called 'two-rowed arrays'.

Our main result is an explicit (albeit, complicated!) formula for the Hilbert function of $K[\mathcal{L}]/I_p(\mathcal{L})$ for any biladder $\mathcal{L}$ and any $p > 1$. This may be viewed as a natural extension of the results of Kulkarni [16] and a refinement of the technique used by Abhyankar and Kulkarni [2] to prove that $I_p(\mathcal{L})$ is hilbertian. A detailed proof of this result shall appear in a forthcoming paper [10]. For this paper, we have a modest two-fold aim. First, to outline some of the main ideas involved in the proof and give the statements of the main lemmas and theorems. Second, to describe applications of the Hilbert function formula for determining, or estimating, the dimension of ladder determinantal varieties. It is hoped that this would make [10], which appears to be a rather long and technical paper, a little more accessible.

This paper is organized as follows. The next section sets up some notation and preliminary notions that we shall use. We consider the case of $I_2(\mathcal{L})$ in Section 3 and the general case in Section 4. Finally, in Section 5, we give results for estimating the degree of the Hilbert function or equivalently the dimension of ladder determinantal varieties. This last section is disjoint from [10] and thus, unlike in the previous sections, complete proofs are given.

2. Preliminaries

By $\mathbb{Z}, \mathbb{N}$, and $\mathbb{N}^+$ we denote the sets of all integers, nonnegative integers, and positive integers respectively. Given any $a, b \in \mathbb{Z}$, we define the closed and semi-closed integral intervals $[a, b]$, $[a, b)$, $(a, b]$ in the obvious way; for example,

$$[a, b) = \{c \in \mathbb{Z} : a \leq c < b\}.$$

Fix a pair $m = (m(1), m(2))$ of positive integers, a field K and an $m(1) \times m(2)$ matrix $X = (X_{ij})$ whose entries are independent indeterminates over K. Given any subset Y of the rectangle

$$[1, m(1)] \times [1, m(2)] = \{(i, j) : 1 \leq i \leq m(1),\ 1 \leq j \leq m(2)\},$$

let $K[Y]$ denote the polynomial ring in the indeterminates $\{X_{ij} : (i, j) \in Y\}$ with coefficients in K. Given any $p \in \mathbb{N}$, we let $I_p(Y)$ denote the ideal of $K[Y]$ generated by all $p \times p$ minors of X in $K[Y]$.

Given any $h \in \mathbb{N}^+$, by a *ladder generating bisequence* (LGB) of *length* h, we mean a map $S : [1, 2] \times [0, h] \to \mathbb{N}$ such that

$$1 = S(1, 0) \leq S(1, 1) < S(1, 2) < \cdots < S(1, h) = m(1)$$

and

$$m(2) = S(2, 0) > S(2, 1) > \cdots > S(2, h-1) \geq S(2, h) = 1.$$

The positive integer h may be denoted by $\operatorname{len}(S)$. We shall find it convenient to also consider the empty bisequence, which we declare to be the unique LGB of length 0. Given any LGB S of length h, we define

$$L(S) = \bigcup_{k=1}^{\operatorname{len}(S)} [S(1, k-1), S(1, k)] \times [1, S(2, k-1)],$$

and

$$L(S)^o = \bigcup_{k=1}^{\operatorname{len}(S)} [S(1, k-1), S(1, k)) \times [1, S(2, k-1)).$$

We call $L(S)$ to be the *ladder* corresponding to S and $L(S)^o$ to be the *interior* of $L(S)$. Note that if $h > 0$, then

$$L(S)^o \subsetneq L(S) \subseteq [1, m(1)] \times [1, m(2)].$$

In case $h = 0$, we have $L(S) = L(S)^0 = \emptyset$, whereas if $h = 1$, then we have $L(S) = [1, m(1)] \times [1, m(2)]$. We shall denote by ∂S or by $\partial L(S)$ the *boundary* of $L(S)$, which is defined by $\partial S = L(S) \setminus L(S)^o$. Points $(S(1, k), S(2, k))$, where $1 \leq k \leq h-1$, are called the *nodes* of S or of the ladder $L(S)$ and we denote by $\mathcal{N}(S)$ the set of all nodes of S.

Given any LGB's S' and S such that $\operatorname{len}(S) \neq 0$ and $L(S') \subseteq L(S)$, we define

$$\mathcal{L}(S', S) = L(S) \setminus L(S') \quad \text{and} \quad \mathcal{L}(S', S)^o = L(S)^o \setminus L(S').$$

We call $\mathcal{L}(S', S)$ to be the *biladder* corresponding to S' and S while $\mathcal{L}(S', S)^o$ to be the *interior* of $\mathcal{L}(S', S)$. Note that since we allow $L(S') = \emptyset$, a ladder is a special case of a biladder. Pictorially, a ladder looks as in Fig. 1 above and a biladder looks as in Fig. 2 (a) or, more generally, as in Fig. 2 (b) below.

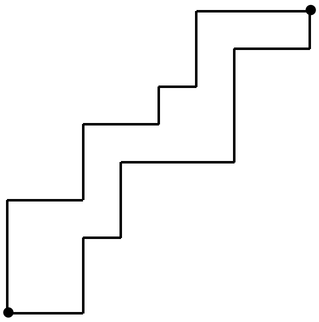

FIGURE 2 (a)

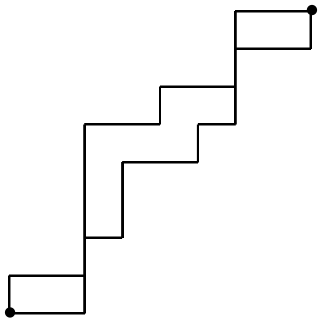

FIGURE 2 (b)

It may be remarked that in these pictures, we adopt the 'matrix notation' rather than that of Coordinate Geometry to represent points. Thus in Fig. 1, the bullet on the top left hand corner indicates the point $(1, 1)$ while the other bullets indicate the 'nodes' $(S(1, k), S(2, k))$, $1 \leq k < h$. In Fig. 2 (a) and Fig. 2 (b), we have only marked the points $(1, m(2))$ and $(m(1), 1)$ corresponding to $h = 0$ and $h = 1$.

Given any biladder $\mathcal{L} = \mathcal{L}(S', S)$, we shall denote by $\Delta(S', S)$ the intersection of the boundaries of $L(S)$ and $L(S')$, and by $\mathcal{N}(S', S)$ the set of common nodes of $\mathcal{N}(S')$ and $\mathcal{N}(S)$, that is,

$$\Delta(S', S) = \partial S' \cap \partial S \quad \text{and} \quad \mathcal{N}(S', S) = \mathcal{N}(S') \cap \mathcal{N}(S).$$

It may be noted that $\Delta(S', S) = \partial S \cap L(S')$.

Observe that ladders as well as biladders are subsets Y of $[1, m(1)] \times [1, m(2)]$ with the property that whenever (i_1, i_2), $(j_1, j_2) \in Y$ with $i_1 < i_2$ and $j_1 < j_2$, we have that $(i_1, j_2) \in Y$ and $(j_1, i_2) \in Y$. Sets Y with this property may be called *generalized ladders* or *saturated sets*. Some authors simply refer to them as ladders. It is not difficult to see that if a generalized ladder is 'connected', then it must be a biladder.

Given any $Y \subseteq [1, m(1)] \times [1, m(2)]$, we let $\text{mon}(Y)$ denote the set of all maps of $Y \to \mathbb{N}$. Given any $\theta \in \text{mon}(Y)$, we let

$$\text{supp}(\theta) = \{(i, j) \in Y : \theta(i, j) \neq 0\}$$

denote the *support* of θ and

$$X^\theta = \prod_{(i,j)\in Y} X_{ij}^{\theta(i,j)}$$

denote the corresponding monomial in $K[Y]$. Following Abhyankar [1], we define the *index* of any subset $T \subseteq [1, m(1)] \times [1, m(2)]$ by

$$\begin{aligned} \text{ind}(T) = \max\{p \in \mathbb{N} \quad : \quad & \exists\ (i_1, j_1), (i_2, j_2), \ldots, (i_p, j_p) \text{ in } T \text{ with} \\ & i_1 < i_2 < \cdots < i_p \text{ and } j_1 < j_2 < \cdots < j_p\}. \end{aligned}$$

For a monomial $\theta \in \text{mon}(Y)$, the index is defined by putting

$$\text{ind}(\theta) = \text{ind}(\text{supp}(\theta)).$$

For every $p \in \mathbb{N}$ we let

$$\text{mon}(Y, p) = \{\theta \in \text{mon}(Y) : \text{ind}(\theta) \leq p\}$$

and, restricting attention to monomials of a specified degree, for every $p \in \mathbb{N}$ and $V \in \mathbb{N}$ we let

$$\text{mon}(Y, p, V) = \{\theta \in \text{mon}(Y, p) : \sum_{y\in Y} \theta(y) = V\}.$$

We now recall a basic result of Abhyankar [1, Thm. 20.10] (see also [8, Thm. 6.7]), which was alluded to in the Introduction.

2.1. Theorem. *Let $Y \subseteq [1, m(1)] \times [1, m(2)]$ be any generalized ladder and let $p \in \mathbb{N}$. Given any $V \in \mathbb{N}$, the set $\{X^\theta : \theta \in \mathrm{mon}(Y, p, V)\}$ forms a free K-basis of the V-th homogeneous component $K[Y]_V/I_{p+1}(Y)_V$ of the residue class ring $K[Y]/I_{p+1}(Y)$. Consequently, the Hilbert function of the residue class ring $K[Y]/I_{p+1}(Y)$ or the corresponding projective variety $\mathcal{V}_{p+1}(Y)$ is given by*

$$\mathcal{H}(V) = |\mathrm{mon}(Y, p, V)|, \qquad (V \in \mathbb{N}).$$

Following Kulkarni [15], we consider the so called radicals and skeletons, which are defined as follows. Fix some $Y \subseteq [1, m(1)] \times [1, m(2)]$. A subset $R \subseteq Y$ is called a *radical* if $\mathrm{ind}(R) \le 1$, and it is called a *skeleton* if for any two distinct elements (i_1, i_2) and (j_1, j_2) of R, we have

$$\text{either: } i_1 < j_1 \text{ and } i_2 > j_2 \quad \text{or: } i_1 > j_1 \text{ and } i_2 < j_2.$$

The set of all radicals (resp: skeletons) in Y is denoted by $\mathrm{rad}(Y)$ (resp: $\mathrm{skel}(Y)$). Note that $\mathrm{skel}(Y) \subseteq \mathrm{rad}(Y)$. More generally, given any $p \in N$, we let $\mathrm{rad}^p(Y)$ denote the set of all $R \subseteq Y$ such that $\mathrm{ind}(R) \le p$. Elements of $\mathrm{rad}^p(Y)$ may be called *p-fold radicals*. Finally, we set for any $p \in \mathbb{N}$ and $d \in \mathbb{N}$,

$$\mathrm{rad}^p(Y, d) = \{R \in \mathrm{rad}^p(Y) : |R| = d\} \quad \text{and} \quad \mathrm{skel}(Y, d) = \{R \in \mathrm{skel}(Y) : |R| = d\}.$$

3. Radicals and Skeletons

It is easy to see that the problem of counting the desired set of monomials can be reduced to the problem of enumerating the p-fold radicals of a given size.

3.1. Lemma. *Given any $Y \subseteq [1, m(1)] \times [1, m(2)]$, $p \in \mathbb{N}$ and $V \in N$, we have*

$$|\mathrm{mon}(Y, p, V)| = \sum_{d \ge 0} \binom{V-1}{V-d} |\mathrm{rad}^p(Y, d)|,$$

where the summation on the right is essentially finite (that is, all except finitely many summands are zero).

Fix any biladder $\mathcal{L} = \mathcal{L}(S', S)$, and let $\mathcal{L}^o$ denote its interior. Let $h = \mathrm{len}(S)$ and $h' = \mathrm{len}(S')$. Also let

$$\delta_0 = |\Delta(S', S)| = |\partial S' \cap \partial S|.$$

Given any $(k, k') \in [1, h] \times [1, h']$, we let

$$\mu_{\mathcal{L}}(k, k') = \mathrm{card}([S(1, k-1), S(1, k)) \cap [S'(1, k'-1), S'(1, k')))$$

and

$$\nu_{\mathcal{L}}(k, k') = \mathrm{card}([S(2, k), S(2, k-1)) \cap (S'(2, k'), S'(2, k'-1)]).$$

Note that these numbers are completely (and easily) determined by $\mathcal{L}$.

As a preliminary step towards calculating $|\mathrm{rad}^p(\mathcal{L}, d)|$, we shall restrict our attention to the case of $p = 1$ so as to determine $|\mathrm{rad}(\mathcal{L}, d)|$. To this end, we use techniques similar to [16] except that now instead of ladders we consider the more general biladders, and so one has to be a little more careful, especially since we are allowing overlaps (as in Fig. 2 (b)) of smaller ladder $L(S')$ with the bigger ladder $L(S)$. As in [15], we reduce the problem to skeletons by constructing two maps

$$\lambda : \mathrm{rad}(\mathcal{L}) \to \mathrm{skel}(\mathcal{L}^o) \quad \text{and} \quad \mu : \mathrm{skel}(\mathcal{L}^o) \to \mathrm{rad}(\mathcal{L})$$

such that λ is surjective, μ is injective and moreover, μ is the inverse of the restriction of λ to maximal subsets of $\mathrm{rad}(\mathcal{L})$. This leads to the following.

3.2. Theorem. *Let $M = m(1) + m(2) - 1$. Given any $d \in \mathbb{N}$, we have*

$$|\mathrm{rad}(\mathcal{L}, d)| = \sum_{\ell \geq 0} \binom{M - \delta_0 - \ell}{d - \ell} |\mathrm{skel}(\mathcal{L}^o, \ell)|,$$

where the summation on the right is essentially finite (that is, all except finitely many summands are zero).

To describe an explicit formula for the number of skeletons in the interior of a biladder, we need some notation.

Given any $\ell \in \mathbb{N}$, let $M_{h,h'}(\mathbb{N}, \ell)$ denote the set of all $h \times h'$ matrices with integral entries such that the sum of all the entries is ℓ. Note that this is a finite set. Given any $\alpha = (\alpha_{kk'}) \in M_{h,h'}(\mathbb{N}, \ell)$ and any $(i, j) \in [1, h] \times [1, h']$, we let

$$\sigma_i(\alpha) = \sum_{k=1}^{i} \sum_{k'=1}^{h'} \alpha_{kk'} \quad \text{and} \quad \tau_j(\alpha) = \sum_{k=1}^{h} \sum_{k'=j}^{h'} \alpha_{kk'}.$$

Given any $\ell \in \mathbb{N}$ and $\alpha, \beta \in M_{h,h'}(\mathbb{N}, \ell)$, we define

$$\sigma(\beta) \leq \sigma(\alpha) \text{ to mean that } \sigma_i(\beta) \leq \sigma_i(\alpha) \text{ for all } i \in [1, h]$$

and

$$\tau(\beta) \leq \tau(\alpha) \text{ to mean that } \tau_j(\beta) \leq \tau_j(\alpha) \text{ for all } j \in [1, h'].$$

Finally, for any $\ell \in \mathbb{N}$, we define

$$\mathcal{S}(\mathcal{L}^o, \ell) = \sum_{\substack{\alpha, \beta \in M_{h\,h'}(\mathbb{N}, l) \\ \sigma(\beta) \leq \sigma(\alpha),\ \tau(\beta) \leq \tau(\alpha)}} \prod_{\substack{1 \leq k \leq h \\ 1 \leq k' \leq h'}} \binom{\mu_{\mathcal{L}}(k, k')}{\alpha_{kk'}} \binom{\nu_{\mathcal{L}}(k, k')}{\beta_{kk'}}.$$

3.3. Theorem. *Given any $\ell \in \mathbb{N}$, we have*

$$|\mathrm{skel}(\mathcal{L}^o, \ell)| = \mathcal{S}(\mathcal{L}^o, \ell).$$

As a consequence of the above results, we obtain the following formula, which may be viewed as an extension of Kulkarni's formula [16, Thm. 11]

3.4. Theorem. *The Hilbert function as well as the Hilbert polynomial of $I_2(\mathcal{L})$ in $K[\mathcal{L}]$ is given by*

$$F(V) = \sum_{\ell \geq 0} \binom{V + M - \delta_0 - 1 - \ell}{M - \delta_0 - 1} \mathcal{S}(\mathcal{L}^o, \ell)$$

where $M = m(1) + m(2) - 1$ and $\mathcal{S}(\mathcal{L}^o, \ell)$ is given by the formula above.

4. General Case

As in Section 3, we fix a biladder $\mathcal{L} = \mathcal{L}(S', S)$, and let $\mathcal{L}^o$ denote its interior. Let h, h', δ_0 and ν be as defined in Section 3. Also, let $M = m(1) + m(2) - 1$.

Given any LGB S^*, we shall write $S^* \leq S$ to mean that $L(S^*) \subseteq L(S)$. Further, given any $p \in \mathbb{N}^+$, we let $\mathcal{D}_p(\mathcal{L})$ denote the set of all $(p-1)$-tuples $\mathbf{S} = (S_1, \ldots, S_{p-1})$ of LGB's such that

$$\mathrm{len}(S_i) \neq 0 \quad \text{and} \quad \Delta(S_{i-1}, S_i) = \Delta(S_{i-1}, S), \text{ for } 1 \leq i \leq p - 1,$$

and

$$S' = S_0 \leq S_1 \leq \cdots \leq S_{p-1} \leq S_p = S.$$

Observe that $\mathcal{D}_p(\mathcal{L})$ is nonempty since it always contains the $(p-1)$-tuple $(S, \ldots, S)$.

The following basic result allows us to tackle the general case recursively by applying the results of Section 3. The map Γ mentioned in the theorem below can be described quite explicitly and it yields a decomposition of $\mathrm{rad}(\mathcal{L})$, which may be viewed as a refinement of the superskeleton decomposition of [2, Thm. 10].

4.1. Theorem. *Let $p \in \mathbb{N}^+$ and $\mathcal{L}$ be as above. Then there exists a LGB S^* such that $S' < S^* \le S$ and there exists an injective map*

$$\Gamma : \mathrm{rad}^p(\mathcal{L}) \to \mathrm{rad}(\mathcal{L}) \times \mathrm{rad}^{p-1}(\mathcal{L}^*),$$

where $\mathcal{L}^*$ denotes the biladder $\mathcal{L}(S^*, S)$.

The map Γ is not surjective, in general. But its image can be characterized reasonably well. This leads to the following enumerative result.

4.2. Theorem. *Given any $d \in \mathbb{N}$ and any $p \in \mathbb{N}^+$, we have*

$$|\mathrm{rad}^p(\mathcal{L}, d)| = \sum_{S^*} \sum_{d_1+d_2=d} \binom{M - \delta_0 - \nu^*}{d_1 - \nu^*} |\mathrm{rad}^{p-1}(\mathcal{L}^*, d_2)|,$$

where the first sum is taken over all LGB's S^* such that

$$S' \le S^* \le S, \ \mathrm{len}(S^*) \neq 0 \text{ and } \Delta(S', S^*) = \Delta(S', S),$$

and the second sum is over all nonnegative integer pairs (d_1, d_2) such that $d_1 + d_2 = d$, and where $\nu^* = |\mathcal{N}(S^*) \setminus \mathcal{N}(S) \cap \mathcal{N}(S^*)|$ and $\mathcal{L}^*$ denotes the biladder $\mathcal{L}(S^*, S)$.

Successive applications of the above result yields the following.

4.3. Theorem. *Given any $d \in \mathbb{N}$ and any $p \in \mathbb{N}^+$, we have*

$$|\mathrm{rad}^p(\mathcal{L}, d)| = \sum_{\mathbf{S} \in \mathcal{D}_p(\mathcal{L})} \sum_{\ell \ge 0} \binom{pM - \delta(\mathbf{S}) - \nu_1 - \cdots - \nu_{p-1} - \ell}{d - \nu_1 - \cdots - \nu_{p-1} - \ell} \mathcal{S}(\mathcal{L}^o_{p-1}, \ell)$$

where given any $\mathbf{S} = (S_1, \ldots, S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$, we have put

$$\delta(\mathbf{S}) = \sum_{i=0}^{p-1} \delta_i \,, \quad \text{where } \delta_i = |\Delta(S_i, S)| = |\partial S_i \cap \partial S|, \text{ for } 0 \le i \le p-1$$

(with the convention that $S_0 = S'$ and $S_p = S$), and $\nu_i = |\mathcal{N}(S_i) \setminus (\mathcal{N}(S_i) \cap \mathcal{N}(S))|$ for $1 \le i \le p-1$, and $\mathcal{L}_{p-1} = \mathcal{L}(S_{p-1}, S)$.

Given $\mathbf{S} = (S_1, \ldots, S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$ and $u \in \mathbb{N}$, we let

$$F_u(\mathbf{S}) = \sum_{\ell \ge 0} \binom{\nu_1 + \cdots + \nu_{p-1} + \ell}{u} \mathcal{S}(\mathcal{L}^o_{p-1}, \ell),$$

where ν_i and $\mathcal{L}_{p-1}$ are as in Theorem 4.3. The main result of [10] is as follows.

4.4. Theorem. *Let $p \in \mathbb{N}^+$ and $\mathcal{L}$ be as above. The Hilbert function as well as the Hilbert polynomial of $K[\mathcal{L}]/I_{p+1}(\mathcal{L})$ is given by*

$$F(V) = \sum_{u \ge 0} \sum_{\mathbf{S} \in \mathcal{D}_p(\mathcal{L})} (-1)^u F_u(\mathbf{S}) \binom{V + pM - \delta(\mathbf{S}) - 1 - u}{pM - \delta(\mathbf{S}) - 1 - u}$$

where $\delta(\mathbf{S})$ is as in Theorem 4.3. In particular, $I_{p+1}(\mathcal{L})$ is a Hilbertian ideal.

4.5. *Remarks.* 1. The first two theorems in this section may motivate the use of biladders although one may only be interested in (one-sided) ladders. Indeed, even if $\mathcal{L}$ were a ladder to begin with, the $\mathcal{L}^*$ that one obtains in Theorem 4.1 is necessarily a biladder. Thus it makes sense to have the results of Section 3 in the general case of biladders.

2. The formulae in Theorem 4.3 and Theorem 4.4 are no doubt complicated and perhaps they may seem unworthy of being called 'explicit', in view of the rather unwieldy summation over $\mathcal{D}_p(\mathcal{L})$. Nevertheless, they can be used to deduce some interesting information about the variety associated to $I_{p+1}(\mathcal{L})$. For example, it is shown in the next section that one can derive fairly simple estimates for the degree of the Hilbert polynomial. Also, as Krattenthaler and Prohaska [14, Sec. 7] seem

to suggest, it appears unlikely that an elegant and simple formula for the Hilbert function of $I_{p+1}(\mathcal{L})$ can be found.

5. Degree Computations

It is well-known that if $\mathcal{H}(V)$ is the Hilbert polynomial of a projective variety, and if

$$\mathcal{H}(V) = \frac{e}{d!}V^d + c_1 V^{d-1} + \cdots + c_d \quad \text{with } e, c_1, \ldots, c_d \in \mathbb{Z} \text{ and } e \neq 0,$$

then the degree d of $\mathcal{H}(V)$ is the dimension of that projective variety and the 'normalized leading coefficient' e is its order or the multiplicity. Note that the dimension d can also be read off from the Hilbert series; indeed, we have

$$\sum_{V=0}^{\infty} \mathcal{H}(V) t^V = \frac{\mathcal{P}(t)}{(1-t)^{d+1}}, \text{ where } \mathcal{P}(t) \in \mathbb{Z}[t] \text{ with } \mathcal{P}(1) \neq 0.$$

Moreover, $d+1$ is the (Krull) dimension of the corresponding homogeneous coordinate ring. With this in view, we now attempt to extract some information about the degree as well as the leading coefficient of the Hilbert polynomial $F(V)$ of the ladder determinantal variety $\mathcal{V}_{p+1}(\mathcal{L})$. First, we need some notation.

Fix some $p \in \mathbb{N}^+$ and a biladder $\mathcal{L} = \mathcal{L}(S', S)$. Let M, $\delta(\mathbf{S})$ and $F(V)$ be as defined in Section 4. Further, we let

$$\delta^*(\mathcal{L}) = \min\{\delta(\mathbf{S}) : \mathbf{S} \in \mathcal{D}_p(\mathcal{L})\}$$

and

$$\mathcal{D}_p^*(\mathcal{L}) = \{\mathbf{S} \in \mathcal{D}_p(\mathcal{L}) : \delta(\mathbf{S}) = \delta^*(\mathcal{L})\}.$$

5.1. Proposition. *The degree of the Hilbert polynomial $F(V)$ of $\mathcal{V}_{p+1}(\mathcal{L})$ equals*

$$pM - \delta^*(\mathcal{L}) - 1 = p(m(1) + m(2) - 1) - \delta^*(\mathcal{L}) - 1$$

and the normalized leading coefficient equals

$$\sum_{\mathbf{S} \in \mathcal{D}_p^*(\mathcal{L})} |\mathrm{skel}(\mathcal{L}_{p-1}^o)| = \sum_{\mathbf{S} \in \mathcal{D}_p^*(\mathcal{L})} \sum_{\ell \geq 0} \mathcal{S}(\mathcal{L}_{p-1}^o, \ell)$$

where for $\mathbf{S} = (S_1, \ldots, S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$, by $\mathcal{L}_{p-1}$ we denote the biladder $\mathcal{L}(S_{p-1}, S)$.

Proof. From Theorem 4.4, we see that $F(V)$ is a sum of terms of the form

$$(-1)^u F_u(\mathbf{S}) \binom{V + pM - \delta(\mathbf{S}) - 1 - u}{pM - \delta(\mathbf{S}) - 1 - u}$$

where the coefficients $(-1)^u F_u(\mathbf{S})$ are independent of V. Clearly, the binomial coefficient above is a polynomial in V of degree $pM - \delta(\mathbf{S}) - 1 - u$, and this degree is maximum when $u = 0$ and $\delta(\mathbf{S}) = \delta^*(\mathcal{L})$. The corresponding leading coefficient

$$\frac{1}{(pM - \delta^*(\mathcal{L}) - 1)!} \sum_{\mathbf{S} \in \mathcal{D}_p^*(\mathcal{L})} F_0(\mathbf{S}) = \sum_{\mathbf{S} \in \mathcal{D}_p^*(\mathcal{L})} \sum_{\ell \geq 0} \mathcal{S}(\mathcal{L}_{p-1}^o, \ell)$$

is clearly positive since $\mathcal{D}_p(\mathcal{L})$ is nonempty. □

It is not difficult to get a simple upper bound for the degree of $F(V)$. To this end, we begin with some elementary observations concerning the boundary of a ladder.

5.2. Proposition. *If S^* is a LGB with* $\mathrm{len}(S^*) \neq 0$*, then*

(1) $$|\partial S^*| = M = m(1) + m(2) - 1$$

and moreover, we can write $\partial S^* = \{P_1, P_2, \ldots, P_M\}$ where

(2) $P_1 = (1, m(2))$, $P_M = (m(1), 1)$; $P_j - P_{j-1} = (1, 0)$ *or* $(0, 1)$ *for* $1 < j \leq M$.

Proof. From the definitions of a ladder and its interior, we easily see that

$$\partial S^* = L(S^*) \setminus L(S^*)^o = \{P_1\} \cup L_1 \cup L_2$$

where $P_1 = (S^*(1,0), S^*(2,0)) = (1, m(2))$ and

$$L_1 = \coprod_{k=1}^{h} (S^*(1,k-1), S^*(1,k)] \times \{S^*(2,k-1)\}$$

and

$$L_2 = \coprod_{k=1}^{h} \{S^*(1,k)\} \times [S^*(2,k), S^*(2,k-1)).$$

This readily implies (1) and (2). □

5.3. Lemma. *Given any* $\mathbf{S} = (S_1, \dots, S_{p-1}) \in \mathcal{D}_p(\mathcal{L})$*, we have*

$$\delta_i = |\partial S_i \cap \partial S| \geq \min\{\delta_0 + 2i, M\}, \quad \textit{for } 1 \leq i \leq p-1, \tag{3}$$

and consequently, if $t = \max\{i \in [1,p] : \delta_0 + 2(i-1) \leq M\}$*, then*

$$\delta(\mathbf{S}) = \sum_{i=0}^{p-1} \delta_i \geq t\delta_0 + t(t-1) + (p-t)M. \tag{4}$$

Further, if we assume that $p < (M - \delta_0 + 1)/2$*, then we have*

$$\delta^*(\mathcal{L}) \geq p\delta_0 + p(p-1) \quad \textit{and} \quad \dim \mathcal{V}_{p+1}(\mathcal{L}) \leq p(m(1) + m(2) - p - \delta_0) - 1. \tag{5}$$

Proof. We prove (3) by induction on i. The case of $i = 0$ being trivial, assume that $i \geq 1$ and that $\delta_{i-1} \geq \min\{\delta_0 + 2i - 2, M\}$. Now since $\partial S_i \cap \partial S_{i-1} = \partial S_{i-1} \cap \partial S$, we have $\partial S_i \cap \partial S \supseteq \partial S_{i-1} \cap \partial S$, and so $\delta_i \geq \delta_{i-1}$. Thus in case $\delta_{i-1} \geq M$, we have $\delta_i \geq M \geq \min\{\delta_0 + 2i, M\}$. Suppose $\delta_{i-1} \leq M - 1$. Then, by (1), $|\partial S_i \cap \partial S_{i-1}| \leq |\partial S_{i-1}| - 1$, and so we can write $\partial S_i = \{P_1, P_2, \dots, P_M\}$ where P_j's are as in (2), and further, we can find $r \in [1, M]$ such that

$$P_r \notin \partial S_{i-1} \quad \text{but} \quad P_j \in \partial S_{i-1} \text{ for } 1 \leq j < r. \tag{6}$$

Also, let us write

$$\partial S_{i-1} = \{Q_1, Q_2, \dots, Q_M\} \quad \text{and} \quad \partial S = \{R_1, R_2, \dots, R_M\}.$$

where Q_j and R_j satisfy the conditions in (2). Using (2) and (6), we see that $P_j = Q_j$ for $1 \leq j < r$. Moreover, since $\partial S_i \cap \partial S_{i-1} = \partial S_{i-1} \cap \partial S$, we have $P_j = R_j$ for $1 \leq j < r$. Now $P_r \neq Q_r$ and $P_{r-1} = R_{r-1}$, and thus in view of (2), we have that $P_r \notin \partial S_{i-1}$ and further, $R_r = P_r$ or $R_r = Q_r$. But if $R_r = Q_r$, then $Q_r \in \partial S_{i-1} \cap \partial S \subseteq \partial S_i$, and this forces that $Q_r = P_r$, which is a contradiction. Thus $R_r = P_r$ and so

$$\delta_i = |\partial S_i \cap \partial S| \geq |\partial S_{i-1} \cap \partial S| + |\{P_r\}| = \delta_{i-1} + 1.$$

In particular, if $\delta_{i-1} = M - 1$, then $\delta_i \geq M$, and hence (3) holds. Now suppose $\delta_{i-1} < M - 1$. Then we can also find some $s \in [1, M]$ such that $r < s < M$ and

$$P_s \notin \partial S_{i-1} \quad \text{but} \quad P_j \in \partial S_{i-1} \text{ for } s < j \leq M.$$

Arguing as in the case of P_r, we obtain that $P_s = R_s$, and thus

$$\delta_i = |\partial S_i \cap \partial S| \geq |\partial S_{i-1} \cap \partial S| + |\{P_r, P_s\}| = \delta_{i-1} + 2.$$

Hence, using the induction hypothesis, we obtain $\delta_i \geq \delta_0 + 2i$. This proves (3). Next, if t is as defined in the Lemma, then by (3), we have

$$\delta(\mathbf{S}) = \sum_{i=0}^{p-1} \delta_i \;\geq\; \sum_{i=0}^{t-1} (\delta_0 + 2i) + \sum_{i=t}^{p-1} M \;=\; t\delta_0 + t(t-1) + (p-t)M.$$

Thus (4) is proved. To prove (5) consider the quadratic function

$$q(t) = t\delta_0 + t(t-1) + (p-t)M.$$

Its derivative with respect to t equals $2(t - t_0)$, where $t_0 = (M - \delta_0 + 1)/2$. Hence $q(t)$ is strictly decreasing for $t < t_0$, and thus if $p < t_0$, then we have $q(t) \geq q(p)$ for all $t \in [1, p]$. This yields (5). $\square$

5.4. *Remark.* It may be noted that the condition $p < (M - \delta_0 + 1)/2$ in Lemma 5.3 is not restrictive. Indeed, the intersection $\partial S' \cap \partial S$ of the boundaries of $L(S)$ and $L(S')$ can be split into 'horizontal overlaps' and 'vertical overlaps' (see Fig. 2 (b)); if δ_h is the cardinality of the former and δ_v is the cardinality of the latter, then we have $\delta_0 = |\partial S' \cap \partial S| \leq \delta_h + \delta_v$. Moreover, in order that that a $(p+1)\times(p+1)$ minor of X has its entries in $\mathcal{L} = L(S) \setminus L(S')$, the columns should avoid the horizontal overlaps and the rows should avoid the vertical overlaps. Thus $p+1 \leq m(2) - \delta_h$ and $p+1 \leq m(1) - \delta_v$. This implies that $2p < m(1) + m(2) - (\delta_h + \delta_v) \leq m(1) + m(2) - \delta_0$, and so $p < (M - \delta_0 + 1)/2$.

5.5. Corollary. *Let $L = L(S)$ be the ladder corresponding to S. Assume that $p + 1 \leq \min\{m(1), m(2)\}$. Then $\dim \mathcal{V}_{p+1}(L) \leq p(m(1) + m(2) - p) - 1$.*

Proof. Apply Lemma 5.3 with S' as the empty LGB, and note that in this case $\delta_0 = 0$ and $t = p$. $\square$

5.6. *Remark.* The upper bound for $\dim \mathcal{V}_{p+1}(L)$ in the Corollary above is, in fact, the dimension of the classical determinantal variety defined by the ideal $I_{p+1}(X)$ generated by the $(p+1) \times (p+1)$ minors of X (see, e.g., [1, Thm. 20.15]). In this particular case of (one-sided) ladders, the same bound also follows from the expression for the Hilbert series in [14, Thm. 2]. However, this bound need not be attained, in general. For instance, in the Example considered in [14, p. 1022], the (projective) dimension is seen to be 103 whereas the value predicted by the above bound as well as by [14, Thm. 2] is 124.

Finally in this section, we illustrate how in some cases the actual value of $\dim \mathcal{V}_{p+1}(\mathcal{L})$ can be determined.

5.7. *Examples.* 1. Consider the most trivial case when $\mathcal{L} = \emptyset$. This corresponds to taking $S' = S$. Hence in this case

$$\mathbf{S} = (S_1, \ldots, S_{p-1}) \in \mathcal{D}_p(\mathcal{L}) \iff S_i = S \ \forall i \implies \nu_i = 0 \ \forall i \text{ and } \delta(S) = pM.$$

Further, $\mathcal{L}_{p-1} = \emptyset$, and so $\mathcal{S}(\mathcal{L}^0_{p-1}, \ell) = 1$ if $\ell = 0$, and 0 otherwise. Hence,

$$F_u(\mathbf{S}) = \sum_{\ell \geq 0} \binom{l}{u} \mathcal{S}(\mathcal{L}^0_{p-1}, \ell) = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{if } u > 0. \end{cases}$$

Consequently, for any $V \in \mathbb{N}$, we have

$$F(V) = \binom{V + pM - \delta(\mathbf{S}) - 1}{pM - \delta(\mathbf{S}) - 1} = \binom{V-1}{-1} = 0.$$

Of course, this is to be expected since $I_{p+1}(\mathcal{L})$ is the zero ideal and $K[\mathcal{L}]/I_{p+1}(\mathcal{L})$ is the trivial ring.

2. Suppose $\mathcal{L}$ is the full rectangle $[1, m(1)] \times [1, m(2)]$, which corresponds to the case when S' is the empty LGB and S is the unique LGB of length 1; also suppose $p < \min\{m(1), m(2)\}$. More generally, let $\mathcal{L}$ be a ladder $L(S)$ (so that S' is the empty LGB) where S is a LGB of length $h > 0$ with $S(1,1) \geq p$ and $S(2, h-1) \geq p$. In this case, by (4), we have

$$\delta^*(\mathcal{L}) = \min\{\delta(\mathbf{S}) : \mathbf{S} \in \mathcal{D}_p(\mathcal{L})\} \geq p(p-1).$$

Moreover, if we take $\mathbf{S} = (S_1, \ldots, S_{p-1})$, where S_i's are the 'hook-like' LGB's determined by the conditions

$$\operatorname{len}(S_i) = 2 \quad \text{and} \quad (S_i(1,1), S_i(2,1)) = (i,i), \quad \text{for } 1 \le i \le p-1,$$

then it is easy to see that for $1 \le i \le p-1$, we have

$$\partial S_i \cap \partial S_{i-1} = \{(1, m(2)), \ldots, (i-1, m(2)), (m(1), 1), \ldots, (m(1), i-1)\} = \partial S_{i-1} \cap \partial S,$$

and therefore $\mathbf{S} \in \mathcal{D}_p(\mathcal{L})$ and $\delta_i = |\partial S_i \cap \partial S| = 2i$ for $0 \le i \le p-1$; hence $\delta(\mathbf{S}) = p(p-1)$. It follows that $\delta^*(\mathcal{L}) = p(p-1)$, and therefore,

$$\dim \mathcal{V}_{p+1}(L) = p(m(1) + m(2) - p) - 1.$$

3. Let $\mathcal{L}$ be a ladder with one corner missing, i.e, S' is the empty LGB and $\mathcal{L} = L(S)$, where S is any LGB of length 2. Assume that $p < \min\{m(1), m(2)\}$. Let $(S(1,1), S(2,1)) = (a, b)$. Then we have

$$\delta^*(\mathcal{L}) = \begin{cases} p(p-1) & \text{if } a \ge p \text{ and } b \ge p \\ p(p-1) + (p-a)(m(2)-b) & \text{if } a < p \text{ and } b \ge p \\ p(p-1) + (p-b)(m(1)-a) & \text{if } a \ge p \text{ and } b < p \\ m(1)(p-b) + m(2)(p-a) + ab - p & \text{if } a < p \text{ and } b < p \end{cases} \tag{7}$$

and consequently,

$$\dim \mathcal{V}_{p+1}(L) = \begin{cases} pM - 1 & \text{if } a \ge p \text{ and } b \ge p \\ pM - 1 + (a-p)(m(2)-b) & \text{if } a < p \text{ and } b \ge p \\ pM - 1 + (b-p)(m(1)-a) & \text{if } a \ge p \text{ and } b < p \\ bm(1) + am(2) - ab & \text{if } a < p \text{ and } b < p. \end{cases} \tag{8}$$

To see this, note that when $a \ge p$ and $b \ge p$, the result follows from the preceding example. In the remaining cases, we can argue as in Lemma 5.3 and the preceding example. Thus, for instance, if $a < p$ and $b \ge p$, then for any $\mathbf{S} \in \mathcal{D}_p(\mathcal{L})$, we must have

$$\delta_i \ge 2i \quad \text{for } 0 \le i < a \quad \text{and} \quad \delta_i \ge 2i + (m(2) - b) \quad \text{for } a \le i \le p-1.$$

Moreover, there exists a configuration $\mathbf{S} \in \mathcal{D}_p(\mathcal{L})$ for which the above inequalities are equalities. Alternatively, we can directly prove (8) in the last three cases from the following simple observations. If $a < p$ and $b \ge p$, then $\mathcal{V}_{p+1}(\mathcal{L})$ is a cylinder over the determinantal variety $\mathcal{V}_{p+1}(Y)$, where Y is the $m(1) \times b$ rectangular submatrix of X obtained by taking the first b columns of X. The case when $a \ge p$ and $b < p$ is similar. Lastly, when $a < p$ and $b < p$, we have $I_{p+1}(\mathcal{L}) = (0)$.

5.8. *Remarks.* 1. The observation about the dimension of $\dim \mathcal{V}_{p+1}(L)$ in the second example above may perhaps explain why in Kulkarni's formula for the Hilbert function of $\mathcal{V}_2(L)$, the degree is the same as that in the case of $\mathcal{V}_2(X)$. Indeed, when $p = 1$, the conditions $S(1,1) \ge p$ and $S(2, h-1) \ge p$ always hold.

2. In the case $\mathcal{L}$ is the full rectangle $[1, m(1)] \times [1, m(2)]$, Proposition 5.1 gives a curious formula for the multiplicity of the classical determinantal ideal $I_{p+1}(X)$. It may be interesting to compare this with the more elegant formulae described in [12, p. 17].

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