CYCLIC HOMOLOGY AND GRAPH HOMOLOGY

SWAPNEEL MAHAJAN

Abstract. We present a way to view the cyclic homology of Connes in terms of the graph homology of Kontsevich, and justify this viewpoint by the following example. A result of Loday-Quillen and Tsygan computes the stable Lie algebra homology of $gl(\mathsf{A})$, matrices over an algebra A, in terms of the cyclic homology of A. As a generalization, for an operad P, we compute the stable Lie algebra homology of vector fields on the standard P-manifold in terms of a graph homology of P. When $\mathsf{P} = \mathsf{A}$, one recovers the previous result by noting that $gl(\mathsf{A})$ is the space of vector fields on the standard A-manifold. The symplectic and orthogonal cases are also briefly discussed.

Contents

Date: 2003.
2000 *Mathematics Subject Classification.* Primary 18D50, 17B65, 05C15; Secondary 53D55.
Key words. species; cyclic/reversible operad; derivation; mating functor; symplectic and orthogonal geometry; Poisson bracket; Lie superalgebra homology; cyclic, dihedral and graph homology.

1. Introduction

The goal of this paper is to show a way to view cyclic homology of algebras in the setting of graph homology of operads. We explain this by the following example.

1.1. **Lie algebra of matrices and cyclic homology.** Cyclic (co)homology first appeared in the work of Connes [3]. Almost immediately after, the cyclic homology of an algebra A was shown to be the primitive part of the Lie algebra homology of matrices by Loday-Quillen [16, 17] and Tsygan [23, 4]. Later Loday-Procesi [15] proved the analogue for symplectic and orthogonal matrices. This material is reviewed in Section 2, with these three results written as three cases of Theorem 1.

1.2. **Lie algebra of symplectic vector fields and graph homology.** A few years later, Kontsevich introduced graph homology [12, 13]. Using similar methods as above, he computed the homology of the Lie algebra of vector fields on certain noncommutative manifolds. He proved that

Theorem. $H_*^{Lie}\begin{pmatrix}\textit{The Lie algebra of symplectic vector}\\ \textit{fields on the standard } \mathsf{P}\textit{-manifold}\end{pmatrix} = \textit{Graph homology of } \mathsf{P},$

for $\mathsf{P} = \mathsf{c}, \mathsf{a}, \mathsf{l}$, the commutative, associative and Lie operads. His method extended to any cyclic/reversible operad P, see Conant-Vogtmann [2] or Mahajan [18]. The graph homology as required in the above result was defined for any cyclic operad P by Markl [19], following a general construction of Getzler-Kapranov [9]. We would also like to mention the work on Ginzburg on symplectic operad geometry [10], which is relevant to the left hand side of the theorem. The result of Loday-Procesi on symplectic matrices mentioned above can be seen as a special case of the above theorem.

At this point, it is natural to ask whether there is an orthogonal, or more simply, a general linear analogue of the above theorem. We present the answer in Section 3, see Theorem 2. Now all three results on cyclic homology mentioned above can be seen as a special case of this general theorem.

In Section 4, we recall the definitions of the graph complexes that we require. In the next two sections, we outline the proof of Theorem 2. Though there are no new ideas in this paper, we hope that it clarifies the relation between cyclic homology and graph homology as also the original proofs.

1.3. **Conventions and references.** For an operad P, it seems customary to assume that $\mathsf{P}[0] = 0$ and also many times that $\mathsf{P}[1] = \mathbb{K}$, the base field of characteristic 0. We do not make these assumptions since they are unnecessary for our purposes. However, we do assume that $\mathsf{P}[j]$ is finite dimensional for all $j \geq 0$.

The main reference for this paper is [18], where the reader will often be referred for skipped details. Apart from the references already mentioned above, the following give useful supplementary material.

- Bergeron-Labelle-Leroux [1] for species.
- Markl-Schnider-Stasheff [20], Ginzburg-Kapranov [11], Getzler-Kapranov [8], Voronov [24], Fresse for operads.
- Fuks [5], Weibel [25, Chapter 7] for Lie algebra homology.
- Loday [14, Chapter 9], Fulton-Harris [6], Weyl [26] for invariant theory.
- McDuff-Salamon [21] for symplectic geometry.
- Gerlits [7] for graph homology.

2. Three Lie algebras for an associative algebra A

In this section, we recall a result on the stable homology of three families of Lie algebras of matrices over an associative algebra. A detailed account can be found in Loday [14, Chapter 10].

Let A be an associative algebra with a unit.

Definition 2.1. $gl_n(\mathsf{A})$ = Lie algebra of $n \times n$ matrices over A.

Now let A be an algebra with an involution $* : \mathsf{A} \to \mathsf{A}$. For $\alpha \in gl_n(\mathsf{A})$, let $\alpha^\dagger \in gl_n(\mathsf{A})$ be defined by $(\alpha^\dagger)_{ij} = (\alpha_{ji})^*$.

Definition 2.2. $sp_{2n}(\mathsf{A})$ = Lie algebra of symplectic $2n \times 2n$ matrices over A. In other words,

$$sp_{2n}(\mathsf{A}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in gl_{2n}(\mathsf{A}) \;\middle|\; \alpha, \beta, \gamma, \delta \in gl_n(\mathsf{A}),\ \alpha = -\delta^\dagger,\ \beta = \beta^\dagger,\ \gamma = \gamma^\dagger \right\}.$$

Definition 2.3. $o_n(\mathsf{A})$ = Lie algebra of orthogonal $n \times n$ matrices over A. In other words,

$$o_n(\mathsf{A}) = \{\alpha \in gl_n(\mathsf{A}) \mid \alpha^\dagger = -\alpha\}.$$

More conceptual definitions of $sp_{2n}(\mathsf{A})$ and $o_n(\mathsf{A})$ are given in Lemmas 2 and 3 in Section 3.

Let $gl(\mathsf{A})$, $sp(\mathsf{A})$ and $o(\mathsf{A})$ denote the limit of the above three Lie algebras as $n \to \infty$. The homology H_*^{Lie} of these Lie algebras can be computed by the following theorem.

Theorem 1. *Let A be an algebra with an unit.*

$$\begin{array}{lrcl} (A) & H_*^{Lie}(gl(\mathsf{A}), \mathbb{K}) & = & \Lambda(HC_{*-1}(\mathsf{A})). \\ (C) & H_*^{Lie}(sp(\mathsf{A}), \mathbb{K}) & = & \Lambda(HD_{*-1}(\mathsf{A})). \\ (B+D) & H_*^{Lie}(o(\mathsf{A}), \mathbb{K}) & = & \Lambda(HD_{*-1}(\mathsf{A})). \end{array}$$

For the last two parts, one needs the algebra A to have an involution. In the right hand side, HC_* and HD_* refer to cyclic and dihedral homology respectively and Λ is the signed symmetric functor. Part (A) of the above theorem is due to Loday-Quillen [16, 17] and Tsygan [23, 4] and Parts (C) and $(B+D)$ are due to Loday-Procesi [15].

3. Three Lie algebras for an operad P

The Lie algebras of Section 2 can be seen as special cases of more general considerations, which we discuss in this section.

3.1. **General linear case.** Let P be an operad with a unit u. Let V_n be a vector space over $\mathbb{K}$ with basis $x_1, \ldots, x_n$. Then the free P-algebra on V_n is given by

$$\mathsf{P} \circ \mathsf{V}_n \;=\; \bigoplus_{j \geq 0} (\mathsf{P}[j] \otimes \mathsf{V}_n^{\otimes j})_{\Sigma_j},$$

where the symmetric group Σ_j acts on $\mathsf{V}_n^{\otimes j}$ by permuting the factors.

It is useful to consider a P-algebra as the space of functions on a P-manifold. For the above example, one says that $\mathsf{P} \circ \mathsf{V}_n$ are the polynomial functions on X_n, the "standard P-manifold of dimension n". The constant functions are $\mathsf{P}[0]$, while V_n are the coordinate functions.

Definition 3.1. $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n)$ = Lie algebra of derivations of $\mathsf{P}\circ\mathsf{V}_n$ = Lie algebra of "vector fields on X_n".

Example 1. Let $\mathsf{P}=\mathsf{c}$, the commutative operad, that is, $\mathsf{c}[n]=\mathbb{K}$ for $n>0$ and $\mathsf{c}[0]=0$. A c-manifold is same as a manifold and the standard c-manifold of dimension n is $\mathbb{R}^n$. The space $\mathsf{c}\circ\mathsf{V}_n$ is the algebra of polynomials in $x_1,\dots,x_n$ with no constant terms and $\mathrm{Der}(\mathsf{c}\circ\mathsf{V}_n)$ are polynomial vector fields on $\mathbb{R}^n$ that vanish at the origin.

If one wants to get all the polynomial functions and vector fields on $\mathbb{R}^n$ then one can consider $\mathsf{P}=1+\mathsf{c}$, that is, $(1+\mathsf{c})[n]=\mathbb{K}$ for all $n\geq 0$.

Example 2. Let A be an associative algebra with a unit. Then A_o is an operad with $\mathsf{A}_o[1]=\mathsf{A}$ and $\mathsf{A}_o[n]=0$ for $n\neq 1$. In this case, $\mathsf{A}_o\circ\mathsf{V}_n=\mathsf{A}\otimes\mathsf{V}_n$, the "polynomial functions on the standard A-manifold". A quick check shows that

Lemma 1. $\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_n)\cong gl_n(\mathsf{A})$.

Hence, $gl_n(\mathsf{A})$ is the Lie algebra of "vector fields on the standard A-manifold". As a special case, set A to be the base field $\mathbb{K}$. Then P is the unit operad u.

Corollary. $\mathrm{Der}(\mathsf{u}(\mathsf{V}_n))\cong gl_n$.

Thus, gl_n is the Lie algebra of "vector fields on the standard u-manifold".

3.2. **Symplectic case.** The reference for this material is [18, Sections 2-7], where the reader can find complete definitions, also see (6.2-6.3). Let P be a reversible operad and $\mathsf{Q}=\mathsf{PP}$ be its associated mated species. It is the image of P under the mating functor

$$\text{Mating Functor} \ : \ \mathcal{P}_r \longrightarrow \mathcal{S},$$

where $\mathcal{P}_r$ and $\mathcal{S}$ are the categories of reversible operads and species respectively. Let V_{2n} be a vector space over $\mathbb{K}$ with basis $p_1,\dots,p_n,q_1,\dots,q_n$. In this setting, it is more natural to consider

$$\mathsf{Q}\circ\mathsf{V}_{2n} \ = \ \bigoplus_{j\geq 0}(\mathsf{Q}[j]\otimes\mathsf{V}_{2n}^{\otimes j})_{\Sigma_j}, \tag{1}$$

instead of $\mathsf{P}\circ\mathsf{V}_{2n}$, as functions on the "standard P-manifold X_{2n}". One can also define the space of differential forms $\Omega(\mathsf{Q}\circ\mathsf{V}_{2n})$ on X_{2n} along with Lie derivative and contraction operators L_ξ, i_ξ for $\xi\in\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n})$, any vector field on X_{2n}. Further, X_{2n} carries the standard (alternating) symplectic form

$$\omega \ = \ \sum_i dp_i\wedge dq_i \ \in \ \Omega^2(\mathsf{Q}\circ\mathsf{V}_{2n}),$$

making it a symplectic P-manifold.

Definition 3.2. $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n},\omega)$ is the Lie algebra of "symplectic vector fields on X_{2n}". More precisely,

$$\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n},\omega)=\{\xi\in\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n}) \bigm| L_\xi\omega=0\}.$$

An alternate description is given in Lemma 7.

Example 3. Returning to Example 1, for $\mathsf{P}=1+\mathsf{c}$, the mated species $\mathsf{Q}=1+\mathsf{c}$. The standard P-manifold X_{2n} is the Euclidean space $\mathbb{R}^{2n}$. The space $\mathsf{Q}\circ\mathsf{V}_{2n}$

consists of polynomial functions on $\mathbb{R}^{2n}$ with $\mathsf{Q}[0]$ being the constants; $\Omega(\mathsf{Q}\circ\mathsf{V}_{2n})$ are differential forms on $\mathbb{R}^{2n}$ with

$$\omega \;=\; \sum_i dp_i\wedge dq_i \;\in\; \Omega^2(\mathbb{R}^{2n}).$$

The space $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n})$ is the Lie algebra of (polynomial) vector fields on $\mathbb{R}^{2n}$, while $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n},\omega)$ is the Lie subalgebra of symplectic vector fields on $\mathbb{R}^{2n}$, with the usual definition of the Lie derivative.

Example 4. Returning to Example 2, let A be an associative algebra with a unit and with an involution $*:\mathsf{A}\to\mathsf{A}$. Then A_o is a reversible operad. The mated species Q is given by $\mathsf{Q}[2]=\mathsf{A}$ and $\mathsf{Q}[n]=0$ for $n\neq 2$. The nontrivial element $\pi\in\Sigma_2$ acts on $\mathsf{Q}[2]$ by $\pi(a)=a^*$.

Lemma 2. $\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_{2n},\omega)\cong sp_{2n}(\mathsf{A})$.

Proof. We know from Example 2 that $\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_{2n})\cong gl_{2n}(\mathsf{A})$. Under this isomorphism, the element $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in gl_{2n}(\mathsf{A})$ corresponds to the derivation $\xi\in\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_{2n})$ given by

$$\xi(p_j)=\sum_i \alpha_{ij}\otimes p_i+\gamma_{ij}\otimes q_i,\quad \xi(q_j)=\sum_i \beta_{ij}\otimes p_i+\delta_{ij}\otimes q_i.$$

Let us compute $L_\xi\omega$.

$$\begin{aligned} L_\xi(\textstyle\sum_j dp_j\wedge dq_j) &= \textstyle\sum_{i,j}\alpha_{ij}\otimes(dp_i\wedge dq_j)+\gamma_{ij}\otimes(dq_i\wedge dq_j)\\ &+\textstyle\sum_{i,j}\beta^*_{ij}\otimes(dp_j\wedge dp_i)+\delta^*_{ij}\otimes(dp_j\wedge dq_i).\end{aligned}$$

Now using the relation $a\otimes(dx\wedge dy)=-a^*\otimes(dy\wedge dx)$, one obtains

$$L_\xi\omega=0\iff \alpha_{ij}+\delta^*_{ji}=0,\ \gamma_{ij}=\gamma^*_{ji},\ \beta_{ij}=\beta^*_{ji}.$$

The lemma follows from Definitions 2.2 and 3.2. □

Hence $sp_{2n}(\mathsf{A})$ is the space of "symplectic vector fields on the standard $2n$ dimensional A-manifold". As a special case, set A to be the base field $\mathbb{K}$. Then P is the unit operad u.

Corollary. $\mathrm{Der}(\mathsf{u}(\mathsf{V}_{2n}),\omega)\cong sp_{2n}$.

Thus, sp_{2n} is the Lie algebra of "symplectic vector fields on the standard $2n$ dimensional u-manifold".

3.3. **Orthogonal case.** This is an odd version of the symplectic case. Let P be a reversible operad and $\mathsf{Q}=\mathsf{PP}$ its associated mated species as before. Let V_n^- be a super vector space over $\mathbb{K}$ of dimension $(0|n)$, with basis $\theta_1,\theta_2,\ldots,\theta_n$. Then the free P-superalgebra on V_n^- is given by

$$\mathsf{P}\circ\mathsf{V}_n^- \;=\; \bigoplus_{j\geq 0}(\mathsf{P}[j]\otimes(\mathsf{V}_n^-)^{\otimes j})_{\Sigma_j},$$

where the symmetric group Σ_j acts on $(\mathsf{V}_n^-)^{\otimes j}$ by permuting the factors via the sign representation.

Here, one can let X_n^- be the "standard P-supermanifold of dimension $(0|n)$". Then $\mathsf{P}\circ\mathsf{V}_n^-$ are the polynomial functions on X_n^-, with V_n^- being the coordinate functions and $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n^-)$ is the Lie superalgebra of "vector fields on X_n^-". One can also define without difficulty, differential forms, Lie derivatives, etc in the super-context (more details in Section 6.2). The supermanifold X_n^- carries a symmetric two tensor

$$\rho \;=\; \sum_i d\theta_i \otimes d\theta_i \;\in\; \Omega^2(\mathsf{Q}\circ \mathsf{V}_n^-).$$

Definition 3.3. $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n^-,\rho)$ is the Lie superalgebra of "orthogonal vector fields on X_n^-". More precisely,

$$\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n^-,\rho) = \{\xi \in \mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n^-) \;\big|\; L_\xi\rho = 0\}.$$

Example 5. Returning to Example 4, let A be an associative algebra with a unit and with an involution $* : \mathsf{A}\to\mathsf{A}$. Then A_o is a reversible operad.

Lemma 3. $\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_n^-,\rho)\cong o_n(\mathsf{A})$.

Proof. We know from Example 2 that $\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_n^-)\cong gl_n(\mathsf{A})$. Replacing V_n by V_n^- does not matter, since $\mathsf{A}_o\circ\mathsf{V}_n^- = \mathsf{A}\otimes\mathsf{V}_n^-$ is concentrated in degree 1; hence $\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_n^-,\rho)$ is a Lie algebra (as opposed to a Lie superalgebra), all derivations being of degree 0.

Under the above isomorphism, the element $\alpha\in gl_n(\mathsf{A})$ corresponds to the derivation $\xi\in\mathrm{Der}(\mathsf{A}_o\circ\mathsf{V}_n^-)$ given by

$$\xi(\theta_j) = \sum_i \alpha_{ij}\otimes\theta_i.$$

Let us compute $L_\xi\rho$.

$$L_\xi\Big(\sum_j d\theta_j\otimes d\theta_j\Big) = \sum_{i,j}\alpha_{ij}\otimes d\theta_i\otimes d\theta_j + \alpha_{ij}^*\otimes d\theta_j\otimes d\theta_i.$$

Now using the relation $a\otimes d\theta\otimes d\psi = a^*\otimes d\psi\otimes d\theta$, one obtains

$$L_\xi\rho = 0 \iff 2(\alpha_{ij}+\alpha_{ji}^*) = 0.$$

The lemma follows from Definitions 2.3 and 3.3. □

Hence $o_n(\mathsf{A})$ is the space of "orthogonal vector fields on the standard $(0|n)$ dimensional A-supermanifold". As a special case, set A to be the base field $\mathbb{K}$. Then P is the unit operad u.

Corollary. $\mathrm{Der}(\mathsf{u}(\mathsf{V}_n^-),\rho)\cong o_n$.

Thus, o_n is the Lie algebra of "orthogonal vector fields on the standard $(0|n)$ dimensional u-supermanifold".

3.4. **Main theorem.** For an operad P, we have defined three families of Lie algebras $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n)$, $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_{2n},\omega)$ and $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n^-,\rho)$, the last being a Lie superalgebra. Let $\mathrm{Der}(gl,\mathsf{P})$, $\mathrm{Der}(sp,\mathsf{P})$ and $\mathrm{Der}(o,\mathsf{P})$, denote the limit of these families as $n\to\infty$. Their homology H_*^{Lie} can be computed as follows.

Theorem 2. *Let P be an operad with an unit.*

$$\begin{array}{rrcl} (A) & H_*^{Lie}(\mathrm{Der}(gl,\mathsf{P}),\mathbb{K}) & = & \Lambda(H_*(\mathcal{C}(gl,\mathsf{P}))).\\ (C) & H_*^{Lie}(\mathrm{Der}(sp,\mathsf{P}),\mathbb{K}) & = & \Lambda(H_*(\mathcal{C}(sp,\mathsf{P}))).\\ (B+D) & H_*^{Lie}(\mathrm{Der}(o,\mathsf{P}),\mathbb{K}) & = & \Lambda(H_*(\mathcal{C}(o,\mathsf{P}))).\end{array}$$

For the last two parts, one needs P to be reversible. The right hand sides are certain graph complexes associated to P (see Section 4) and Λ is the signed symmetric functor. For Part (A), one can say the following.

Corollary. *If* $\mathsf{P}[0] = 0$ *then* $H_*^{Lie}(\mathrm{Der}(gl, \mathsf{P}), \mathbb{K}) = \Lambda(HC_{*-1}(\mathsf{P}[1]))$.

Proof. If P is an operad with a unit then $\mathsf{P}[1]$ is an associative algebra with a unit; hence it makes sense to talk of the cyclic homology of $\mathsf{P}[1]$. And if $\mathsf{P}[0] = 0$ then the graph complex $\mathcal{C}(gl, \mathsf{P})$ coincides with Connes' complex for computing the cyclic homology of $\mathsf{P}[1]$, see Lemma 4. □

If $\mathsf{P} = \mathsf{A}$ for A an associative algebra (Example 2) then in Theorem 2, the left hand sides specialize to $gl(\mathsf{A})$, $sp(\mathsf{A})$ and $o(\mathsf{A})$ respectively (see Lemmas 1, 2 and 3) while the right hand sides specialize to cyclic and dihedral homology of A respectively; thus one recovers Theorem 1. Part (A) of Theorem 1 is more transparent from the above corollary.

As already mentioned, Kontsevich proved Theorem 2, Part (C) for $\mathsf{P} = \mathsf{c}, \mathsf{a}, \mathsf{l}$, the commutative, associative and Lie operads. In [18, Theorem 4, Proposition 4], the theorem is proved for any reversible operad in the category of Sets, and full credit is given to Kontsevich for the ideas involved. In [2, Corollary 5], it is proved for any cyclic operad without restriction, with a different but isomorphic Lie algebra in the left hand side. In [2, 18], it is assumed that $\mathsf{P}[0] = 0$ and a further reduction is done on the graph complex $\mathcal{C}(sp, \mathsf{P})$, see [2, Proposition 14], or [18, Proposition 5].

4. Graph homology

In this section, we give the definitions of the graph complexes that occur in Theorem 2. More details on some of it can be found in [18, Sections 8-9], where plenty of examples are discussed.

Definition 4.1. A graph is a 1 dimensional CW complex. For a graph Γ, we denote the set of vertices by $V(\Gamma)$, the set of edges by $E(\Gamma)$, the set of ends of an edge e by $V(e)$ and the set of edges incident at a vertex v by $E(v)$.

Definition 4.2. For a set S, let $\mathbb{K}\,\mathsf{S}$ be the vector space over $\mathbb{K}$ which has the elements of S as a basis. Let $\mathsf{S} = \mathsf{S}_0 \sqcup \mathsf{S}_1$ be a super set of cardinality $(k|l)$. Then $\mathsf{W} = \mathsf{W}_0 \oplus \mathsf{W}_1 = \mathbb{K}\,\mathsf{S}$ is a super vector space of dimension $(k|l)$, with $\mathsf{W}_0 = \mathbb{K}\,\mathsf{S}_0$ and $\mathsf{W}_1 = \mathbb{K}\,\mathsf{S}_1$. Define the super determinant $\det \mathsf{W}$ to be the one dimensional quotient of $\mathsf{W}^{\otimes(k+l)}$ with the relations:

- A $(k+l)$ tensor of elements of S is zero if there is repetition of elements.
- Switching adjacent factors s, s' in a $(k+l)$ tensor of elements of S incurs a minus sign except when both $s, s' \in \mathsf{S}_1$, in which case the sign is positive.

Note that if S_1 is empty then $\det \mathsf{W} = \Lambda^k \mathsf{W}$. We will be in this case, except for Definition 4.7, where the super version is necessary.

4.1. **P-graph.** Let P be an operad. A P-graph is a directed graph Γ such that for every vertex v, there is exactly one outgoing edge and a P-structure is specified on the set of incoming edges at v.

Figure 1 shows a P-graph with 5 vertices and 5 edges. The edges are drawn broken to emphasize that the graph is made of 5 operad elements with $p_1, p_2 \in \mathsf{P}[0]$, $p_3, p_4 \in \mathsf{P}[2]$ and $p_5 \in \mathsf{P}[1]$. Recall from [18, Section 2.2], that the generic picture

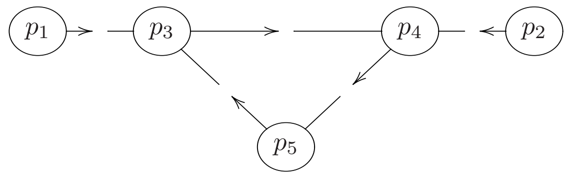
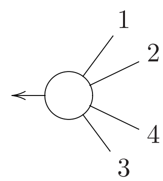
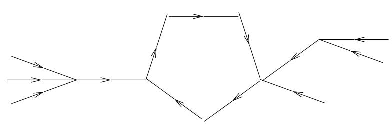


FIGURE 1. P-graph.

for an element in, say, $\mathsf{P}[4]$ is



Remark. The underlying graph Γ of a P-graph is a polygon with trees attached to each vertex.



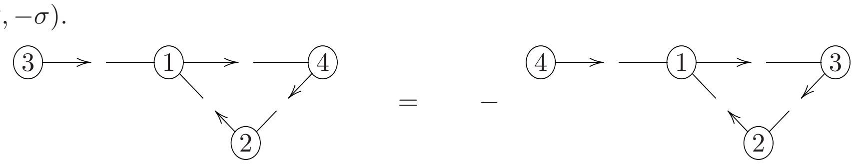
If $\mathsf{P}[0] = 0$ then there are no valence 1 vertices and the trees are necessarily empty. Hence Γ, in this case, is just a polygon and the P-graph uses only the $\mathsf{P}[1]$ part of the operad P.

4.2. **Oriented P-graph.** We will use the letter Γ to denote a P-graph as well as its underlying graph.

Definition 4.3. An orientation σ of a graph Γ is an element of the one-dimensional vector space $\det \mathbb{K}V(\Gamma)$. We say that (Γ, σ) is an oriented P-graph. A way to represent an orientation σ is to order the vertices of Γ. An odd permutation of the labels on the vertices reverses the orientation to $-\sigma$.

4.3. **Graph complex $\mathcal{C}(gl, \mathsf{P})$.** We now define the chain complex $\mathcal{C}(gl, \mathsf{P})$.

Definition 4.4. The kth chain group of $\mathcal{C}(gl, \mathsf{P})$, which we denote $\mathcal{C}_k(gl, \mathsf{P})$, is the vector space over $\mathbb{K}$ generated by all oriented connected graphs (Γ, σ) with k vertices, upto automorphism, subject to vertex linearity and the relation $(\Gamma, \sigma) = -(\Gamma, -\sigma)$.



This is illustrated in the picture above.

Definition 4.5. The boundary map $\partial_E : \mathcal{C}_k(gl, \mathsf{P}) \to \mathcal{C}_{k-1}(gl, \mathsf{P})$ is defined using edge contractions. We do not contract loops. More precisely, we have

$$\partial_E(\Gamma, \sigma) = \sum_{e \in E(\Gamma)} (\Gamma/e, \sigma/e),$$

where Γ/e is the graph Γ with the edge e contracted using operad substitution, and σ/e is obtained the following way: let v_1 and v_2 be the ends of the edge e. Choose a representative of σ where v_1 and v_2 have labels 1 and 2 respectively, and e points from v_1 to v_2. Give the new vertex arising from the contraction of e the label 1, and subtract 1 from the label of each of the other vertices.

An equivalent way to describe σ/e is the following: if the labels on the endpoints of e are $i < j$, collapse e, label the resulting vertex i, decrease the labels greater than j by one, and multiply this orientation by $(-1)^j$ if e points from i to j, and by $(-1)^{j+1}$ if it points from j to i.

The associativity property of operad substitution and the choice of sign imply that $\partial_E^2 = 0$. This defines the chain complex $\mathcal{C}(gl, \mathsf{P}) = (\mathcal{C}_*(gl, \mathsf{P}), \partial_E)$.

Lemma 4. *If $\mathsf{P}[0] = 0$ then the complex $\mathcal{C}(gl, \mathsf{P})$, upto a shift in grading, is isomorphic to Connes' complex for the cyclic homology of $\mathsf{P}[1]$.*

4.4. **Q-graph.** Let Q be any species. Define a Q-graph to be a graph Γ such that for every vertex v of Γ, a Q-structure is specified on the set of half-edges incident to v. Figure 2 shows a Q-graph with 4 vertices and 7 edges, drawn using the generic

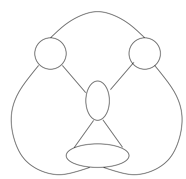


FIGURE 2. Q-graph.

picture of a species as explained in [18, Section 2.1].

4.5. **Oriented Q-graph.** We define two different notions of orientation for a Q-graph.

Definition 4.6. An orientation σ of a Q-graph Γ is an element of the one dimensional vector space $\det \mathbb{K}V(\Gamma) \otimes \bigotimes_{e \in E(\Gamma)} \det \mathbb{K}V(e)$. We say that (Γ, σ) is an oriented Q-graph. There is another notion of orientation of a graph equivalent to the above; see Thurston [22] for details.

A way to represent an orientation is to order the vertices and orient each edge of the graph. An odd permutation of the labels on the vertices reverses the orientation, and a single change of the orientation of one edge reverses it as well. An even number of these transformations produces an orientation equivalent to the original one.

Definition 4.7. Consider $V(\Gamma)$ as a super set with vertices of even (resp. odd) degree as the even (resp. odd) part. An odd orientation σ^- of a Q-graph Γ is an element of the one-dimensional vector space $\det \mathbb{K}V(\Gamma) \otimes \bigotimes_{v \in V(\Gamma)} \det \mathbb{K}E(v)$. We say that (Γ, σ^-) is an odd oriented Q-graph.

A way to represent an odd orientation is to order the vertices and for every vertex, order the edges incident on it. Switching adjacent labels on the vertices, reverses the orientation unless both vertices have odd degree. And for a vertex, an odd permutation of the labels on the edges incident to it reverses the orientation.

4.6. **Graph complexes $\mathcal{C}(sp,\mathsf{P})$ and $\mathcal{C}(o,\mathsf{P})$.** We assume that Q is a mated species, that is $\mathsf{Q}=\mathsf{PP}$ for a reversible operad P.

Definition 4.8. The kth chain group of $\mathcal{C}(sp,\mathsf{P})$, which we denote $\mathcal{C}_k(sp,\mathsf{P})$, is the vector space over $\mathbb{K}$ generated by all connected oriented Q-graphs (Γ,σ) with k vertices, upto automorphism, subject to vertex linearity and the relation $(\Gamma,\sigma)=-(\Gamma,-\sigma)$. The boundary map $\partial_E:\mathcal{C}_k(sp,\mathsf{P})\to\mathcal{C}_{k-1}(sp,\mathsf{P})$ is defined exactly as in Definition 4.5, except that an edge is contracted using a mating, rather than an operad substitution. This is where one uses that Q is a mated species.

Definition 4.9. The chain complex $\mathcal{C}(o,\mathsf{P})$ is defined similarly to $\mathcal{C}(sp,\mathsf{P})$, using connected odd oriented Q-graphs. The induced orientation σ^-/e is obtained the following way: let v_1 and v_2 be the ends of the edge e. Choose a representative of σ^- where v_1 and v_2 have labels 1 and 2 respectively, the edge e has label 1 for both v_1 and v_2; give the new vertex arising from the contraction of e the label 1, and subtract 1 from the label of each of the other vertices; shift up the labels on the edges that were incident to v_2, and multiply by the sign $(-1)^{|v_1|}$.

Lemma 5. *For the operad $\mathsf{P}=\mathsf{A}$ as in Example 4, the complexes $\mathcal{C}(sp,\mathsf{P})$ and $\mathcal{C}(o,\mathsf{P})$ are isomorphic. And upto a shift in grading, they are isomorphic to the complex $(C(\mathsf{A})/(1-t,1-y),b)$, see Loday [14, Section 5.2.8], that computes the dihedral homology of A.*

This is a simple check. At some point in the sequel, we will need to work with disconnected graphs. We will denote the corresponding chain complexes by $\mathcal{G}(gl,\mathsf{P})$, $\mathcal{G}(sp,\mathsf{P})$ and $\mathcal{G}(o,\mathsf{P})$.

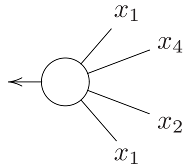
5. Proof of Theorem 2, part (A)

Recall that V_n is the vector space over $\mathbb{K}$ with basis $x_1,x_2,\dots,x_n$ and $\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n)$ is the Lie algebra of derivations of the free P-algebra $\mathsf{P}\circ\mathsf{V}_n$. Put $\mathfrak{g}_n=\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n)$ and $\mathfrak{g}=\mathrm{Der}(gl,\mathsf{P})$ for the limit as $n\to\infty$. We restate the result that we are trying to prove.

Theorem 3. $H^{Lie}_*(\mathfrak{g},\mathbb{K})=\Lambda(H_*(\mathcal{C}(gl,\mathsf{P})))$, *with $\mathcal{C}(gl,\mathsf{P})$ as defined in (4.3).*

Proof. We repeat the proof in Loday-Quillen [17] idea for idea, rewriting it in a way that is serves as a toy model for the proof in Kontsevich [12] for Theorem 2, Part (C). Before starting the actual proof, we need a little preparation.

5.1. **Lie algebra $\mathfrak{g}_n$.** As in [18, Section 5.1], we represent a monomial in the free P-algebra $\mathsf{P}\circ\mathsf{V}_n$ by a picture of the form

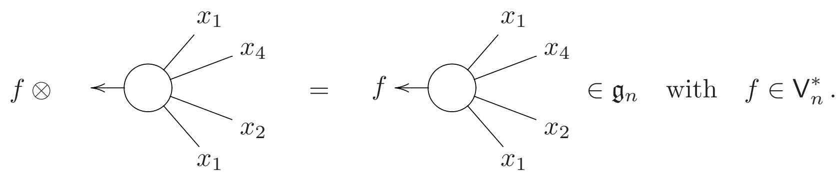


$\in\mathsf{P}\circ\mathsf{V}_n$ with $p\in\mathsf{P}[4]$ and $x_i\in\mathsf{V}_n$.

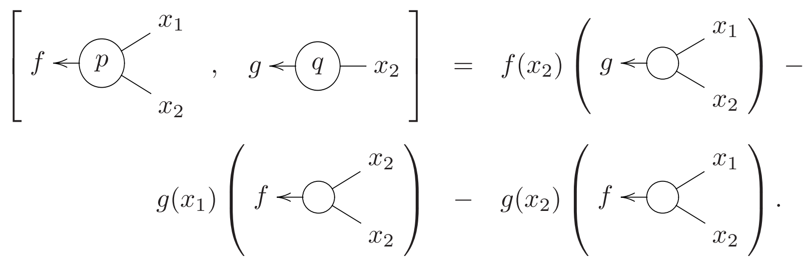
To get a general element of $\mathsf{P}\circ\mathsf{V}_n$, we take linear combinations of monomials.

Now $\mathfrak{g}_n=\mathrm{Der}(\mathsf{P}\circ\mathsf{V}_n)\cong\mathrm{Hom}(\mathsf{V}_n,\mathsf{P}\circ\mathsf{V}_n)\cong\mathsf{V}_n^*\otimes\mathsf{P}\circ\mathsf{V}_n$. Hence to get an element of $\mathfrak{g}_n$, we take an operad element p and label its inputs by elements of V_n

and its output by an element of V_n^*, as shown below.



And to get a general element of $\mathfrak{g}_n$, we take linear combinations of these. One can describe the bracket on $\mathfrak{g}_n$ in this notation. We illustrate by an example.



In the first term on the right, p is substituted into q and in the next two, q is substituted into each input of p. For each term, we pick a coefficient given by contracting an element of V^* with an element of V, along with the appropriate sign.

5.2. **Lie subalgebra gl_n.** Since the free P-algebra $\mathsf{P} \circ \mathsf{V}_n$ is graded,

$$\mathfrak{g}_n \;=\; \bigoplus_{j \geq 0} \mathsf{V}_n^* \otimes (\mathsf{P}[j] \otimes \mathsf{V}_n^{\otimes j})_{\Sigma_j} \;=\; \bigoplus_{j \geq 0} \mathfrak{g}_n^j. \tag{2}$$

is a graded Lie algebra, the $(j-1)$st graded piece being $\mathfrak{g}_n^j$. Note that the grading begins in degree -1. The space of degree 0 (linear) derivations of $\mathsf{P} \circ \mathsf{V}_n$, namely $\mathsf{V}_n^* \otimes \mathsf{P}[1] \otimes \mathsf{V}_n$, is a Lie subalgebra of $\mathfrak{g}_n$. Since the operad P has a unit u,

$$gl_n = \mathrm{Hom}(\mathsf{V}_n, \mathsf{V}_n) = \mathsf{V}_n^* \otimes u[1] \otimes \mathsf{V}_n$$

is a Lie subalgebra of the space of linear derivations of $\mathsf{P} \circ \mathsf{V}_n$.

Proposition 1. *The adjoint action of gl_n on $\mathfrak{g}_n$ coincides with the one induced by the usual action of gl_n on V_n and trivial action on the $\mathsf{P}[j]$'s.*

The proof is a straightforward check.

5.3. **Lie algebra homology and (co)invariant theory.** We now start the proof of the theorem. The Lie algebra homology $H_*^{Lie}(\mathfrak{g}_n, \mathbb{K})$ can be computed using the Chevalley-Eilenberg complex $(\Lambda^* \mathfrak{g}_n, \partial)$, where $\Lambda^k \mathfrak{g}_n$ is the kth exterior power of $\mathfrak{g}_n$. The reductive Lie subalgebra gl_n acts on $\mathfrak{g}_n$ (adjoint action) and hence on $\Lambda^k \mathfrak{g}_n$. It is well-known that the adjoint action commutes with the boundary operator ∂.

Proposition 2. *The maps φ and ψ in the diagram*

$$(\Lambda^* \mathfrak{g}_n)^{gl_n} \xrightarrow{\phi} \Lambda^* \mathfrak{g}_n \xrightarrow{\psi} (\Lambda^* \mathfrak{g}_n)_{gl_n}$$

are both quasi-isomorphisms, that is, they induce an isomorphism on homology.

The proof is a standard argument that we skip. This proposition is the main tool in the proof. As vector spaces, $(\Lambda^k \mathfrak{g}_n)^{gl_n} \cong (\Lambda^k \mathfrak{g}_n)_{gl_n}$. The next step is to construct explicitly a space $\mathcal{G}_k(gl, \mathsf{P})$ isomorphic to the above spaces, along with an explicit description of the diagram

$$\mathcal{G}_k(gl, \mathsf{P}) \xrightarrow{\phi} \Lambda^k \mathfrak{g}_n \xrightarrow{\psi} \mathcal{G}_k(gl, \mathsf{P}). \tag{3}$$

In the remainder of this section, we will denote $\mathcal{G}_k(gl, \mathsf{P})$ simply by $\mathcal{G}_k$.

5.4. **Basis for the space of (co)invariants $\mathcal{G}_k$.** This problem can be solved by looking at a simpler problem first.

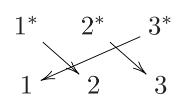
5.4.1. *The first stage.* Consider the gl_n module $M = (\mathsf{V}_n^*)^{\otimes i} \otimes (\mathsf{V}_n)^{\otimes j}$. We would like an explicit understanding of the diagram

$$M^{gl_n} \xrightarrow{i} M \xrightarrow{p} M_{gl_n}. \tag{4}$$

From classical invariant theory of gl_n, one knows that if $i \neq j$ then $M^{gl_n} = 0$. Hence from now on, we only consider the case $i = j$. And for $\dim \mathsf{V}_n > i$, the space $M^{gl_n} \cong \mathbb{K}\Sigma_i$.

The map $i : \mathbb{K}\Sigma_i \hookrightarrow M$.

An element $\pi \in \Sigma_i$ specifies a bijection between the i copies of V_n^* and the i copies of V_n in M. We represent this by a directed chord diagram with $2i$ vertices labelled $1, 2, \ldots, i, 1^*, 2^*, \ldots, i^*$ and i directed edges connecting them. The edges are directed away from the $*$ vertices. For example, for $\pi = (123) \in \Sigma_3$, written in the cycle notation, we draw



.

Each vertex in the chord diagram represents a tensor factor, in the order given by the vertex labelling. For each edge, we put a x_i at the head of the arrow and a x_i^* at the tail. We then sum over all possibilities to get the invariant. In the above example, the invariant is

$$\sum_{1 \leq i,j,k \leq n} x_i^* \otimes x_j^* \otimes x_k^* \otimes x_k \otimes x_i \otimes x_j.$$

The map $p : M \twoheadrightarrow \mathbb{K}\Sigma_i$.

One can describe the map p in diagram (4) by dualising i to get a map $M^* \to (\mathbb{K}\Sigma_i)^*$ and then using the identifications $M \cong M^*$ (remember $i = j$) and $(\mathbb{K}\Sigma_i)^* \cong \mathbb{K}\Sigma_i$. The resulting map $p : M \twoheadrightarrow \mathbb{K}\Sigma_i$ is given by

$$p(m) = \sum_{\pi \in \Sigma_i} \langle m, \pi \rangle \; \pi,$$

where $\langle m, \pi \rangle$ is obtained by writing the tensor factors of m on the corresponding vertices of the chord diagram for π and contracting elements of V^* with V along an edge. For example,

$$\langle f \otimes g \otimes h \otimes x \otimes y \otimes z, (123) \rangle = f(y)g(z)h(x).$$

5.4.2. *Second stage.* Now we go back to the problem of describing diagram (3). With notation as in equation (2), observe that

$$\Lambda^k \mathfrak{g}_n = \bigoplus_{\substack{k_1+k_2+\ldots+k_r=k,\, k_i\geq 1 \\ 0\leq j_1<j_2<\ldots<j_r}} \left(\Lambda^{k_1}\mathfrak{g}_n^{j_1}\otimes\cdots\otimes\Lambda^{k_r}\mathfrak{g}_n^{j_r}\right).$$

For a fixed choice of the numbers k_t, j_t for $1\leq t\leq r$, the summand on the right hand side is L_Σ, the quotient of the space

$$L = \bigotimes_{t=1}^{r} (\mathsf{V}_n^* \otimes \mathsf{P}[j_t] \otimes \mathsf{V}_n^{\otimes j_t})^{\otimes k_t}$$

by the group

$$\Sigma = \times_{t=1}^{r}((\Sigma_{j_t}\times\ldots\times\Sigma_{j_t})\times\Sigma_{k_t}),$$

with the Σ_{k_t}'s permuting the factors in the respective tensor summand via the sign representation (because of the wedges).

From Proposition 1, the adjoint action of gl_n on $L\subset\Lambda^k\mathfrak{g}_n$ coincides with the action of gl_n on L induced from the usual action on V_n (and V_n^*) and the trivial action on the $\mathsf{P}[j]$'s. One can check that the actions of gl_n and Σ on L commute. Hence,

$$(L_\Sigma)^{gl_n} \cong (L^{gl_n})_\Sigma \quad \text{and} \quad (L_\Sigma)_{gl_n} \cong (L_{gl_n})_\Sigma.$$

Note that

$$L^{gl_n}\cong L_{gl_n}\neq 0 \iff \sum_{t=1}^{r} k_t(j_t-1)=0 \iff \begin{array}{l}\text{The number of copies of } \mathsf{V} \text{ and } \mathsf{V}^* \\ \text{occurring in } L \text{ are equal.}\end{array}$$

In this case, a typical element of $(L^{gl_n})_\Sigma$ is obtained by tensoring a chord diagram (as in the first stage) by

$$\bigotimes_{r=1}^{t} \mathsf{P}[j_t]^{\otimes k_t}$$

and then moding out by the action of Σ. This is precisely an oriented P-graph (4.2). Hence, one sees that $(L^{gl_n})_\Sigma \cong (L_{gl_n})_\Sigma$ is spanned by oriented P-graphs, each having k vertices with k_t vertices of degree j_t+1, for $1\leq t\leq r$. It follows that

$$\mathcal{G}_k \cong (\Lambda^k\mathfrak{g}_n)^{gl_n} \cong (\Lambda^k\mathfrak{g}_n)_{gl_n}$$

is the span of oriented P-graphs with k vertices. The description of the maps $\varphi:\mathcal{G}_k\hookrightarrow\Lambda^k\mathfrak{g}_n$ and $\psi:\Lambda^k\mathfrak{g}_n\twoheadrightarrow\mathcal{G}_k$ given below follow from those of i and p in the first stage.

The map $\varphi:\mathcal{G}_k\hookrightarrow\Lambda^k\mathfrak{g}_n$.

Let $(\Gamma,\sigma)\in\mathcal{G}_k$ be an oriented P-graph with k vertices. Choose a representative for σ, that is, order the vertices of Γ. Each vertex of the graph represents a wedge factor, in the order given by the vertex labelling. For each edge, we put a x_i at the head of the arrow and a x_i^* at the tail. This is called a state of the edge. And a *state* of the graph is a choice of a state for every edge. Summing over all states of Γ gives the required element in $\Lambda^k(\mathfrak{g}_n)$.

The map $\psi:\Lambda^k\mathfrak{g}_n\twoheadrightarrow\mathcal{G}_k$.

Let $g_1\wedge\ldots\wedge g_k\in\Lambda^k(\mathfrak{g}_n)$, with $g_i\in\mathfrak{g}_n^{m_i}$. Choose a bijection between the copies of V^* and the copies of V. It exists only if $\sum_{i=1}^{k}(m_i+1)=0$. If this is not the case then the element maps to zero. Using the picture representation for elements of $\mathfrak{g}_n$

suggested before, one sees that such a bijection gives an oriented P-graph with a coefficient given by contracting elements of V^* with V. We sum over all bijections to get the element of $\mathcal{G}_k$.

5.5. **Conclusion of the proof.** Proposition 2 says that $H_*^{Lie}(\mathfrak{g}_n, \mathbb{K})$ can be computed using either the chain complex of gl_n invariants or coinvariants. So far, we have treated them on an equal footing and obtained a description of the chain groups $\mathcal{G}_k$, which are same in both cases and the maps i and p. However, from now on, coinvariants are much nicer to work with. Let $\mathcal{G}(gl, \mathsf{P}) = (\mathcal{G}_*, \partial)$ be the chain complex with chain groups $\mathcal{G}_k$ and boundary operator $\partial : \mathcal{G}_k \to \mathcal{G}_{k-1}$ defined at the end of Section 4.

Lemma 6. *The map $\psi : (\Lambda^* \mathfrak{g}_n, \partial) \to (\mathcal{G}, \partial)$ is a chain map, that is, the following diagram commutes.*

$$\begin{tikzcd} \Lambda^k \mathfrak{g}_n \arrow[r, "\psi"] \arrow[d, "\partial"] & \mathcal{G}_k \arrow[d, "\partial"] \\ \Lambda^{k-1} \mathfrak{g}_n \arrow[r, "\psi"] & \mathcal{G}_{k-1} \end{tikzcd} .$$

This follows directly from the definitions of the maps involved. Using Proposition 2, we conclude that $H_*^{Lie}(\mathfrak{g}_n, \mathbb{K})$ stabilises as $n \to \infty$ and

$$H_*^{Lie}(\mathfrak{g}_n, \mathbb{K}) \cong H_*(\mathcal{G}_*, \partial).$$

Observe that $(\mathcal{G}_*, \partial)$ is a differential graded commutative and cocommutative Hopf algebra with connected oriented P-graphs as the space of primitive elements and product given by disjoint union. It is easy to check that the induced structure on homology is same as the Hopf algebra structure on $H_*^{Lie}(\mathfrak{g}_n, \mathbb{K})$. This finishes the proof.

$\square$

Remark. The map $\varphi : (\mathcal{G}_*, \partial) \to (\Lambda^* \mathfrak{g}_n, \partial)$ is not a chain map. There is a different boundary map for $\mathcal{G}$, which depends on n, for which it is a chain map. Hence φ is less useful than ψ for stability purposes and plays no role in the proof.

6. Proof of the rest of Theorem 2

In this section, we outline the proof of Theorem 2, parts (C) and $(B + D)$. Our main goal is to point out the similarities and differences with the proof of part (A) and for part (C), clarify some steps in the proofs given in [2, 18].

6.1. Proof of Theorem 2, Part (C).

6.1.1. *Lie algebra $\mathfrak{g}_{2n}^{\omega}$.* One would like to describe the Lie algebra $\mathfrak{g}_{2n}^{\omega} = \mathrm{Der}(\mathsf{P} \circ \mathsf{V}_{2n}, \omega)$ using pictures, in a way similar to $\mathfrak{g}_n$. This is done by using ideas from symplectic operad geometry, as below. There are some details in (6.2-6.3).

Lemma 7. *There is a split short exact sequence of Lie algebras*

$$0 \to \mathsf{Q}[0] \hookrightarrow \mathsf{Q} \circ \mathsf{V}_{2n} \twoheadrightarrow \mathrm{Der}(\mathsf{P} \circ \mathsf{V}_{2n}, \omega) \to 0, \tag{5}$$

TABLE 1.

	A	C	B+D
Linear Lie algebra h	gl_n	sp_{2n}	o_n
h module M	$(\mathsf{V}_n^*)^{\otimes i} \otimes (\mathsf{V}_n)^{\otimes j}$	$(\mathsf{V}_{2n})^{\otimes i}$	$(\mathsf{V}_n^-)^{\otimes i}$
Chord diagrams	$\mathbb{K}\Sigma_i$	$\mathbb{K}C_i$	$\mathbb{K}C_i^-$
Lie algebra	$\mathfrak{g}_n$	$\mathfrak{g}_{2n}^{\omega}$	$\mathfrak{g}_n^{\rho}$
(Co)invariants	$\mathcal{G}_k(gl, \mathsf{P})$	$\mathcal{G}_k(sp, \mathsf{P})$	$\mathcal{G}_k(o, \mathsf{P})$

with the bracket on the space of functions $\mathsf{Q} \circ \mathsf{V}_{2n}$ on X_{2n} (see equation (1)) given by the Poisson bracket

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \otimes \frac{\partial H}{\partial q_i} - \frac{\partial F}{\partial q_i} \otimes \frac{\partial H}{\partial p_i}, \quad \text{for} \quad F, H \in \mathsf{Q} \circ \mathsf{V}_{2n}\,. \tag{6}$$

The precise meaning of this formula is explained in [18, Sections 5.1-5.3]. *The above map $\mathsf{Q} \circ \mathsf{V}_{2n} \twoheadrightarrow \mathrm{Der}(\mathsf{P} \circ \mathsf{V}_{2n}, \omega)$ is given by $H \mapsto \xi_H$, where $\xi_H(p_i) = \frac{\partial H}{\partial q_i}$ and $\xi_H(q_i) = -\frac{\partial H}{\partial p_i}$. With these conventions, it is a Lie algebra anti-homomorphism.*

The reader may be familiar with the above short exact sequence in (commutative) symplectic geometry (Example 3), where it says that the Lie algebra of Hamiltonian functions on $(\mathbb{R}^{2n}, \omega)$ with the Poisson bracket is a central extension of the Lie algebra of Hamiltonian vector fields on $(\mathbb{R}^{2n}, \omega)$ by the constants. Note that since $H^1(\mathbb{R}^{2n}) = 0$, the space of symplectic and Hamiltonian vector fields coincide. This fact continues to hold in the operad setting, see equation (7).

6.1.2. *Symplectic Lie algebra sp_{2n}.* Since P has an unit u, by the Corollary to Lemma 2 and arguments as in (5.2), the symplectic Lie algebra sp_{2n} is a Lie subalgebra of $\mathfrak{g}_{2n}^{\omega}$. For more clarity, one can draw the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Der}(u(\mathsf{V}_{2n})) = gl_{2n} \arrow[r, hook] & \mathfrak{g}_{2n} = \mathrm{Der}(\mathsf{P} \circ \mathsf{V}_{2n}) \\
\mathrm{Der}(u(\mathsf{V}_{2n}), \omega) = sp_{2n} \arrow[r, hook] \arrow[u, hook] & \mathfrak{g}_{2n}^{\omega} = \mathrm{Der}(\mathsf{P} \circ \mathsf{V}_{2n}, \omega) \arrow[u, hook]
\end{tikzcd}\ .$$

Applying Lemma 7 to the unit operad u, which satisfies $u[0] = 0$, one obtains:

Corollary. *There is a Lie algebra anti-isomorphism $uu(\mathsf{V}_{2n}) \xrightarrow{\cong} sp_{2n}$.*

Proposition 3. *The sequence in equation (5) is a sequence of sp_{2n} modules with the adjoint action on $\mathfrak{g}_{2n}^{\omega}$ and the action on $\mathsf{Q} \circ \mathsf{V}_{2n}$ induced by the usual action of sp_{2n} on V_{2n} and the trivial action on the $\mathsf{Q}[j]$'s.*

For a detailed discussion, see [18, Section 5.4].

6.1.3. *The rest of the proof.* The homology of the Lie algebra $\mathsf{Q} \circ \mathsf{V}_{2n}$, which upto the abelian part $\mathsf{Q}[0]$, is the Lie algebra $\mathfrak{g}_{2n}^{\omega}$, can be computed using the method in (5.3-5.5). Table 1 explains the analogy.

From classical invariant theory of sp_{2n}, for n large enough, $\mathbb{K}C_i \cong (\mathsf{V}_{2n}^{\otimes i})^{sp_{2n}} \cong (\mathsf{V}_{2n}^{\otimes i})_{sp_{2n}}$ is the span of oriented chord diagrams on i vertices. For a definition and also a description of the map $i : \mathbb{K}C_i \to M$, see [18, Sections 12.3.1-12.3.2]. Now consider the non-degenerate anti-symmetric bilinear form on V_{2n} given by

$$(p_i, q_i) = 1 = -(q_i, p_i),$$

other pairings on basis elements being zero. This induces an sp_{2n} module isomorphism $\mathsf{V}_{2n} \cong (\mathsf{V}_{2n})^*$ given by the bilinear form, which further induces $M \cong M^*$. One can now describe the map $p : M \to \mathbb{K}C_i$ by dualising i to get a map $M^* \to (\mathbb{K}C_i)^*$ and then using the identifications $M \cong M^*$ and $(\mathbb{K}C_i)^* \cong \mathbb{K}C_i$. The resulting map p is given by

$$p(m) = \sum_{\pi \in C_i} \langle m, \pi \rangle \; \pi,$$

where $\langle m, \pi \rangle$ is obtained by writing the tensor factors of m on the corresponding vertices of the oriented chord diagram for π and contracting elements along an edge using the bilinear form, in the order specified by the edge direction.

Following the procedure as in (5.4.2), one obtains

$$\mathcal{G}_k(sp, \mathsf{P}) \cong (\Lambda^k \mathfrak{g}_n^\omega)^{sp_{2n}} \cong (\Lambda^k \mathfrak{g}_n^\omega)_{sp_{2n}}$$

as the span of oriented Q-graphs with k vertices and maps $\varphi : \mathcal{G}_k(sp, \mathsf{P}) \to \Lambda^k \mathfrak{g}_n^\omega$ and $\psi : \Lambda^k \mathfrak{g}_n^\omega \to \mathcal{G}_k(sp, \mathsf{P})$. One then shows that ψ is a chain map and the rest of the proof works as before.

Remark. The word "coinvariant" and the map p is missing in [2, 18]. The maps i and φ, related to invariants are desribed in [18, Sections 12.3.1-12.3.2,12.4.1] and [2, Sections 2.5.1,2.5.4]. As we know, the proof cannot proceed without the map ψ. In [18, Sections 13.1-13.3], ψ is defined using φ^* and certain nondegenerate pairings M' on $\Lambda^k \mathfrak{g}_n^\omega$ and A on $\mathcal{G}_k(sp, \mathsf{P})$ (which we now realise is similar to the way the map p is defined from i). Unfortunately, this method works only for operads in the category of Sets, and so the proof holds only for this case. In [2, Section 2.5.2], ψ is defined directly and coincides with the map obtained above. In either paper, since ψ is not seen as a map to coinvariants, more work is necessary to show that is a quasi-isomorphism.

6.2. **Operad supergeometry.** To understand the orthogonal case, one needs to do supermathematics, which we briefly present here. It is the superversion of the material in [18, Section 6], where more details can be found.

Let P be an operad with a unit u. Let W be a super vector space of dimension $(k|l)$ and $\mathsf{P}(\mathsf{W})$ be the free P-superalgebra on W. One can regard $\mathsf{P}(\mathsf{W})$ as functions on X, which is the "standard P-supermanifold of dimension $(k|l)$".

Definition 6.1. The space $\Omega(\mathsf{P}(\mathsf{W}))$ is the free differential P-superalgebra on W. More explicitly,

$$\Omega(\mathsf{P}(\mathsf{W})) = \mathsf{P}(\mathsf{W} \oplus \Pi \mathsf{W}) = \text{The free } \mathsf{P}\text{-superalgebra on } \mathsf{W} \oplus \Pi \mathsf{W},$$

where Π is the functor on super vector spaces that switches parity and the odd superderivation $d : \Omega(\mathsf{P}(\mathsf{W})) \to \Omega(\mathsf{P}(\mathsf{W}))$ sends W isomorphically onto $\Pi \mathsf{W}$ and $\Pi \mathsf{W}$ to 0. It follows that $d^2 = 0$.

Definition 6.2. For any P-superalgebra A, let $\mathrm{Der}(\mathsf{A})$ be the Lie superalgebra of derivations of A. In particular, this defines $\mathrm{Der}(\mathsf{P}(\mathsf{W}))$ and $\mathrm{Der}(\Omega(\mathsf{P}(\mathsf{W})))$.

For $\xi \in \mathrm{Der}(\mathsf{P}(\mathsf{W}))$, we define the Lie derivative $L_\xi \in \mathrm{Der}(\Omega(\mathsf{P}(\mathsf{W})))$ and the contraction operator $i_\xi \in \mathrm{Der}(\Omega(\mathsf{P}(\mathsf{W})))$ by the formulas

$$L_\xi(w) = (-1)^{|\xi|}\xi(w),\ L_\xi(dw) = d\xi(w) \ \text{ and } \ i_\xi(w) = 0,\ i_\xi(dw) = (-1)^{|\xi|}\xi(w),$$

for every $w \in \mathsf{W}$ and where $|\xi|$ denotes the superdegree of ξ. The operators L_ξ and i_ξ have superdegrees $|\xi|$ and $|\xi| + 1$ respectively. For $\xi, \eta \in \mathrm{Der}(\mathsf{P}(\mathsf{W}))$, the following commutation relations hold.

$$[d, i_\xi] = L_\xi,\ [i_\xi, i_\eta] = 0,\ [L_\xi, i_\eta] = i_{[\xi,\eta]},\ [L_\xi, L_\eta] = L_{[\xi,\eta]}.$$

Apart from the supergrading, $\Omega(\mathsf{P}(\mathsf{W}))$ has a $\mathbb{Z}$ grading given by the number of occurrences of elements of $\Pi\,\mathsf{W}$. Denote $\Omega^i(\mathsf{P}(\mathsf{W}))$ for the ith graded part. Then $\Omega^0(\mathsf{P}(\mathsf{W})) = \mathsf{P}(\mathsf{W})$. With respect to this grading, the operators L_ξ, i_ξ and d have degrees 0, -1 and 1 respectively. Using the above formulas, one can show that

$$H^i(\Omega(\mathsf{P}(\mathsf{W})), d) = \begin{cases} 0 & \text{if } i > 0, \\ \mathsf{P}[0] & \text{if } i = 0. \end{cases} \tag{7}$$

6.3. **Reversible operad supergeometry.** Let P be a reversible operad and $\mathsf{Q} = \mathsf{P}\mathsf{P}$ be its associated mated species. Then one defines

$$\mathsf{Q}(\mathsf{W}) = \bigoplus_{j\geq 0}(\mathsf{Q}[j] \otimes \mathsf{W}^{\otimes j})_{\Sigma_j},\ \ \Omega(\mathsf{Q}(\mathsf{W})) = \bigoplus_{j\geq 0}(\mathsf{Q}[j] \otimes (\mathsf{W} \oplus \Pi\,\mathsf{W})^{\otimes j})_{\Sigma_j}. \tag{8}$$

These objects can also be seen as the images of $\mathsf{P}(\mathsf{W})$ and $\Omega(\mathsf{P}(\mathsf{W}))$ under the mating functor

$$\{\mathsf{P}\text{-superalgebras}\} \to \{\text{Super vector spaces}\}.$$

This is the superversion of the functor λ defined by Getzler-Kapranov [8]. For $\xi \in \mathrm{Der}(\mathsf{P}(\mathsf{W}))$, one gets super linear maps $L_\xi, i_\xi, d : \Omega(\mathsf{Q}(\mathsf{W})) \to \Omega(\mathsf{Q}(\mathsf{W}))$ of superdegrees $|\xi|$, $|\xi| + 1$ and 1 respectively, with $d^2 = 0$ by functoriality. Also, $\Omega(\mathsf{Q}(\mathsf{W}))$ has a $\mathbb{Z}$ grading with $\Omega^0(\mathsf{Q}(\mathsf{W})) = \mathsf{Q}(\mathsf{W})$, and the operators L_ξ, i_ξ and d have degrees 0, -1 and 1 respectively, with respect to this grading. Explicit definitions can be given as in [18, Section 6]. The same proof as before shows that equation (7) holds with P replaced by Q. The importance of having the objects in equation (8) is that one can then do symplectic and orthogonal geometry as below.

6.3.1. *Symplectic operad geometry.* Set $\mathsf{W} = \mathsf{V}_{2n}$, a super vector space of dimension $(2n|0)$ with basis $p_1, \ldots, p_n, q_1, \ldots, q_n$. Since the operad P has a unit u, one can define

$$\omega = \sum_i dp_i \wedge dq_i \in \Omega^2(\mathsf{Q} \circ \mathsf{V}_{2n}).$$

There is an isomorphism $\mathrm{Der}(\mathsf{P} \circ \mathsf{V}_{2n}) \xrightarrow{\cong} \Omega^1(\mathsf{Q} \circ \mathsf{V}_{2n})$ between vector fields and 1 forms given by $\xi \to i_\xi\omega$. By usual arguments and using equation (7), with Q for P, one can prove Lemma 7.

6.3.2. *Orthogonal operad geometry.* Set $\mathsf{W} = \mathsf{V}_n^-$, a super vector space of dimension $(0|n)$ with basis $\theta_1, \dots, \theta_n$. One can define

$$\rho = \sum_i d\theta_i \otimes d\theta_i \in \Omega^2(\mathsf{Q}(\mathsf{V}_n^-)).$$

There is an isomorphism $\mathrm{Der}(\mathsf{P}(\mathsf{V}_n^-)) \xrightarrow{\cong} \Omega^1(\mathsf{Q}(\mathsf{V}_n^-))$ given by $\xi \to \frac{(-1)^{|\xi|}}{2}\, i_\xi \rho$. By arguments, as in the symplectic case, one then derives Lemma 8 and so forth, see below. The conventions are made such that $dH = \sum_i d\theta_i \otimes \frac{\partial H}{\partial \theta_i}$.

6.4. **Proof of Theorem 2, Part** $(B + D)$**.** With the disccussion in (6.2-6.3), it is fairly clear that one can repeat the proof of Part (C) with appropriate sign corrections.

6.4.1. *Lie algebra* $\mathfrak{g}_n^\rho$*.* We write down the analogue of Lemma 7 and also describe the Lie algebra $\mathsf{Q}(\mathsf{V}_n^-)$.

Lemma 8. *There is a split short exact sequence of Lie superalgebras*

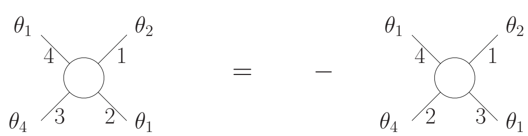
$$0 \to \mathsf{Q}[0] \hookrightarrow \mathsf{Q}(\mathsf{V}_n^-) \twoheadrightarrow \mathrm{Der}(\mathsf{P}(\mathsf{V}_n^-), \rho) \to 0, \tag{9}$$

with the Poisson bracket on $\mathsf{Q}(\mathsf{V}_n^-)$ *given by*

$$\{F, H\} = (-1)^{|F|} \sum_{i=1}^{n} \frac{\partial F}{\partial \theta_i} \otimes \frac{\partial H}{\partial \theta_i}, \quad \textit{for} \quad F, H \in \mathsf{Q}(\mathsf{V}_n^-). \tag{10}$$

The above map $\mathsf{Q}(\mathsf{V}_n^-) \twoheadrightarrow \mathrm{Der}(\mathsf{P}(\mathsf{V}_n^-), \rho)$ *is given by* $H \mapsto \xi_H$*, where* $\xi_H(\theta_i) = \frac{\partial H}{\partial \theta_i}$.

We represent a monomial in $\mathsf{Q}(\mathsf{V}_n^-)$ by the picture

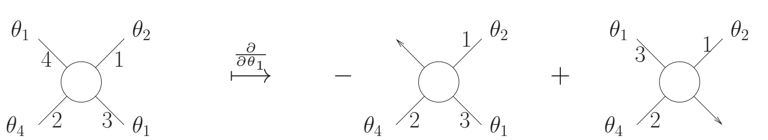
(11) 

$\in \mathsf{Q}(\mathsf{V}_n^-)$.

In other words, we attach a θ_i to each input of an element of Q and also order the inputs in the sense of orientation, that is, an even permutation of the order leaves an element unchanged while an odd permutation gives its negative. To get a general element of $\mathsf{Q}(\mathsf{V}_n^-)$, we take linear combinations of monomials. A similar pictorial description can be given for the elements of $\mathsf{P}(\mathsf{V}_n^-)$.

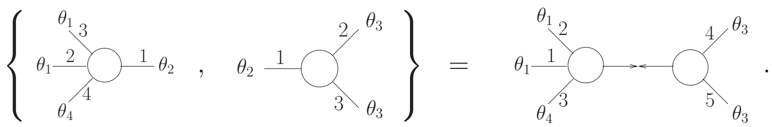
Cutting and Mating.

As for the symplectic case [18, Section 5], the Poisson bracket on $\mathsf{Q}(\mathsf{V}_n^-)$ in equation (10) can be described pictorially by a cutting and mating process. We define $\frac{\partial}{\partial \theta_i} : \mathsf{Q}(\mathsf{V}_n^-) \longrightarrow \mathsf{P}(\mathsf{V}_n^-)$ by showing how it works on a schematic example.



Namely, to define $\frac{\partial}{\partial \theta_1}$, we cut the inputs with label θ_1, one at a time, first reordering the labels so that the input being cut has label 1 and then shifting down the labels of the remaining inputs.

Now, we illustrate the Poisson bracket by an example.



In the above example, there is only one mating possible. It is shown by an edge with two opposing arrowheads in the centre. Note that the labels on the inputs coming from the second term got pushed up.

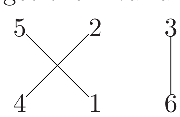
6.4.2. *Orthogonal Lie algebra o_n.* Since P has an unit u, by the Corollary to Lemma 3 and as in (5.2) and (6.1.2), the orthogonal Lie algebra o_n is a Lie subalgebra of $\mathfrak{g}_n^g$. Applying Lemma 8 to the unit operad u, which satisfies $u[0] = 0$, one obtains:

Corollary. *There is a Lie algebra isomorphism $uu(\mathsf{V}_n^-) \xrightarrow{\cong} o_n$.*

Proposition 4. *The sequence in equation (9) is a sequence of o_n modules with the adjoint action on $\mathfrak{g}_n^\rho$ and the action on $\mathsf{Q}(\mathsf{V}_n^-)$ induced by the usual action of o_n on V_n^- and the trivial action on the $\mathsf{Q}[j]$'s.*

6.4.3. *The rest of the proof.* The homology of the Lie superalgebra $\mathsf{Q}(\mathsf{V}_n^-)$, which upto the abelian part $\mathsf{Q}[0]$, is the Lie superalgebra $\mathfrak{g}_n^\rho$, can be computed using the method in (5.3-5.5). Note that one now needs to start with the super version of the Chevalley-Eilenberg complex. We will use the notation as in Table 1.

From classical invariant theory of o_n, for n large enough, $\mathbb{K}C_i^- \cong ((\mathsf{V}_n^-)^{\otimes i})^{o_n} \cong ((\mathsf{V}_n^-)^{\otimes i})_{o_n}$ is the span of chord diagrams on i vertices. The map $i : \mathbb{K}C_i^- \to M$ is as follows. Each vertex in the chord diagram represents a tensor factor, in the order given by the vertex labelling. For each edge, we put a θ_i at either end and then sum over all possibilities to get the invariant. For example,



gives the invariant

$$\sum_{1 \le i,j,k \le n} \theta_i \otimes \theta_j \otimes \theta_k \otimes \theta_j \otimes \theta_i \otimes \theta_k.$$

Now consider the non-degenerate symmetric bilinear form on V_n^- given by

$$(\theta_i, \theta_i) = 1,$$

other pairings on basis elements being zero. This induces an o_n module isomorphism $\mathsf{V}_n^- \cong (\mathsf{V}_n^-)^*$ given by the bilinear form, which further induces $M \cong M^*$. One can now describe the map $p : M \to \mathbb{K}C_i^-$ by dualising i to get a map $M^* \to (\mathbb{K}C_i^-)^*$ and then using the identifications $M \cong M^*$ and $(\mathbb{K}C_i^-)^* \cong \mathbb{K}C_i^-$. The resulting map p is given by

$$p(m) = \sum_{\pi \in C_i^-} \langle m, \pi \rangle \; \pi,$$

where $\langle m, \pi \rangle$ is obtained by writing the tensor factors of m on the corresponding vertices of the chord diagram for π and contracting elements along an edge using the bilinear form.

Following the procedure as in (5.4.2), one obtains

$$\mathcal{G}_k(o, \mathsf{P}) \cong (\Lambda^k \mathfrak{g}_n^\rho)^{o_n} \cong (\Lambda^k \mathfrak{g}_n^\rho)_{o_n}$$

as the span of odd oriented Q-graphs with k vertices, and the rest of the proof is similar.

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