DIOPERAD GEOMETRY AND GRAPH HOMOLOGY

SWAPNEEL MAHAJAN

Abstract. In this paper, we introduce dioperads and present their basic theory. Roughly, dioperads generalize operads in the same way that dialgebras generalize algebras. We explain the notion of a dioperad algebra and show that the space of derivations of a dioperad algebra forms a Leibniz algebra.

As a primary motivation, we propose a connection between Leibniz homology of the space of derivations of the free Leibniz algebra and a certain graph complex associated to a dioperad. This provides a generalized setting for work of Loday-Cuvier [3] and Frabetti [6]. We also propose a similar setting in symplectic and orthogonal dioperad geometry for the homology of the appropriate Leibniz algebras. This is a conjectured dioperad analogue of the main result in Mahajan [15], which builds on work of Loday-Quillen [11, 12], Tsygan [18, 5], Loday-Procesi [10], Kontsevich [8, 7], Conant-Vogtmann [2] and Mahajan [14].

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Key words. species; operad bimodule; cyclic/reversible dioperad; dialgebra; dimonoid; dioperad algebra; derivation; mating functor; symplectic and orthogonal geometry; Poisson bracket; Leibniz superalgebra homology; cyclic, dihedral, Hochschild and graph homology.

1. Introduction

In this paper, we introduce dioperads and present their basic theory. Roughly, dioperads generalize operads in the same way that dialgebras generalize algebras [9]. In this section, we give our motivation for studying this concept. It is not required to understand the rest of the paper.

1.1. **The Lie algebra of vector fields on operad manifolds.** Let P be an operad, $\mathbb{K}$ be a field of characteristic 0, and V be a vector space over $\mathbb{K}$ of dimension n. Let $\mathrm{Der}(gl_n, \mathsf{P})$ be the Lie algebra of derivations of the free P algebra $\mathsf{P} \circ \mathsf{V}$, and let $\mathrm{Der}(gl, \mathsf{P})$ be the colimit as n goes to ∞. One may interpret $\mathrm{Der}(gl_n, \mathsf{P})$ as the Lie algebra of vector fields on the standard P manifold of dimension n. Recall the following theorem from [15].

Theorem 1. *Let P be an operad with an unit.*

$$\begin{array}{ccccc} (A) & H_*^{Lie}(\mathrm{Der}(gl, \mathsf{P}), \mathbb{K}) & = & \Lambda(H_*(\mathcal{C}(gl, \mathsf{P}))). \\ (C) & H_*^{Lie}(\mathrm{Der}(sp, \mathsf{P}), \mathbb{K}) & = & \Lambda(H_*(\mathcal{C}(sp, \mathsf{P}))). \\ (B+D) & H_*^{Lie}(\mathrm{Der}(o, \mathsf{P}), \mathbb{K}) & = & \Lambda(H_*(\mathcal{C}(o, \mathsf{P}))). \end{array}$$

In this statement, H_*^{Lie} denotes Lie homology, $\mathcal{C}(gl, \mathsf{P})$ is a certain graph complex associated to P, and Λ is the signed symmetric functor. Parts (C) and $(B+D)$, for which one needs P to be cyclic/reversible, are the symplectic and orthogonal cases. The isomorphism in this theorem is not just a vector space isomorphism but an isomorphism of graded Hopf algebras. Part (C) of the above theorem was initiated by Kontsevich [8, 7], and completed by Conant-Vogtmann [2] and Mahajan [14].

Every associative algebra A gives rise to an operad A_o defined by $\mathsf{A}_o[I] = A$ if I is a singleton and zero otherwise. Further $\mathrm{Der}(gl, \mathsf{A}_o)$ is the Lie algebra of matrices $gl(A)$. The above theorem then specialises to the following.

Theorem 2. *Let A be an algebra with an unit.*

$$\begin{array}{ccccc} (A) & H_*^{Lie}(gl(A), \mathbb{K}) & = & \Lambda(HC_{*-1}(A)). \\ (C) & H_*^{Lie}(sp(A), \mathbb{K}) & = & \Lambda(HD_{*-1}(A)). \\ (B+D) & H_*^{Lie}(o(A), \mathbb{K}) & = & \Lambda(HD_{*-1}(A)). \end{array}$$

In this statement, HC_* and HD_* refer to cyclic and dihedral homology respectively. Parts (C) and $(B+D)$, for which one needs A to have an involution, are the symplectic and orthogonal cases. Part (A) of the above theorem is due to Loday-Quillen [11, 12] and Tsygan [18, 5] and Parts (C) and $(B+D)$ are due to Loday-Procesi [10].

1.2. **From Lie to Leibniz algebras.** A Leibniz algebra is a vector space equipped with a bracket $[\ ,\ ]$ which satisfies the Leibniz relation

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

We do not impose anti-symmetry; thus Leibniz algebras generalize Lie algebras. They have a homology theory, denoted H_*^{Leib}, where the chain complex is constructed using tensor powers of the Leibniz algebra. The historical motivation for Leibniz algebras comes from the following result due to Loday-Cuvier [3, 4].

Theorem 3. *Let A be an algebra with an unit.*

$$(A) \quad H_*^{Leib}(gl(A), \mathbb{K}) \quad = \quad T(HH_{*-1}(A)).$$

where HH_* refers to Hochschild homology, and T is the tensor power functor.

In other words, replacing Lie by Leibniz has the effect of replacing cyclic by Hochschild homology. Different proofs of the above computation can be found in Lodder [13], Oudom [17] and Frabetti [6]. In light of this discussion, it is natural to ask for a Leibniz analogue of Theorem 1. Accordingly, we propose the following.

Conjecture 1. *Let* P *be an operad with an unit.*

$$\begin{array}{rrcl} (A) & H_*^{Leib}(\mathrm{Der}(gl,\mathsf{P}),\mathbb{K}) & = & T(H_*(\hat{\mathcal{C}}(gl,\mathsf{P}))). \\ (C) & H_*^{Leib}(\mathrm{Der}(sp,\mathsf{P}),\mathbb{K}) & = & T(H_*(\hat{\mathcal{C}}(sp,\mathsf{P}))). \\ (B+D) & H_*^{Leib}(\mathrm{Der}(o,\mathsf{P}),\mathbb{K}) & = & T(H_*(\hat{\mathcal{C}}(o,\mathsf{P}))). \end{array}$$

In this statement the graph complexes $\hat{\mathcal{C}}_*$ are an appropriate lift of the complexes $\mathcal{C}_*$ in Theorem 1. In particular, the symplectic and orthogonal cases of Theorem 3 which are not in the literature, are also taken care of.

1.3. **From operads to dioperads.** The categories of associative, diassociative, Lie and Leibniz algebras assemble into a commutative diagram as below.

$$\begin{tikzcd} \mathbf{As} \arrow[r] \arrow[d] & \mathbf{Lie} \arrow[d] \\ \mathbf{Dias} \arrow[r] & \mathbf{Leib} \end{tikzcd}$$

This diagram indicates that Theorem 3 needs to be generalized. Namely, Leibniz homology must be related to an appropriate homology theory of dialgebras rather than algebras. This was done by Frabetti [6], where more details can be found. Her result is as follows.

Theorem 4. *Let* D *be an dialgebra with a bar unit.*

$$(A) \quad H_*^{Leib}(gl(D),\mathbb{K}) \quad = \quad T(HHS_{*-1}(D)),$$

where HHS_* *is one generalization of Hochschild homology to dialgebras.*

One would now like to appropriately generalize Conjecture 1. Namely, we would like to replace P by another object Y such that the space, say $\mathrm{Der}(gl,\mathsf{Y})$, is a Leibniz algebra which is not necessarily a Lie algebra. This leads us to the concept of a dioperad, which is explained in Sections 2, 3 and 4. The main conjecture can be stated as follows.

Conjecture 2. *Let* Y *be a dioperad with a bar unit.*

$$\begin{array}{rrcl} (A) & H_*^{Leib}(\mathrm{Der}(gl,\mathsf{Y}),\mathbb{K}) & = & T(H_*(\hat{\mathcal{C}}(gl,\mathsf{Y}))). \\ (C) & H_*^{Leib}(\mathrm{Der}(sp,\mathsf{Y}),\mathbb{K}) & = & T(H_*(\hat{\mathcal{C}}(sp,\mathsf{Y}))). \\ (B+D) & H_*^{Leib}(\mathrm{Der}(o,\mathsf{Y}),\mathbb{K}) & = & T(H_*(\hat{\mathcal{C}}(o,\mathsf{Y}))). \end{array}$$

The Leibniz algebras in the statement above are defined in Sections 5 and 6, while the graph complexes are defined in Section 7. The isomorphism in this theorem is an isomorphism of groups in the category of graded Zinbeil coalgebras. For Parts (C) and $(B+D)$, one needs Y to be reversible.

Remark. By following the pattern of proof in Theorem 1, which is adequately explained in [15], one can reduce the Leibniz homologies to homologies of certain disconnected graph complexes. However at this step, there is a gap in my understanding. I do not understand how one identifies the primitive elements in the disconnected graph complex with the connected piece. This is the reason why

Conjectures 1 and 2 are stated as conjectures and not theorems. The claim [6, Proposition 2.13] by Frabetti just appears false to me.

2. Species, operads and dioperads

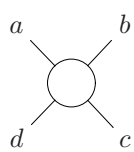
In this section, we give a brief introduction to species and operads and then define dioperads. The main reference for the first part is [14]. We also recall the notion of operad bimodules which is central to the construction of dioperads. We write $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{Y}$ for the categories of species, operads and dioperads respectively.

2.1. **Species.** A detailed treatment on species can be found in the book by Bergeron, Labelle and Leroux [1]. We write $\mathcal{S}$ for the category of species. Let $\mathtt{Set}$ be the category of finite sets with bijections as the morphisms. Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathtt{Vect}$ be the category of vector spaces over $\mathbb{K}$.

Definition 2.1. A species Q is a functor

$$\mathtt{Set} \longrightarrow \mathtt{Vect}\,.$$

We denote the image of a set I by $\mathsf{Q}[I]$ and say that $\mathsf{Q}[I]$ is the space of Q-structures on the set I. An element of $\mathsf{Q}[I]$ can be schematically drawn as



$\in \mathsf{Q}[I]$ for $I = \{a, b, c, d\}$.

The arms attached to the disc are labelled by elements of I, while the disc specifies the Q-structure on its arms. By definition, a bijection $I \to J$ induces an isomorphism $\mathsf{Q}[I] \to \mathsf{Q}[J]$. In the schematic notation, this map relabels the arms using elements of J.

An alternative definition of species is given in Definition 5.1.

Definition 2.2. A morphism between the species P and Q is a natural transformation between the functors P and Q.

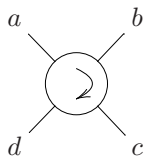
2.2. **Examples of species.** We now give some examples of species. In each of them, $\mathsf{Q}[\emptyset] = 0$.

- $\mathsf{u}[I] = k$ if I is a singleton and zero otherwise.
- $\mathsf{c}[I] = k$.
- $\mathsf{a}[I]$ is the $\mathbb{K}$-span of the set of linear orders on the set I.
- $\mathsf{s}[I]$ is the $\mathbb{K}$-span of the set of nonempty subsets of the set I.
- $\mathsf{Perm}[I]$ is the $\mathbb{K}$-span of the set I.
- $\mathsf{p}[I]$ is the $\mathbb{K}$-span of the set of partitions of the set I.

From the species u, c, a and s, one can construct related species as below. This connection will be made clear in Section 4.

- $\mathsf{uu}[I] = k$ if I has two elements and zero otherwise.
- $\mathsf{cc}[I] = k$ if I has two or more elements and zero otherwise.
- $\mathsf{aa}[I]$ is the $\mathbb{K}$-span of the set of cyclic orders on the set I.
- $\mathsf{ss}[I]$ is the $\mathbb{K}$-span of the set of partitions of the set I into two parts.

The species aa in the picture notation is as follows, with the arc indicating the cyclic order.



$\in \mathsf{aa}[I]$ for $I = \{a, b, c, d\}$.

The reader may devise similar pictures for the other examples.

2.3. **Operads.** A detailed treatment on operads can be found in the book by Markl, Shnider and Stasheff [16]. We write $\mathcal{P}$ for the category of operads.

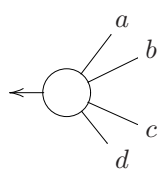
Definition 2.3. An operad P is a species with a substitution rule

$$\mathsf{P}[I] \otimes \mathsf{P}[J] \to \mathsf{P}[(I \setminus \{y\}) \cup J] \quad \text{for} \quad y \in I,$$

denoted $\underset{y}{\longleftarrow}$ which is

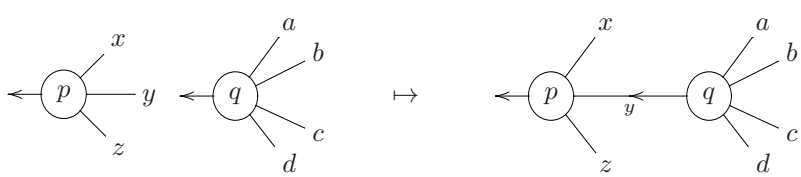
(i) compatible with the morphisms in the source category, and
(ii) associative.

An element of $\mathsf{P}[I]$ can be schematically drawn as



$\in \mathsf{P}[I]$ for $I = \{a, b, c, d\}$,

with the arrow indicating the presence of a substitution rule. One thinks of a, b, c, d as four inputs and the arrow as an output. The substitution rule allows us to substitute the output of one element q into a specified input, say y, of another element p. This can be shown as



For shorthand, we say that $p \underset{y}{\text{—}} \!\!\leftarrow q$ maps to $p \underset{y}{\longleftarrow} q$. It is convenient many times to drop y from the notation, and say that $p \text{—} \!\!\leftarrow q$ maps to $p \leftarrow q$. We do not distinguish between $q \to p$ and $p \leftarrow q$.

The first condition says that substitution commutes with relabelling of the inputs. The second condition says that if we perform two substitutions, one after the other, then the order in which we do them does not matter. Equivalently,

$$p \longleftarrow q \longleftarrow r, \quad \text{and} \quad \begin{tikzcd} & q \arrow[dl] \\ p & \\ & r \arrow[ul] \end{tikzcd} \tag{1}$$

are well defined. As a shorthand, we say that

$$p \text{—} q \text{—} r \tag{2}$$

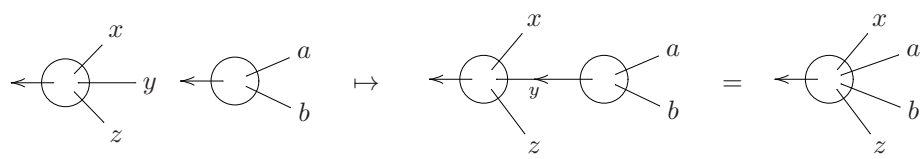
is well defined, without specifying the arrowheads. The convention is that the arrowheads either both point to the left or to the right or to the centre.

Definition 2.4. A unit in an operad P is a map $\mathsf{u} \to \mathsf{P}$ of species such that $p \leftarrow u \;=\; p$ and $u \leftarrow p \;=\; p$ for $p \in \mathsf{P}[I]$ for any finite set I.

Definition 2.5. A morphism between operads P and Q is a map of species which commutes with the respective substitution maps.

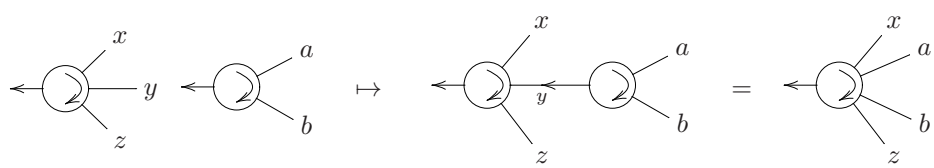
2.4. **Examples of operads.** We now give examples of operads using some of the examples of species in (2.2).

- The unit operad is the species u with substitution given by the canonical map $k \otimes k \xrightarrow{\cong} k$.
- The commutative operad is the species c with substitution defined in the same way as for u.



Each figure represents the unit element $1 \in c[I] \cong k$, for appropriate I.

- The associative operad is the species a of linear orders with the following substitution rule.



- The permutative operad is the species Perm with the following substitution rule. Let $i \in \mathsf{Perm}[I]$ and $j \in \mathsf{Perm}[J]$ be elements of I and J respectively. Define
$$i \xleftarrow{y} j = \begin{cases} i & \text{if } i \neq y, \\ j & \text{if } i = y. \end{cases}$$
More information on this operad can be found on [9, page 105].

2.5. **Operad bimodules.** Now we discuss bimodules over an operad along with some examples. These will be useful for constructing dioperads.

Definition 2.6. For an operad P, a P-bimodule is a species M with maps
$$\begin{array}{ccc} \mathsf{P}[I] \otimes \mathsf{M}[J] & \searrow & \\ & & \mathsf{M}[I \setminus \{y\} \cup J] \qquad \text{for} \quad y \in I, \\ \mathsf{M}[I] \otimes \mathsf{P}[J] & \nearrow & \end{array}$$
both denoted $\underset{y}{\longleftarrow}$ which are

(i) compatible with the morphisms in the source category, and
(ii) associative, in the sense explained below.

We denote the image of $p \otimes m$ in the first map by $p \longleftarrow m$, and the image of $m \otimes p$ in the second map by $m \longleftarrow p$, dropping y from the notation as usual. Then the second condition above says that

$$p \text{ — } m \text{ — } q \quad \text{and} \quad p \text{ — } q \text{ — } m$$

are well defined, the convention on the arrowheads being as in Equation (2).

2.6. **Examples of operad bimodules.** By definition, P is a bimodule over itself. One can use trees to construct other interesting examples of bimodules. For the terminology on trees, we follow [16].

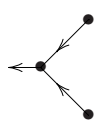
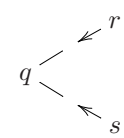
Definition 2.7. A tree is a finite connected contractible graph. We will modify the standard convention according to which all edges in a graph have two adjacent vertices and delete some of the vertices with only one adjacent edge. This means that some edges will have only one adjacent vertex. We call these edges external edges. The edges which are adjacent to two vertices will be called internal edges. A rooted tree is a tree with a distinguished external edge called the root. The remaining external edges are called leaves. We direct a rooted tree by pointing the edges towards the root.

Definition 2.8. A standard tree is a tree with no external edges. Similarly, a standard rooted tree is a tree with exactly one external edge called the root. For a rooted tree T, let $\mathrm{st}(T)$ be the standard rooted tree obtained by deleting all the external edges of T, except the root. And let $\overline{\mathrm{st}}(T)$ be the standard tree obtained by deleting all the external edges of T.

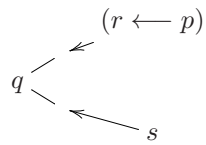
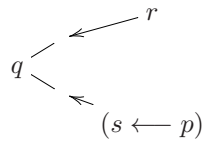
Definition 2.9. A tree of type $T(\mathsf{P})$ is the rooted tree T, with a P-structure specified on the set of incoming edges at each vertex of T. We now define a species $T(\mathsf{P})$. For any finite set I, let $T(\mathsf{P})[I]$ be the space spanned by all trees of type $T(\mathsf{P})$, with leaves labelled by elements of I, subject to the linearity condition at each vertex.

Note that for c, the commutative operad, $T(c)[I]$ is the space spanned by T, with leaves labelled by elements of I.

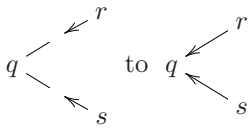
Example 1. Now fix a standard rooted tree t and an operad P. Define $t(\mathsf{P})$ to be the species, where $t(\mathsf{P})[I]$ is the direct sum over all $T(\mathsf{P})[I]$ such that $\mathrm{st}(T) = t$. Then $t(\mathsf{P})$ is a P-bimodule as below.

We illustrate with $t =$  . Let $m =$  $\in t(\mathsf{P})$, with q, r and s being the P-structures at the three vertices of t. Then

$p \longleftarrow m \quad := \quad (p \longleftarrow q)$  and

$m \longleftarrow p \quad := \quad (q \longleftarrow p)$  or  or 

depending on the input of m in which we substitute p.

Note that $\leftarrow\bullet(\mathsf{P}) = \mathsf{P}$. This gives the usual bimodule structure on P. Now let t' be a standard rooted tree obtained by contracting some of the internal edges of t. This induces a map $t(\mathsf{P}) \to t'(\mathsf{P})$ of P-bimodules. In particular, contracting all the internal edges gives a map $t(\mathsf{P}) \to \mathsf{P}$ of P-bimodules. In the example above, this map sends 

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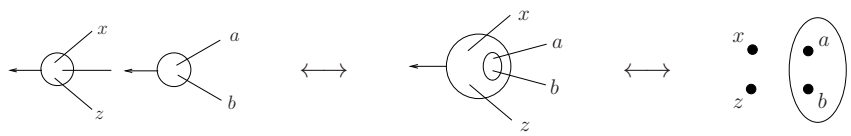
Remark. In the above construction, instead of a standard rooted tree, one can fix a standard tree $\bar{t}$, and define a P-bimodule $\bar{t}(\mathsf{P})$ by using the map $\overline{\mathrm{st}}$ instead of st.

Example 2. We apply the previous example to $\mathsf{P} = \mathsf{c}$, the commutative operad. We note that

$$\leftarrow\bullet\leftarrow\bullet(c) \quad = \quad \bullet\!\!-\!\!\bullet(c) \quad = \quad \mathsf{s},$$

the subset species. This should be clear from the following picture.



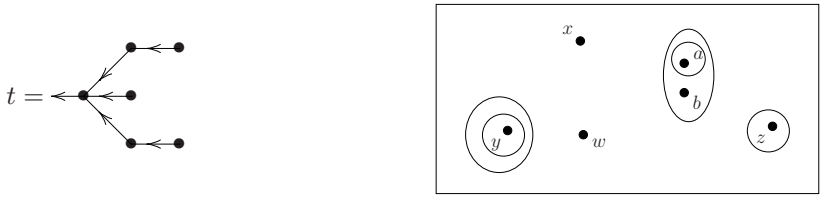
The last two diagrams show the subset species in picture notation. We leave it to the reader to make the c-bimodule structure on s explicit. The map $\mathsf{s} \to \mathsf{c}$ is the forgetful map. We now explain the general case.

Definition 2.10. Let $Tree_k$, respectively $\overline{Tree}_k$ be the set of standard rooted trees, respectively standard trees, with $k+1$ vertices. Let $Conf_k$, respectively $\overline{Conf}_k$ be the combinatorial configuration of k disjoint circles in the plane, respectively on the sphere.

An element of $Conf_k$ divides the plane into $k+1$ regions. Put a vertex in each region, and draw an edge between vertices in adjacent regions. This gives a standard rooted tree, with the root corresponding to the vertex lying in the unique unbounded region. In fact, this procedure gives a bijection

$$Tree_k \longleftrightarrow Conf_k, \qquad \text{and} \qquad \overline{Tree}_k \longleftrightarrow \overline{Conf}_k. \tag{3}$$

For a standard rooted tree t, let $C(t)$ be the corresponding configuration. We now describe $t(\mathsf{c})$. An element of $t(\mathsf{c})[I]$ consists of $C(t)$ along with points labelled by elements of I, such that the points do not lie on any of the circles and each of the disc-like regions contains at least one point.



For example, the box above shows an element of $t(\mathsf{c})[\{a, b, x, y, z, w\}]$, for t as above. Apart from the unbounded region with two points, there are three disc-like regions each with one point, and two annulur regions, one of which is empty.

Example 3. For an operad P with a unit, using the idea in Example 1, we note that $\mathsf{P}\circ\mathsf{P}$ is a P-bimodule, with the substitution map $\mathsf{P}\circ\mathsf{P}\to\mathsf{P}$ being a map of P-bimodules. This idea can also be extended to higher iterates, namely $\mathsf{P}\circ\mathsf{P}\circ\mathsf{P}$, and so on.

Example 4. We note that $\mathsf{c}\circ\mathsf{c}=\mathsf{p}$, the partition species. Hence p is a c-bimodule with the map $\mathsf{p}\to\mathsf{c}$ being the forgetful map.

2.7. **Dioperads.** The concept of a dioperad merges the notion of an operad and a dialgebra. We write $\mathcal{Y}$ for the category of dioperads.

Definition 2.11. A dioperad Y is a species with two substitution rules

$$\mathsf{Y}[I]\otimes\mathsf{Y}[J]\to\mathsf{Y}[I\setminus\{y\}\cup J]\quad\text{for}\quad y\in I,$$

denoted $\xleftarrow[y]{\dashv}$ and $\xleftarrow[y]{\vdash}$ which

(i) are compatible with the morphisms in the source category, and
(ii) satisfy the following associative axioms.

$$p\dashv(q\vdash r)=p\dashv(q\dashv r)=(p\dashv q)\dashv r$$

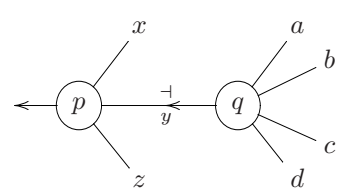
$$p\vdash(q\dashv r)=(p\vdash q)\dashv r$$

$$p\vdash(q\vdash r)=(p\vdash q)\vdash r=(p\dashv q)\vdash r$$

Note that as written these are exactly the axioms for a dialgebra. However the above notation requires some explanation.

We write $p\xleftarrow[y]{\dashv}q$, respectively $p\xleftarrow[y]{\vdash}q$, for the image of $p\otimes q$ under the rule $\xleftarrow[y]{\dashv}$, respectively $\xleftarrow[y]{\vdash}$. For a picture, we draw

$p\xleftarrow[y]{\dashv}q\quad=$



As with operads, it is convenient to drop y from the notation. We follow the convention that

$$p\xleftarrow{\vdash}q\;=\;q\xrightarrow{\dashv}p. \tag{4}$$

The notation, say $p\dashv q$, is then ambiguous and can mean either $p\xleftarrow{\dashv}q$ or $p\xrightarrow{\dashv}q$. The convention used in the second condition above is as follows.

In each identity, the arrows either all point to the left or to the right or to the centre. For example, the identity $p\dashv(q\vdash r)=p\dashv(q\dashv r)$ gives

$$p\xleftarrow{\dashv}(q\xleftarrow{\vdash}r)=p\xleftarrow{\dashv}(q\xleftarrow{\dashv}r)$$

$$p\xrightarrow{\dashv}(q\xrightarrow{\vdash}r)=p\xrightarrow{\dashv}(q\xrightarrow{\dashv}r)$$

$$p\xrightarrow{\dashv}(q\xleftarrow{\vdash}r)=p\xrightarrow{\dashv}(q\xleftarrow{\dashv}r)$$

Note that these axioms are identical to those obtained from the identity $(r\vdash q)\vdash p=(r\dashv q)\vdash p$.

From the axioms obtained by using arrows that point to the centre, we see that among the expressions

(5)
$$\begin{tikzcd} & q \arrow[dl, "\dashv"'] \\ p & \\ & r \arrow[ul, "\dashv"] \end{tikzcd} \qquad \begin{tikzcd} & q \arrow[dl, "\dashv"'] \\ p & \\ & r \arrow[ul, "\vdash"] \end{tikzcd} \qquad \begin{tikzcd} & q \arrow[dl, "\vdash"'] \\ p & \\ & r \arrow[ul, "\vdash"] \end{tikzcd}$$

the first two are well defined whereas the third one is not. We also have the axiom

$$\begin{tikzcd} & q \arrow[dl, "\dashv"'] \\ p \arrow[r, phantom, "1"] \arrow[r, phantom, shift right=3ex, "2"] & \\ & r \arrow[ul, "\vdash"] \end{tikzcd} \quad = \quad \begin{tikzcd} & q \arrow[dl, "\vdash"'] \\ p \arrow[r, phantom, "1"] \arrow[r, phantom, shift right=3ex, "2"] & \\ & r \arrow[ul, "\vdash"] \end{tikzcd}$$

where the letters 1 and 2 specify the order of substitution.

Definition 2.12. A bar unit in a dioperad Y is a map $\mathsf{u} \to \mathsf{Y}$ of species such that $p \xleftarrow{\dashv} u \;=\; p$ and $u \xleftarrow{\vdash} p \;=\; p$ for $p \in \mathsf{Y}$. So it is only assumed that u substitutes trivially from the bar side.

Remark. It is clear that a dioperad for which $\xleftarrow{\dashv}$ and $\xleftarrow{\vdash}$ coincide is same as an operad.

Definition 2.13. A morphism of dioperads is a map $\mathsf{Y} \to \mathsf{Z}$ of species, which commutes with the substitution rules $\xleftarrow{\dashv}$ and $\xleftarrow{\vdash}$.

2.8. **Examples of dioperads.** We now give some examples of dioperads.

Example 5. Let $K \in \mathsf{s}[I]$ and $L \in \mathsf{s}[J]$ be nonempty subsets of I and J respectively. Define

$$K \xleftarrow[y]{\vdash} L \;=\; L, \qquad K \xleftarrow[y]{\dashv} L \;=\; \begin{cases} K & \text{if } y \notin K, \\ K \cup J & \text{if } y \in K. \end{cases}$$

This gives a dioperad structure on the subset species s defined in (2.2). This is part of a general construction explained below in Example 9.

Example 6. Every dialgebra D is a dioperad, which we denote by D_{do}, defined by $\mathsf{D}_{do}[I] = \mathsf{D}$ if I is a singleton and zero otherwise. More precisely, we have a functor $do : \mathcal{D} \to \mathcal{Y}$, where $\mathcal{D}$ is the category of diagebras. This functor has an adjoint $da : \mathcal{Y} \to \mathcal{D}$ that sends a dioperad Y to the dialgebra $\mathsf{Y}[1]$.

This is the analogue of the pair of adjoint functors $o : \mathcal{A} \to \mathcal{P}$ and $a : \mathcal{P} \to \mathcal{A}$, defined in the same way as above, where $\mathcal{A}$ is the category of associative algebras.

Example 7. Let F be any ideal in Y, that is, a subspecies of Y with maps

$$\begin{tikzcd} \xleftarrow{\vdash}, \xleftarrow{\dashv} : \mathsf{Y}[I] \otimes \mathsf{F}[J] \arrow[dr] & \\ & \mathsf{F}[I \setminus \{y\} \cup J] \qquad \text{for} \quad y \in I, \\ \xleftarrow{\vdash}, \xleftarrow{\dashv} : \mathsf{F}[I] \otimes \mathsf{Y}[J] \arrow[ur] & \end{tikzcd}$$

restricted from Y. Then Y/F is a dioperad with the maps $\overset{\dashv}{\longleftarrow}$ and $\overset{\vdash}{\longleftarrow}$ induced from the quotient map $\mathsf{Y} \to \mathsf{Y}/\mathsf{F}$.

Example 8. Every operad P is a dioperad, which we denote by P_{do}, for which $\overset{\dashv}{\longleftarrow}$ and $\overset{\vdash}{\longleftarrow}$ coincide. More precisely, we have a functor $do : \mathcal{P} \to \mathcal{Y}$.

Let Y be a dioperad and set

$$\mathsf{Y}_o := \mathsf{Y}/\mathsf{F}(\mathsf{Y}), \qquad \mathsf{F}(\mathsf{Y}) := \langle p \overset{\dashv}{\longleftarrow} q \ - \ p \overset{\vdash}{\longleftarrow} q \mid p \in \mathsf{Y}[I], q \in \mathsf{Y}[J] \rangle,$$

where $\mathsf{F}(\mathsf{Y})$ is an ideal in Y. Then Y_o is an operad, since by construction the induced maps $\overset{\dashv}{\longleftarrow}$ and $\overset{\vdash}{\longleftarrow}$ coincide. This defines a functor $o : \mathcal{Y} \to \mathcal{P}$, which is the left adjoint to the functor do.

Example 9. Let A be an associative algebra, M a A-bimodule, and $\mathsf{M} \to \mathsf{A}$ a map of A-bimodules. Then M has the structure of a dialgebra. This construction is due to Loday. In [9, page 72], Frabetti shows that all dialgebras can be achieved in this way. We now extend these ideas to dioperads.

Let P be an operad, M be a P-bimodule and $\varphi : \mathsf{M} \to \mathsf{P}$ be a map of P-bimodules. Then M is a dioperad defined as follows.

$$m \overset{\dashv}{\longleftarrow} n \ := \ m \longleftarrow \varphi(n), \quad m \overset{\vdash}{\longleftarrow} n \ := \ \varphi(m) \longleftarrow n \quad \text{for } m, n \in \mathsf{M}\,.$$

Further a diagram $\begin{tikzcd} \mathsf{M} \arrow[rr] \arrow[dr] & & \mathsf{N} \arrow[dl] \\ & \mathsf{P} & \end{tikzcd}$ of P-bimodules induces a map $\mathsf{M} \to \mathsf{N}$ of dioperads.

We now show that all dioperads arise in this way. Let Y be a dioperad and Y_o be its associated operad. Then Y is a Y_o-bimodule via

$$p \longleftarrow [q] \ := \ p \overset{\dashv}{\longleftarrow} q, \quad [p] \longleftarrow q \ := \ p \overset{\vdash}{\longleftarrow} q \quad \text{for } p, q \in \mathsf{Y},$$

and with this structure $\mathsf{Y} \to \mathsf{Y}_o$ is a map of Y_o-bimodules. Note that if we apply our construction to $\mathsf{M} = \mathsf{Y}$ and $\mathsf{P} = \mathsf{Y}_o$ then we recover the dioperad Y.

Applying the construction to Example 1:

Lemma 1. *For P a operad and t a standard rooted tree, the species $t(\mathsf{P})$ is a dioperad. Further the edge contraction map $t(\mathsf{P}) \to t'(\mathsf{P})$ is a map of dioperads.*

The same statement holds for any standard tree.

Applying this lemma to Example 2 shows that the subset species s is a dioperad. This dioperad structure was discussed in Example 5.

Similarly, applying the construction to Example 3 shows that $\mathsf{P} \circ \mathsf{P}$ is a dioperad. As a special case, Example 4 shows that the partition species p is a dioperad.

Exercise 1. Make the dioperad structure on p explicit.

3. Operads and dioperads revisited

In this section, we give the categorical perspective on operads and dioperads. This involves the notion of a monoid in a monoidal category, which is adequately explained in [16, Chapter 1, Section 1.1]. The categorical viewpoint will be useful later for defining dioperad algebras.

3.1. **Operads as monoids.** Recall that $\mathcal{S}$ is the category of species.

Definition 3.1. Let P and Q be species such that $\mathsf{Q}[\emptyset] = 0$. Define $\mathsf{P} \circ \mathsf{Q}$ to be the species

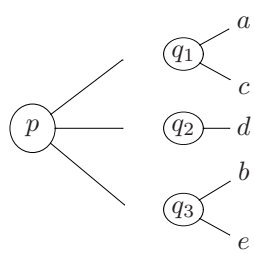
$$(\mathsf{P} \circ \mathsf{Q})[I] = \bigoplus_{\pi \text{ a partition of } I} \mathsf{P}[\pi] \otimes \left(\bigotimes_{J \in \pi} \mathsf{Q}[J] \right).$$

This definition can be extended to the case when $\mathsf{Q}[\emptyset] \neq 0$; see [16, Definition 1.63]. We denote an element of $(\mathsf{P} \circ \mathsf{Q})[I]$ by

$$p \otimes \{q_1, \ldots, q_n\},$$

by which we understand that there is a partition $I = J_1 \sqcup \cdots \sqcup J_n$, with $q_i \in \mathsf{Q}[J_i]$ and $p \in \mathsf{P}[\{J_1, \ldots, J_n\}]$. It is also useful to have a picture in mind for $\mathsf{P} \circ \mathsf{Q}$. We draw, for example,

$p \otimes \{q_1, q_2, q_3\} \;=$



$\in (\mathsf{P} \circ \mathsf{Q})[\{a, b, c, d, e\}].$

Observe that the product $\circ$ defines a monoidal structure on $\mathcal{S}$, with the species u as the unit.

The following is an equivalent formulation of operads.

Definition 3.2. An operad is a monoid in $\mathcal{S}$ for the $\circ$ product. Explicitly, an operad is a species P with

- a map $\Longleftarrow : \mathsf{P} \circ \mathsf{P} \to \mathsf{P}$ of species such that

$$\begin{tikzcd}
\mathsf{P} \circ (\mathsf{P} \circ \mathsf{P}) \arrow[r, "\cong"] \arrow[d, "\mathrm{id} \circ (\Longleftarrow)"] & (\mathsf{P} \circ \mathsf{P}) \circ \mathsf{P} \arrow[r, "(\Longleftarrow) \circ \mathrm{id}"] & \mathsf{P} \circ \mathsf{P} \arrow[d, "\Longleftarrow"] \\
\mathsf{P} \circ \mathsf{P} \arrow[rr, "\Longleftarrow"] & & \mathsf{P}
\end{tikzcd}$$

  commutes. We use the shorthand

$$\Longleftarrow (\Longleftarrow) \;=\; (\Longleftarrow) \Longleftarrow \tag{6}$$

  for the above commutative diagram. We denote the image of $p \otimes \{q_1, \ldots, q_n\}$ under this map by $p \Longleftarrow \{q_1, \ldots, q_n\}$.
- a map $\mathsf{u} \xrightarrow{i} \mathsf{P}$ such that the composite maps

$$\mathsf{P} \xrightarrow{\cong} \mathsf{u} \circ \mathsf{P} \xrightarrow{i \circ \mathrm{id}} \mathsf{P} \circ \mathsf{P} \xrightarrow{\Longleftarrow} \mathsf{P} \quad \text{and} \quad \mathsf{P} \xrightarrow{\cong} \mathsf{P} \circ \mathsf{u} \xrightarrow{\mathrm{id} \circ i} \mathsf{P} \circ \mathsf{P} \xrightarrow{\Longleftarrow} \mathsf{P} \tag{7}$$

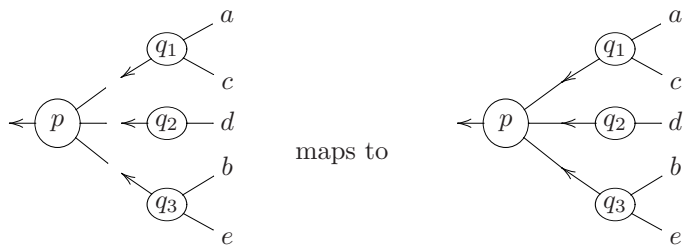
  are the identity.

Lemma 2. *For a unital operad, Definition (2.3+2.4) and Definition 3.2 are equivalent.*

Proof. If P is an operad in the sense of Definition 2.3 then the substitution rule in P defines a map $\Longleftarrow : \mathsf{P} \circ \mathsf{P} \to \mathsf{P}$ of species that sends $p \otimes \{q_1, \ldots, q_n\}$ to

$$p \Longleftarrow \{q_1, \ldots, q_n\} \ := \ (\ldots((p \underset{J_1}{\longleftarrow} q_1) \underset{J_2}{\longleftarrow} q_2) \ldots \underset{J_n}{\longleftarrow} q_n).$$

From Equation (1), the order of substitution of the q_i's does not matter, as required. For a picture notation, we say



$p \otimes \{q_1, q_2, q_3\}$ $\qquad$ $p \Longleftarrow \{q_1, q_2, q_3\}$

Further Equation (1) also shows that Equation (6) holds. And the map $\mathsf{u} \xrightarrow{i} \mathsf{P}$ from Definition 2.4 shows that the composite maps in Equation (7) are the identity. This shows that P is a monoid in $\mathcal{S}$ as claimed in Definition 3.2.

Conversely, starting with Definition 3.2, we use the unit map $\mathsf{u} \to \mathsf{P}$ from the monoid structure of P to define a substitution rule as follows. For $p \in \mathsf{P}[I]$ and $q \in \mathsf{P}[J]$, we set,

$$p \underset{y}{\longleftarrow} q \quad := \quad p \Longleftarrow \{u, \ldots, u, q\},$$

where in the right hand side, we use the partition of $I \setminus \{y\} \sqcup J$, whose one part is J and the rest are singletons. It is then straightforward to verify Definition (2.3+2.4). $\square$

3.2. **Dioperads as similar to dimonoids.** In analogy with operads, it is tempting to say that a dioperad Y is a dimonoid in $\mathcal{S}$ for the $\circ$ product, that is, we have two maps

$$\overset{\dashv}{\Longleftarrow}, \overset{\vdash}{\Longleftarrow} : \ \mathsf{Y} \circ \mathsf{Y} \to \mathsf{Y}$$

which satisfy the dialgebra axioms, see [9, page 11]. However the situation is more complicated. The difficulty is as follows.

Starting with Definition 2.11, given $p \otimes \{q_1, \ldots, q_n\}$, we need to define $p \overset{\dashv}{\Longleftarrow} \{q_1, \ldots, q_n\}$ and $p \overset{\vdash}{\Longleftarrow} \{q_1, \ldots, q_n\}$. The logical thing is to set them to be

$$(\ldots((p \underset{J_1}{\overset{\dashv}{\longleftarrow}} q_1) \underset{J_2}{\overset{\dashv}{\longleftarrow}} q_2) \ldots \underset{J_n}{\overset{\dashv}{\longleftarrow}} q_n) \text{ and } (\ldots((p \underset{J_1}{\overset{\vdash}{\longleftarrow}} q_1) \underset{J_2}{\overset{\vdash}{\longleftarrow}} q_2) \ldots \underset{J_n}{\overset{\vdash}{\longleftarrow}} q_n)$$

respectively. However Equation (5) says that only the first expression above is well defined.

We propose the following solution to this problem.

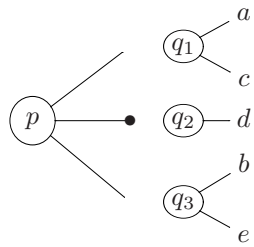
Definition 3.3. We recall [16, Definition 3.27]. Let P, Q and R be three species. Define $\mathsf{P} \circ (\mathsf{Q}, \mathsf{R})$ to be the subspecies of $\mathsf{P} \circ (\mathsf{Q} \oplus \mathsf{R})$ consisting of terms with just one tensor factor from R. And using this, define

$$\mathsf{P} \diamond \mathsf{Q} := \mathsf{P} \circ (\mathsf{Q}, \mathsf{Q}).$$

We denote an element of $(\mathsf{P}\diamond\mathsf{Q})[I]$ by

$$p\otimes\{q_1,\dots,\textcircled{q_i},\dots.q_n\},$$

where the circled entry belongs to the second factor of Q in $\mathsf{P}\circ(\mathsf{Q},\mathsf{Q})$. For a picture notation, we draw, for example,

$p\otimes\{q_1,\textcircled{q_2},q_3\}\ =$ 

$\in(\mathsf{P}\diamond\mathsf{Q})[\{a,b,c,d,e\}],$

with the bullet indicating the second factor of Q.

We make a few observations.

- There is no isomorphism $\mathsf{P}\diamond(\mathsf{Q}\diamond\mathsf{R})\to(\mathsf{P}\diamond\mathsf{Q})\diamond\mathsf{R}$, see Definition 3.4 below. However there is an isomorphism $\mathsf{P}\diamond(\mathsf{Q}\circ\mathsf{R})\to(\mathsf{P}\diamond\mathsf{Q})\circ\mathsf{R}$.
- There is a natural transformation $\mathsf{P}\diamond\mathsf{Q}\to\mathsf{P}\circ\mathsf{Q}$ of species which forgets the bullet.
- We have $\mathsf{P}\diamond\mathsf{Q}=(\mathsf{P}\times\mathsf{Perm})\circ\mathsf{Q}$, where Perm is the permutative operad defined in (2.4). In this language, the natural transformation above forgets the Perm factor.
- The unit species u is only a left unit for the $\diamond$ product. More precisely, we have $\mathsf{u}\diamond\mathsf{P}=\mathsf{P}$ while $\mathsf{P}\diamond u\ =\ \mathsf{P}\times\mathsf{Perm}$.

Definition 3.4. We define an associator $\mathsf{P}\diamond(\mathsf{Q}\diamond\mathsf{R})\to(\mathsf{P}\diamond\mathsf{Q})\diamond\mathsf{R}$ that sends, for example,

$$p\otimes\big\{q_1\otimes\{r_{11},\textcircled{r_{12}}\},\textcircled{q_2}\otimes\{\textcircled{r_{21}},r_{22},r_{23}\},q_3\otimes\{\textcircled{r_{31}},r_{32}\}\big\}$$

to

$$\big(p\otimes\{q_1\otimes\textcircled{q_2}\otimes q_3\}\big)\otimes\{r_{11},r_{12},\textcircled{r_{21}},r_{22},r_{23},r_{31},r_{32}\}.$$

For definiteness, we have taken $p\in\mathsf{P}[3]$, $q_1\in\mathsf{Q}[2]$, $q_2\in\mathsf{Q}[3]$ and $q_3\in\mathsf{Q}[2]$. Also we have written $\textcircled{q_2}\otimes\{\textcircled{r_{21}},r_{22},r_{23}\}$ instead of

$$\textcircled{q_2\otimes\{\textcircled{r_{21}},r_{22},r_{23}\}}$$

for notational simplicity. The associator satisfies the Maclane pentagon condition, see [16, page 37]. However it is neither one to one nor onto. Hence the $\diamond$ product does not define a monoidal category. However one can still define monoids and dimonoids using the associator.

We can now give an equivalent formulation of dioperads. It turns out to be stronger than the notion of a dimonoid in $\mathcal{S}$ with the $\diamond$ product.

Definition 3.5. A dioperad is a species Y with two maps of species

$$\overset{\dashv}{\Longleftarrow},\overset{\vdash}{\Longleftarrow}:\ \mathsf{Y}\diamond\mathsf{Y}\to\mathsf{Y},$$

which we abbreviate to $\dashv$ and $\vdash$, such that

(i) the map $\dashv$ factors through the quotient $\mathsf{Y}\diamond\mathsf{Y}\to\mathsf{Y}\circ\mathsf{Y}$, and

(ii) the following axioms hold.

$$\dashv(\dashv,\vdash) \;=\; \dashv(\vdash,\dashv) \;=\; (\dashv)\dashv$$

$$\vdash(\vdash,\dashv) \;=\; (\vdash)\dashv$$

$$\vdash(\dashv,\vdash) \;=\; (\vdash)\vdash \;=\; (\dashv)\vdash$$

The notation used above is similar to the one used in Equation (6). For example, the map $(\vdash)\dashv$ is the composite map

$$(\mathsf{Y}\diamond\mathsf{Y})\diamond\mathsf{Y} \xrightarrow{\vdash\circ\,\mathrm{id}} \mathsf{Y}\diamond\mathsf{Y} \xrightarrow{\dashv} \mathsf{Y}\,.$$

However the notation of the form $\dashv(\dashv,\vdash)$ still requires some explanation. It stands for the composite map

$$\mathsf{Y}\diamond(\mathsf{Y}\diamond\mathsf{Y}) = \mathsf{Y}\circ(\mathsf{Y}\diamond\mathsf{Y},\mathsf{Y}\diamond\mathsf{Y}) \xrightarrow{\mathrm{id}\circ(\dashv,\vdash)} \mathsf{Y}\circ(\mathsf{Y},\mathsf{Y}) \xrightarrow{\dashv} \mathsf{Y},$$

Note that in condition (ii), whenever there are two maps inside a bracket, they are always different. We denote the image of $p\otimes\{q_1,\ldots,q_i,\ldots,q_n\}$ under the map, say $\overset{\vdash}{\Longleftarrow}$, by the notation $p\overset{\vdash}{\Longleftarrow}\{q_1,\ldots,q_i,\ldots,q_n\}$.

Definition 3.6. A bar unit in a dioperad Y is a map $\mathsf{u}\xrightarrow{i}\mathsf{Y}$ of species such that the composite maps

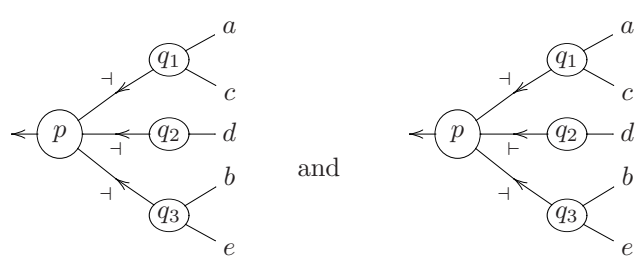
$$\mathsf{Y}\xrightarrow{\cong} u\diamond\mathsf{Y}\xrightarrow{i\diamond\mathrm{id}}\mathsf{Y}\diamond\mathsf{Y}\xrightarrow{\vdash}\mathsf{Y}\quad\text{and}\quad(\mathsf{Y}\times\mathsf{Perm})\xrightarrow{\cong}\mathsf{Y}\diamond u\xrightarrow{\mathrm{id}\diamond i}\mathsf{Y}\diamond\mathsf{Y}\xrightarrow{\dashv}\mathsf{Y}$$

are the identity and the projection on the first factor respectively. So it is only assumed that u substitutes trivially from the bar side.

Remark. It is clear that condition (i) of Definition 3.5 implies that a dioperad for which $\overset{\dashv}{\Longleftarrow}$ and $\overset{\vdash}{\Longleftarrow}$ coincide is same as an operad.

Lemma 3. *For a bar unital dioperad, Definitions (2.11+2.12) and (3.5+3.6) are equivalent.*

Proof. If Y is a dioperad in the sense of Definition 2.11 then the substitution rules in Y define two maps $\overset{\dashv}{\Longleftarrow},\overset{\vdash}{\Longleftarrow}:\ \mathsf{Y}\diamond\mathsf{Y}\to\mathsf{Y}$ of species that send $p\otimes\{q_1,q_2,q_3\}$, for example, to



$p\overset{\dashv}{\Longleftarrow}\{q_1,q_2,q_3\}$ $\qquad$ $p\overset{\vdash}{\Longleftarrow}\{q_1,q_2,q_3\}$

respectively. From Equation (5), these are well defined, that is, the order of substitution of the q_i's does not matter. It is clear from the definition that the map

$\overset{\dashv}{\Longleftarrow}$ factors through the quotient $\mathsf{Y}\diamond\mathsf{Y} \to \mathsf{Y}\circ\mathsf{Y}$. The axioms in condition (ii) in Definition 3.5 are routine to check. For example, the axiom

$$\overset{\dashv}{\Longleftarrow}(\overset{\dashv}{\Longleftarrow},\overset{\vdash}{\Longleftarrow}) \;=\; \overset{\dashv}{\Longleftarrow}(\overset{\vdash}{\Longleftarrow},\overset{\dashv}{\Longleftarrow})$$

follows from repeated shuffling of the tensor factors and use of the axiom

$$p \overset{\dashv}{\longleftarrow} (q \overset{\vdash}{\longleftarrow} r) = p \overset{\dashv}{\longleftarrow} (q \overset{\dashv}{\longleftarrow} r).$$

Further the bar unit map in Definition 2.12 satisfies the bar unit axioms in Definition 3.6. This shows that Y is a dioperad in the sense of Definition (3.5+3.6).

Conversely, starting with Definition (3.5+3.6), we use the bar unit map $\mathsf{u} \to \mathsf{Y}$ to define a substitution rule as follows. For $p \in \mathsf{Y}[I]$ and $q \in \mathsf{Y}[J]$, we set,

$$\begin{aligned} p \underset{y}{\overset{\dashv}{\longleftarrow}} q \quad &:= \quad p \overset{\dashv}{\Longleftarrow} \{u,\dots,u,\textcircled{q}\} \quad = \;\dots\; = \quad p \overset{\dashv}{\Longleftarrow} \{\textcircled{u},\dots,u,q\} \\ p \underset{y}{\overset{\vdash}{\longleftarrow}} q \quad &:= \quad p \overset{\vdash}{\Longleftarrow} \{u,\dots,u,\textcircled{q}\}, \end{aligned}$$

where in the right hand sides, we use the partition of $I \setminus \{y\} \sqcup J$, whose one part is J and the rest are singletons. Note that in the first definition, the position of the circle does not matter, while in the second definition, the circle is on q. The main part now is to check the axioms in condition (ii) in Definition 2.11. We give two typical illustrations. For definiteness, let $p \in \mathsf{Y}[3]$, $q \in \mathsf{Y}[3]$ and $r \in \mathsf{Y}[2]$.

The axiom $\dashv(\dashv,\vdash) \;=\; \dashv(\vdash,\dashv)$ says that

$$\begin{aligned} p \overset{\dashv}{\Longleftarrow} \left\{ u \overset{\dashv}{\Longleftarrow} \textcircled{u}, \textcircled{q} \overset{\vdash}{\Longleftarrow} \{\textcircled{r},u,u\}, u \overset{\dashv}{\Longleftarrow} \textcircled{u} \right\} \\ = \quad p \overset{\dashv}{\Longleftarrow} \left\{ u \overset{\vdash}{\Longleftarrow} \textcircled{u}, \textcircled{q} \overset{\dashv}{\Longleftarrow} \{\textcircled{r},u,u\}, u \overset{\vdash}{\Longleftarrow} \textcircled{u} \right\}, \end{aligned}$$

which we may also write as

$$p \overset{\dashv}{\Longleftarrow} \left\{ \begin{array}{ccc} u & \overset{\dashv}{\Longleftarrow} & \textcircled{u} \\ \textcircled{q} & \overset{\vdash}{\Longleftarrow} & \left\{ \begin{array}{c} \textcircled{r} \\ u \\ u \end{array} \right\} \\ u & \overset{\dashv}{\Longleftarrow} & \textcircled{u} \end{array} \right\} \quad = \quad p \overset{\dashv}{\Longleftarrow} \left\{ \begin{array}{ccc} u & \overset{\vdash}{\Longleftarrow} & \textcircled{u} \\ \textcircled{q} & \overset{\dashv}{\Longleftarrow} & \left\{ \begin{array}{c} \textcircled{r} \\ u \\ u \end{array} \right\} \\ u & \overset{\vdash}{\Longleftarrow} & \textcircled{u} \end{array} \right\}.$$

This implies that $p \overset{\dashv}{\longleftarrow} (q \overset{\vdash}{\longleftarrow} r) = p \overset{\dashv}{\longleftarrow} (q \overset{\dashv}{\longleftarrow} r)$.

We now illustrate the first part of Equation (5). Note that

$$p \overset{\dashv}{\Longleftarrow} (q,u,r) \quad = \quad p \overset{\dashv}{\Longleftarrow} (\textcircled{q},u,r) \quad = \quad p \overset{\dashv}{\Longleftarrow} (q,u,\textcircled{r}).$$

Using the axiom $\dashv(\vdash,\dashv) \;=\; (\dashv)\dashv$, the middle term is equal to

$$\begin{aligned} p \overset{\dashv}{\Longleftarrow} \left\{ \textcircled{q} \overset{\dashv}{\Longleftarrow} \{u,\textcircled{u},u\}, u \overset{\vdash}{\Longleftarrow} \textcircled{u}, u \overset{\vdash}{\Longleftarrow} \textcircled{r} \right\} \\ = \quad \left\{ p \overset{\dashv}{\Longleftarrow} \{\textcircled{q},u,u\} \right\} \overset{\dashv}{\Longleftarrow} \{u,\textcircled{u},u,u,r\} \\ = \quad \left\{ p \overset{\dashv}{\Longleftarrow} \{q,u,u\} \right\} \overset{\dashv}{\Longleftarrow} \{u,u,u,u,r\} \end{aligned}$$

Again using the same axiom, or by symmetry, the last term is equal to

$$\begin{aligned} p \overset{\dashv}{\Longleftarrow} \Big\{ u \overset{\vdash}{\Longleftarrow} \textcircled{q}, u \overset{\vdash}{\Longleftarrow} \textcircled{u}, \textcircled{r} \overset{\dashv}{\Longleftarrow} \{\textcircled{u}, u\} \Big\} & \\ = \quad & \Big\{ p \overset{\dashv}{\Longleftarrow} \{u, u, \textcircled{r}\} \Big\} \overset{\dashv}{\Longleftarrow} \{q, u, \textcircled{u}, u\} \\ = \quad & \Big\{ p \overset{\dashv}{\Longleftarrow} \{u, u, r\} \Big\} \overset{\dashv}{\Longleftarrow} \{q, u, u, u\} \end{aligned}$$

Comparing the last two expressions shows that $r \overset{\vdash}{\longrightarrow} p \overset{\dashv}{\longleftarrow} q$ is well defined as required. □

3.3. **Comparison between dioperads and dimonoids.**

Lemma 4. *A bar unital dioperad satisfies the following additional axioms.*

$$\dashv (\vdash, \vdash) \;=\; \dashv (\dashv, \dashv) \;=\; (\dashv) \dashv$$

$$\vdash (\dashv, \dashv) \;=\; (\vdash) \dashv$$

$$\vdash (\vdash, \vdash) \;=\; (\vdash) \vdash \;=\; (\dashv) \vdash$$

Note that whenever there are two maps inside a bracket, they are identical.

Equivalently, every dioperad is a dimonoid in $\mathcal{S}$ for the $\diamond$ product.

The converse to the above lemma is false.

4. Reversible dioperads

In this section, we recall briefly the notion of a reversible operad and the construction of the mating functor. The main reference for this part is [14]. Then we discuss the corresponding notions for a dioperad along with examples. We write $\mathcal{P}_r$ and $\mathcal{Y}_r$ for the categories of reversible operads and reversible dioperads respectively.

4.1. **Reversible operads.**

To define a reversible operad, one needs to treat the output of an operad element on par with its inputs. This requires a slight change of perspective. We replace the category $\mathtt{Set}$ with the equivalent category of pointed sets $\mathtt{Set}_p$ defined as follows. An object of $\mathtt{Set}_p$ is a pair of disjoint sets (I, U), where U is a singleton. A morphism between (I, U) and (J, V) is a bijection between I and J.

Definition 4.1. A species Q is equivalently a functor

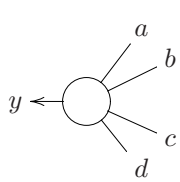
$$\mathtt{Set}_p \longrightarrow \mathtt{Vect}\,.$$

We denote the image of (I, U) by $\mathsf{Q}[I, U]$.

An operad P is a species P with a substitution rule

$$\mathsf{P}[I, U] \otimes \mathsf{P}[J, V] \to \mathsf{P}[(I \setminus V) \cup J, U] \quad \text{for} \quad V \subseteq I,$$

with the same constraints as in Definition 2.3. An element of $\mathsf{P}[I, U]$ can be schematically drawn as



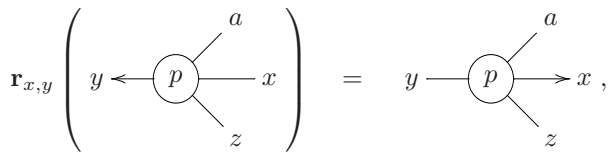
$\in \mathsf{P}[I, U]$ for $I = \{a, b, c, d\}$ and $U = \{y\}$,

with the output labelled by the unique element of U. So the notation $p \underset{y}{\longleftarrow} q$ now means that the label of the output of q is y and it is fed to an input of p whose label is again y.

Definition 4.2. A reversible operad is a operad P such that for each object $(I, \{y\})$ in $\mathtt{Set}_p$ and $x \in I$, there exist reversal maps

$$\mathbf{r}_{x,y} : \mathsf{P}[I, \{y\}] \to \mathsf{P}[(I \setminus \{x\}) \cup \{y\}, \{x\}],$$

which satisfy the two conditions below. One visualises the reversal rule as



namely, it allows one to switch the output with any given input.

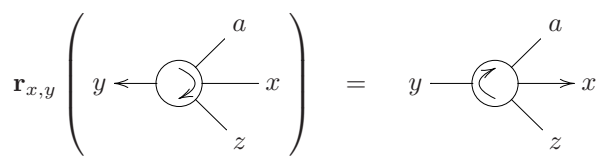
(1) We require $\mathbf{r}_{y,x} \circ \mathbf{r}_{x,y} = \mathrm{id}$ and $\mathbf{r}_{x,y} \circ \mathbf{r}_{y,z} = \mathbf{r}_{x,z}$. It follows from these two relations that the composite of a sequence of reversals is either the identity map or a single reversal.

(2) Reversal is compatible with substitution. That is,

$$\mathbf{r}_{x,z}(p \underset{y}{\longleftarrow} q) = \begin{cases} \mathbf{r}_{x,z}(p) \underset{y}{\longleftarrow} q & \text{if } x \text{ is an input of } p, \\ \mathbf{r}_{x,y}(p) \underset{y}{\longrightarrow} \mathbf{r}_{y,z}(q) & \text{if } x \text{ is an input of } q. \end{cases}$$

Here y labels one of the inputs of p as well as the output of q and z labels the output of p.

Among the examples in (2.4), the operads u, c and a are reversible, whereas Perm is not reversible. The reversal rule for the associative operad a is as follows.



Definition 4.3. A morphism between reversible operads P and Q is a map of operads between P and Q which commutes with the respective reversal maps.

4.2. **Reversible dioperads.** As suggested by the present discussion, in order to define a reversible dioperad, we must also modify our perspective on dioperads. Accordingly, a dioperad Y is a species with two substitution rules

$$\mathsf{Y}[I, U] \otimes \mathsf{Y}[J, V] \to \mathsf{Y}[I \setminus V \cup J, U] \quad \text{for} \quad V \subseteq I,$$

with the same constraints as in Definition 2.11.

Definition 4.4. A reversible dioperad is a dioperad Y such that for each object $(I, \{y\})$ in $\mathtt{Set}_p$ and $x \in I$, there exist reversal maps

$$\mathbf{r}_{x,y} : \mathsf{Y}[I, \{y\}] \to \mathsf{Y}[(I \setminus \{x\}) \cup \{y\}, \{x\}],$$

which satisfy the two conditions below.

(1) We require $\mathbf{r}_{y,x} \circ \mathbf{r}_{x,y} = \mathrm{id}$ and $\mathbf{r}_{x,y} \circ \mathbf{r}_{y,z} = \mathbf{r}_{x,z}$.

(2) Reversal is compatible with substitution. That is,

$$\mathbf{r}_{x,z}(p \xleftarrow[y]{\dashv} q) = \begin{cases} \mathbf{r}_{x,z}(p) \xleftarrow[y]{\dashv} q & \text{if } x \text{ is an input of } p, \\ \mathbf{r}_{x,y}(p) \xrightarrow[y]{\dashv} \mathbf{r}_{y,z}(q) & \text{if } x \text{ is an input of } q. \end{cases}$$

And similarly with $\dashv$ replaced by $\vdash$. Here y labels one of the inputs of p as well as the output of q and z labels the output of p.

It may appear at first glance that the conditions for $\dashv$ and $\vdash$ are independent. However this is not true, as is clear from the substitution convention defined in Equation (4).

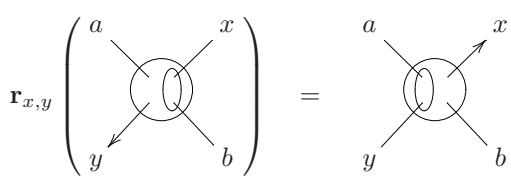
Definition 4.5. A morphism between reversible dioperads Y and Z is a map of dioperads between Y and Z which commutes with the respective reversal maps.

4.3. **Examples of reversible dioperads.** We now look at the examples of dioperads in (2.8) and see which of them are reversible.

Example 10. The subset dioperad discussed in Example 5 is reversible. For $K \subseteq I$ and $x \in I$, define

$$\mathbf{r}_{x,y}(K) = \begin{cases} (I \setminus K) \cup \{y\} & \text{if } x \in K, \\ K & \text{if } x \notin K. \end{cases}$$

One can check that this satisfies the required conditions.



The above picture illustrates the first case.

Example 11. An involution $\mathbf{r}$ on a dialgebra D is a map $\mathbf{r} : \mathsf{D} \to \mathsf{D}$ such that

$$\mathbf{r}(a \dashv b) = \mathbf{r}(b) \vdash \mathbf{r}(a), \quad \text{and} \quad \mathbf{r}(a \vdash b) = \mathbf{r}(b) \dashv \mathbf{r}(a).$$

Now let D_{do} be the dioperad associated to D, as in Example 6. Then D_{do} is reversible if and only if D has an involution.

This is the analogue to the fact that the operad A_o is reversible if and only if the associative algebra A has an involution.

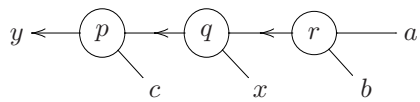
Example 12. We now refer to Example 8. If P is a reversible operad then the associated dioperad P_{do}, for which $\xleftarrow{\dashv}$ and $\xleftarrow{\vdash}$ coincide, is a reversible dioperad. This is clear from the definitions. Conversely, let Y be a reversible dioperad. Then the reversal maps of Y preserve the ideal $\mathsf{F}(\mathsf{Y})$ and hence the quotient Y_o is a reversible operad. Thus we obtain an induced pair of adjoint functors

$$do : \mathcal{P}_r \to \mathcal{Y}_r \quad \text{and} \quad o : \mathcal{Y}_r \to \mathcal{P}_r.$$

Example 13. We now discuss reversibility for the dioperad constructed using a standard tree; see the remark after Example 9. It will be clear from the discussion below that a standard rooted tree is not relevant here.

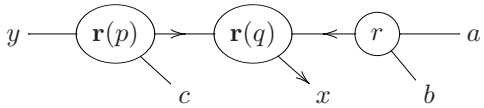
Lemma 5. *Let $\bar{t}$ be a standard tree and P be a reversible operad. Then the dioperad $\bar{t}(\mathsf{P})$ is reversible.*

Proof. We illustrate with $\bar{t}$ = •—•—•. Let $(I, U) = (\{a, b, c, x\}, \{y\})$ and



$\in\ \bar{t}(\mathsf{P})[I, U]$,

where p, q and r refer to elements of P. Then the reversal map $\mathbf{r}_{x,y}$ applied to the above element gives



$\in\ \bar{t}(\mathsf{P})[(I \setminus \{x\}) \cup \{y\}, \{x\}]$.

Namely, using the reversal maps of P, we reverse those elements of P, which lie on the path joining x and y. The root of the tree so produced is labelled by x. It is now straightforward to check that the dioperad $\bar{t}(\mathsf{P})$ is reversible. □

Applying this lemma to $\bar{t}$ = •—• and $\mathsf{P} = c$, we get the reversal rules for the subset dioperad s written in Example 10.

Now we describe explicitly the reversal rules for $\bar{t}(\mathsf{c})$, for any standard tree $\bar{t}$. Let $C(\bar{t})$ be the configuration of circles on the sphere corresponding to $\bar{t}$ under the bijection of Equation (3). An element of $\bar{t}(\mathsf{c})[I, U]$ then consists of $C(\bar{t})$ along with points labelled by elements of $I \cup U$, such that the points do not lie on any of the circles and each of the disc-like regions contains at least one point. Further the point labelled by the element of U is marked as special. The reversal map $\mathbf{r}_{x,y}$, with $y \in U$, applied to the above element, changes the marking from y to x, keeping the rest of the configuration untouched.

4.4. **The mating functor.** We now discuss a crucial ingredient for (di)operad geometry.

Definition 4.6. Define a species Q starting with a reversible operad P as follows. For any finite set K, let

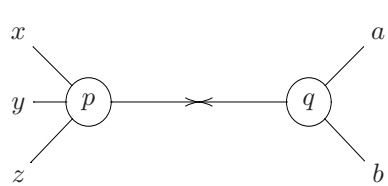
$$\mathsf{Q}[K] = \bigoplus_{I \sqcup J = K} \mathsf{P}[I, U] \otimes \mathsf{P}[J, U],$$

subject to the two relations below.

$$p \otimes q \ = \ q \otimes p \tag{R1}$$

$$(p \to q) \otimes r \ = \ p \otimes (\mathbf{r}(q) \leftarrow r). \tag{R2}$$

The set U is any singleton, all choices being considered equivalent. We interpret the tensor sign as a mating, and say that $p \otimes q$ is a mating of p and q. We show it as

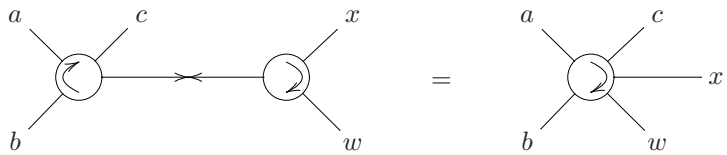


with two opposing arrowheads in the centre. For a shorthand we write $p \rightarrowtail\!\!\!\!\leftarrow q$. The symmetry of relation $(R1)$ is built into this notation.

This defines the mating functor

$$\mathcal{P}_r \longrightarrow \mathcal{S}.$$

It maps the operads u, c and a to the species uu, cc and aa respectively.



The above picture illustrates the associative case. In general, we use the notation that the operad P maps to the species $\mathsf{Q} = \mathsf{PP}$. We refer to PP as the mated species of P.

Definition 4.7. Define a species Z starting with a reversible dioperad Y as follows. For any finite set K, let

$$\mathsf{Z}[K] = \bigoplus_{I \sqcup J = K} \mathsf{Y}[I, U] \otimes \mathsf{Y}[J, U] \otimes k\{\dashv, \vdash\},$$

subject to the two relations below. The set U is any singleton, all choices being considered equivalent. We denote $p \otimes q \otimes \dashv$ by the notation $p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow q$ and $p \otimes q \otimes \vdash$ by the notation $p \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow q$, in analogy with the operad situation.

(R1) $$p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow q \;=\; q \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow p$$

(R2) $$\begin{aligned} (p \xrightarrow{\dashv} q) \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow r \;&=\; (p \xrightarrow{\vdash} q) \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow r \;=\; p \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (\mathbf{r}(q) \xleftarrow{\vdash} r). \\ (p \xrightarrow{\vdash} q) \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow r \;&=\; p \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (\mathbf{r}(q) \xleftarrow{\dashv} r). \\ (p \xrightarrow{\dashv} \mathbf{r}(q)) \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow r \;&=\; p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (q \xleftarrow{\dashv} r) \;=\; p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (q \xleftarrow{\vdash} r). \end{aligned}$$

It follows from relation $(R1)$ that the first and third lines of relation $(R2)$ are equivalent. However as written the similarity with the dialgebra axioms is evident.

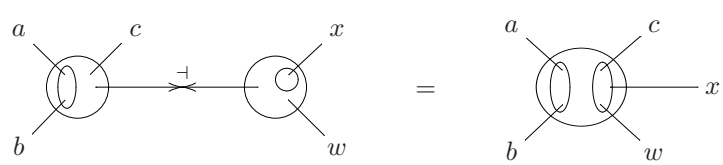
This defines the mating functor

$$\mathcal{Y}_r \longrightarrow \mathcal{S}.$$

As an example, it maps the dioperad s to the species ss. The isomorphism is as follows. Let $K \in \mathsf{s}[I]$ and $L \in \mathsf{s}[J]$ be nonempty subsets of I and J respectively. Then

$$K \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow L \;=\; \{K, (I \cup J) \setminus K\}.$$

Note that the right hand side is a partition of $I \cup J$ into two parts, which is an element of $\mathsf{ss}[I \cup J]$. As an example,



In general, we use the notation that the dioperad Y maps to the species $\mathsf{Z} = \mathsf{YY}$. We refer to YY as the mated species of Y.

Exercise 2. For a standard tree $\bar{t}$ and a species Q, define the species $\bar{t}(\mathsf{Q})$ appropriately such that for a reversible operad P, the mating functor maps the dioperad $\bar{t}(\mathsf{P})$ to the species $\bar{t}(\mathsf{PP})$. In particular, for $\bar{t} = \bullet$, we obtain that the dioperad P_{do} maps to the species PP.

4.5. **The partial derivative.** An element of $\mathsf{Z}[K]$ can be expressed as a mating in many ways, using the relations $(R1)$ and $(R2)$. A mating of the form

$$p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u,$$

with the element of the unit operad on the bar side is called a trivial mating. Such matings play an important role in the concept of a partial derivative for a mated species, which we define below.

Proposition 1. *Let Y be a reversible dioperad and $\mathsf{Z} = \mathsf{YY}$ be its mated species. For $w \in K$ and $q \in \mathsf{Z}[K]$, there is a unique element $p = \frac{\partial q}{\partial w} \in \mathsf{Y}[K \setminus a]$ such that*

$$q \;=\; \frac{\partial q}{\partial w} \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u,$$

where $u \in \mathsf{u}[\{w\}]$.

This is the dioperad analogue of [14, Proposition 1]. We refer to $\frac{\partial q}{\partial w}$ as the operad element obtained by cutting q at the input w.

Proof. There are two parts to the proposition. The first one is the existence of $\frac{\partial q}{\partial w}$, which means that every mating can be expressed as a trivial mating. To prove this, say $q \;=\; r \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} s$ and w is an input of s. Then

$$\begin{aligned}
q \;&=\; r \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} s \\
&=\; r \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} (s \xleftarrow{\dashv} u) && (u \text{ is a bar unit}) \\
&=\; (r \xleftarrow{\vdash} s) \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u && (\text{Relation } (R2)) \\
&=\; p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u && (\text{Setting } p \;=\; r \xleftarrow{\vdash} s)
\end{aligned}$$

The case $q \;=\; r \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} s$ is handled similarly.

The second part is to show the uniqueness of $\frac{\partial q}{\partial w}$. Suppose that

$$q \;=\; p_1 \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u \;=\; p_2 \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u.$$

This means that one can obtain $p_2 \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u$ from $p_1 \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u$ by successive applications of relation $(R2)$. Now the reduction lemma below says that this can be done in one step. This implies that $p_1 = p_2$. □

Reduction lemma. *Let Y be a reversible dioperad. The result obtained by two successive applications of relation $(R2)$ to the trivial mating $p \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\xleftarrow{} u$ can, in fact, be obtained by a single application of $(R2)$.*

Proof. This involves a straightforward case analysis, which we omit. The reader may consult the proof of the operad version of this lemma in [14, Section 3.4]. □

5. Dioperad algebras

In this section, we briefly recall the notions of an operad algebra, its derivations, and the free operad algebra. We then discuss the corresponding notions for a dioperad.

5.1. **Operad algebras.** Let Σ_n be the symmetric group on n letters. For any species Q, we write $\mathsf{Q}[n]$ as a shorthand for $\mathsf{Q}[\{1, 2, \ldots, n\}]$. Since Q is a functor, each element π of Σ_n induces a map $\mathsf{Q}(\pi) : \mathsf{Q}[n] \to \mathsf{Q}[n]$. In fact, one sees that $\mathsf{Q}[n]$ is a Σ_n-module.

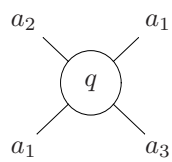
Definition 5.1. A species can be equivalently defined as a sequence $\mathsf{Q}[0]$, $\mathsf{Q}[1]$, $\mathsf{Q}[2]$, ..., where $\mathsf{Q}[n]$ is a Σ_n-module.

Definition 5.2. For Q a species and A a vector space, define

$$Q \circ \mathsf{A} \;=\; \bigoplus_{j \geq 0} \left(\mathsf{Q}[j] \otimes \mathsf{A}^{\otimes j}\right)_{\Sigma_j},$$

the space of coinvariants, where Σ_j acts on $\mathsf{Q}[j]$ as above and on $\mathsf{A}^{\otimes j}$ by permuting the tensor factors.

This definition matches the $\circ$ product in $\mathcal{S}$ if we regard A as a species concentrated in degree 0. Hence we may denote an element of $\mathsf{Q} \circ \mathsf{A}$ by $q \otimes \{a_1, \ldots, a_j\}$. For a picture, we draw



$\in \mathsf{Q} \circ \mathsf{A}\,.$

Definition 5.3. Let P be an operad. A P-algebra is a vector space A with a map $\Longleftarrow : \mathsf{P} \circ \mathsf{A} \to \mathsf{A}$ such that the equation

$$\Longleftarrow (\Longleftarrow) \;=\; (\Longleftarrow) \Longleftarrow$$

holds as a map from $\mathsf{P} \circ \mathsf{P} \circ \mathsf{A}$ to A. The above notation is as in Equation (6). We denote the image of $p \otimes \{a_1, \ldots, a_n\}$ under this map by $p \Longleftarrow \{a_1, \ldots, a_n\}$. For a picture, we draw

We also require that the composite map

$$\mathsf{A} \xrightarrow{\cong} \mathsf{u} \circ \mathsf{A} \longrightarrow \mathsf{P} \circ \mathsf{A} \xrightarrow{\Longleftarrow} \mathsf{A}$$

is the identity.

A morphism between P-algebras A and B is a linear map $\mathsf{A} \to \mathsf{B}$ which commutes with the respective $\Longleftarrow$ maps.

We denote by P-`alg` the category of P-algebras.

Definition 5.4. A derivation ξ of the P-algebra A is a linear map $\xi : \mathsf{A} \to \mathsf{A}$ such that

$$\xi\left(q \Longleftarrow \{a_1, \ldots, a_j\}\right) \;=\; \sum_{i=1}^{j} q \Longleftarrow \{a_1, \ldots, \xi(a_i), \ldots, a_j\}.$$

Alternatively, for any linear map $\xi : \mathsf{A} \to \mathsf{A}$, we define $\tilde{\xi} : \mathsf{P} \circ \mathsf{A} \to \mathsf{P} \circ \mathsf{A}$ by

$$\tilde{\xi}\,(q \otimes \{a_1, \ldots, a_j\}) \;=\; \sum_{i=1}^{j} q \otimes \{a_1, \ldots, \xi(a_i), \ldots, a_j\}.$$

A derivation ξ of the P-algebra A is a linear map $\xi : \mathsf{A} \to \mathsf{A}$ such that

$$\begin{tikzcd} \mathsf{P} \circ \mathsf{A} \arrow[r, "\tilde{\xi}"] \arrow[d, "\Longleftarrow"'] & \mathsf{P} \circ \mathsf{A} \arrow[d, "\Longleftarrow"] \\ \mathsf{A} \arrow[r, "\xi"'] & \mathsf{A} \end{tikzcd}$$

commutes. A good shorthand for this diagram is

$$(\;\Longleftarrow\;)\;\xi \;=\; \Longleftarrow\;(\;\tilde{\xi}\;). \tag{8}$$

The space of derivations of A, denoted $\mathrm{Der}(\mathsf{A})$, forms a Lie algebra for the bracket

$$[\xi, \eta] := \xi \circ \eta - \eta \circ \xi. \tag{9}$$

One needs to use the identity $[\tilde{\xi}, \tilde{\eta}] = \widetilde{[\xi, \eta]}$ to prove this fact.

5.2. **The free operad algebra.**

Definition 5.5. For P an operad and V a vector space, let $\mathsf{P} \circ \mathsf{V}$ be the P-algebra given by

$$\mathsf{P} \circ (\mathsf{P} \circ \mathsf{V}) \xrightarrow{\cong} (\mathsf{P} \circ \mathsf{P}) \circ \mathsf{V} \xrightarrow{\Longleftarrow \circ \mathrm{id}} \mathsf{P} \circ \mathsf{V}\,.$$

This defines a functor

$$\mathsf{P} \circ (-) \;:\; \mathtt{Vect} \longrightarrow \mathsf{P}\text{-}\mathtt{alg},$$

which is called the free functor.

We denote the Lie algebra $\mathrm{Der}(\mathsf{P} \circ \mathsf{V})$ by the notation $\mathrm{Der}(gl_n, \mathsf{P})$, where n is the dimension of V.

Proposition 2. *The free functor is the left adjoint to the forgetful functor*

$$\mathsf{P}\text{-}\mathtt{alg} \longrightarrow \mathtt{Vect}\,.$$

In other words, $\mathsf{P} \circ \mathsf{V}$ *is the free* P*-algebra on* V*.*

One may phrase the above proposition by the following universal property.

Given a P-algebra A and a linear map $F : \mathsf{V} \to \mathsf{A}$, there is a unique map of P-algebras $\check{F} : \mathsf{P} \circ \mathsf{V} \to \mathsf{A}$ which extends F.

It is defined as the composite map

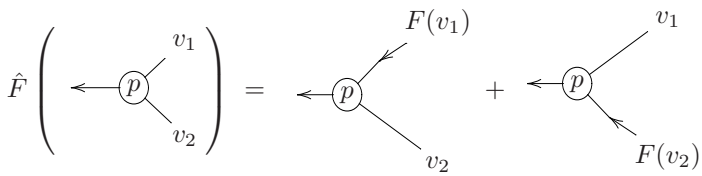
$$\check{F} :\; \mathsf{P} \circ \mathsf{V} \xrightarrow{\mathrm{id} \circ F} \mathsf{P} \circ \mathsf{A} \xrightarrow{\Longleftarrow} \mathsf{A}\,.$$

It is a good exercise to check that this is a map of algebras.

Proposition 3. *A derivation on* $\mathsf{P} \circ \mathsf{V}$ *is uniquely determined by its value on* V*. That is,*

$$\mathrm{Hom}(\mathsf{V}, \mathsf{P} \circ \mathsf{V}) \xrightarrow{\cong} \mathrm{Der}(\mathsf{P} \circ \mathsf{V}).$$

Proof. The above map extends $F \in \mathrm{Hom}(\mathsf{V}, \mathsf{P} \circ \mathsf{V})$ to a derivation $\hat{F}$ of $\mathsf{P} \circ \mathsf{V}$ as follows. We illustrate it on an example.



And by definition, $\hat{F}$ sends $\mathsf{P}[0]$, the degree 0 part of $\mathsf{P} \circ \mathsf{V}$, to 0. The inverse map restricts a derivation ξ to

$$\xi\,|_{\mathsf{V}} : \ \mathsf{V} \xrightarrow{\cong} u \circ \mathsf{V} \longrightarrow \mathsf{P} \circ \mathsf{V} \xrightarrow{\xi} \mathsf{P} \circ \mathsf{V}\,.$$

□

Definition 5.6. The Lie structure on $\mathrm{Hom}(\mathsf{V}, \mathsf{P} \circ \mathsf{V})$ induced by the above isomorphism is given by

$$[F, G] = \hat{F} \circ G - \hat{G} \circ F. \tag{10}$$

A pictorial description of this can be given by writing $\mathrm{Hom}(\mathsf{V}, \mathsf{P} \circ \mathsf{V}) \cong \mathsf{V}^* \otimes (\mathsf{P} \circ \mathsf{V})$, as explained at the end of this section.

5.3. **Dioperad algebras.**

Definition 5.7. Let Y be an dioperad. A Y-algebra is a vector space A with two maps

$$\overset{\dashv}{\Longleftarrow}, \overset{\vdash}{\Longleftarrow} : \ \mathsf{Y} \diamond \mathsf{A} \to \mathsf{Y},$$

which we abbreviate to $\dashv$ and $\vdash$, such that

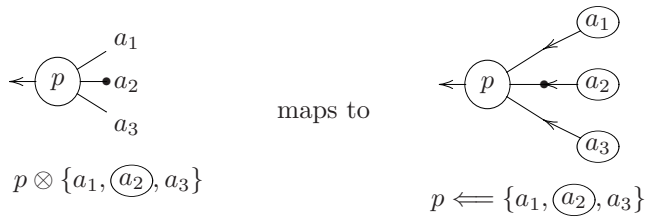
(i) the map $\dashv$ factors through the quotient $\mathsf{Y} \diamond \mathsf{A} \to \mathsf{Y} \circ \mathsf{A}$, and
(ii) the following axioms hold as a map from $\mathsf{Y} \diamond (\mathsf{Y} \diamond \mathsf{A})$ to A.

$$\dashv (\vdash, \vdash) \ = \dashv (\dashv, \dashv) \ = \ (\dashv) \dashv$$

$$\vdash (\dashv, \dashv) \ = \ (\vdash) \dashv$$

$$\vdash (\vdash, \vdash) \ = \ (\vdash) \vdash = \ (\dashv) \vdash$$

The notation used above is same as the one used in Definition 3.5. We denote the image of $p \otimes \{a_1, \ldots, \textcircled{a_i}, \ldots, a_n\}$ under the map, say $\overset{\vdash}{\Longleftarrow}$, by the notation $p \overset{\vdash}{\Longleftarrow} \{a_1, \ldots, \textcircled{a_i}, \ldots, a_n\}$. For a picture, we draw



We also require that the composite map

$$\mathsf{A} \xrightarrow{\cong} \mathsf{u} \diamond \mathsf{A} \longrightarrow \mathsf{Y} \diamond \mathsf{A} \xrightarrow{\vdash} \mathsf{A}$$

is the identity.

A morphism between Y-algebras A and B is a linear map $\mathsf{A} \to \mathsf{B}$ which commutes with the respective $\dashv$ and $\vdash$ maps.

We denote by Y-`alg` the category of Y-algebras.

Remark. The unit condition implies that the map $\overset{\vdash}{\Longleftarrow}$ is surjective. It is not clear whether the map $\overset{\vdash}{\Longleftarrow}$ is surjective in general. But if it is surjective then one can derive useful results; see Lemma 6.

Definition 5.8. A derivation $(\xi_\vdash, \xi_\dashv)$ of the Y-algebra A is a pair of linear maps

$$\xi_\vdash, \xi_\dashv : \mathsf{A} \to \mathsf{A}$$

which satisfy the conditions below. First we define $\tilde{\xi}_\vdash, \tilde{\xi}_\dashv : \mathsf{Y} \diamond \mathsf{A} \to \mathsf{Y} \diamond \mathsf{A}$ by

$$\begin{aligned} \tilde{\xi}_\vdash \left(q \otimes \{a_1, \ldots, a_l, \ldots, a_j\}\right) &= \textstyle\sum_{i=1}^{j} \; q \otimes \{a_1, \ldots, \xi(a_i), \ldots, a_j\}. \\ \tilde{\xi}_\dashv \left(q \otimes \{a_1, \ldots, a_l, \ldots, a_j\}\right) &= \textstyle\sum_{i=1}^{j} \; q \otimes \{a_1, \ldots, \xi(a_i), \ldots, a_l, \ldots, a_j\}. \end{aligned}$$

Note that $\tilde{\xi}_\vdash$ factors through the quotient $\mathsf{Y} \diamond \mathsf{A} \to \mathsf{Y} \circ \mathsf{A}$, whereas $\tilde{\xi}_\dashv$ does not. We require that the following five diagrams commute.

$$\begin{tikzcd} \mathsf{Y} \diamond \mathsf{A} \arrow[r, "\tilde{\xi}_\dashv"] \arrow[d, "\dashv"'] & \mathsf{Y} \diamond \mathsf{A} \arrow[d, "\dashv"] \\ \mathsf{A} \arrow[r, "\xi_\dashv"'] & \mathsf{A} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Y} \diamond \mathsf{A} \arrow[r, "\tilde{\xi}_\dashv"] \arrow[d, "\tilde{\xi}_\vdash"'] & \mathsf{Y} \diamond \mathsf{A} \arrow[d, "\dashv"] \\ \mathsf{Y} \diamond \mathsf{A} \arrow[r, "\dashv"'] & \mathsf{A} \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Y} \diamond \mathsf{A} \arrow[r, "\tilde{\xi}_\dashv"] \arrow[d, "\vdash"'] & \mathsf{Y} \diamond \mathsf{A} \arrow[d, "\vdash"] \\ \mathsf{A} \arrow[r, "\xi_\dashv"'] & \mathsf{A} \end{tikzcd}$$

$$\begin{tikzcd} \mathsf{Y} \diamond \mathsf{A} \arrow[r, "\tilde{\xi}_\vdash"] \arrow[d, "\vdash"'] & \mathsf{Y} \diamond \mathsf{A} \arrow[d, "\vdash"] \\ \mathsf{A} \arrow[r, "\xi_\vdash"'] & \mathsf{A} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Y} \diamond \mathsf{A} \arrow[r, "\dashv"] \arrow[d, "\vdash"'] & \mathsf{A} \arrow[d, "\xi_\vdash"] \\ \mathsf{A} \arrow[r, "\xi_\vdash"'] & \mathsf{A} \end{tikzcd}$$

In the shorthand of Equation (8), we may write

$$\dashv \; (\; \tilde{\xi}_\vdash \;) \; = \dashv (\; \tilde{\xi}_\dashv \;) \; = \; (\; \dashv \;) \; \xi_\dashv$$

$$\vdash \; (\; \tilde{\xi}_\dashv \;) \; = \; (\; \vdash \;) \; \xi_\dashv$$

$$\vdash \; (\; \tilde{\xi}_\vdash \;) \; = \; (\; \vdash \;) \; \xi_\vdash \; = \; (\; \dashv \;) \; \xi_\vdash$$

for these diagrams. They bring out the similarity with the dialgebra axioms.

The space of derivations of A, denoted $\mathrm{Der}(\mathsf{A})$, is a vector space via

$$a \cdot (\xi_\dashv, \xi_\vdash) := (a \cdot \xi_\dashv, a \cdot \xi_\vdash) \quad \text{for } a \in k.$$

Further it forms a Leibniz algebra for the bracket

$$[(\xi_\dashv, \xi_\vdash), (\eta_\dashv, \eta_\vdash)] := ([\xi_\dashv, \eta_\dashv], [\xi_\vdash, \eta_\dashv]), \tag{11}$$

where in the right hand side, [,] is the usual commutator, as in Equation (9). The fact that a bracket of this kind defines a Leibniz algebra is straightforward. The more interesting part is to check that

$$(\xi_{\dashv}, \xi_{\vdash}), (\eta_{\dashv}, \eta_{\vdash}) \in \mathrm{Der}(\mathsf{A}) \implies ([\xi_{\dashv}, \eta_{\dashv}], [\xi_{\vdash}, \eta_{\dashv}]) \in \mathrm{Der}(\mathsf{A}).$$

One needs to use the identities $[\tilde{\xi}_{\dashv}, \tilde{\eta}_{\dashv}] = \widetilde{[\xi_{\dashv}, \eta_{\dashv}]}$ and $[\tilde{\xi}_{\vdash}, \tilde{\eta}_{\dashv}] = \widetilde{[\xi_{\vdash}, \eta_{\dashv}]}$ to prove this fact.

Remark. We use the same notation [,] for the Lie and Leibniz commutator. The meaning should be understood from the context.

Lemma 6. *If the map* $\mathsf{Y} \diamond \mathsf{A} \xrightarrow{\dashv} \mathsf{A}$ *is surjective then for* $\xi, \eta \in \mathrm{Der}(\mathsf{A})$*, we have*

$$\xi_{\vdash} \circ \eta_{\dashv} \;=\; \xi_{\vdash} \circ \eta_{\vdash}.$$

Proof. The proof follows from the following commutative diagram.

$$\begin{tikzcd}
 & \mathsf{A} \arrow[rr, "\eta_{\dashv}"] & & \mathsf{A} \arrow[dr, "\xi_{\vdash}"] & \\
\mathsf{Y} \diamond \mathsf{A} \arrow[ur, "\dashv"] \arrow[rr, "\tilde{\eta}_{\vdash}"] \arrow[dr, "\dashv"'] & & \mathsf{Y} \diamond \mathsf{A} \arrow[ur, "\dashv"] \arrow[dr, "\dashv"'] & & \mathsf{A} \\
 & \mathsf{A} \arrow[rr, "\eta_{\vdash}"'] & & \mathsf{A} \arrow[ur, "\xi_{\vdash}"'] &
\end{tikzcd}$$

The derivation axioms imply that each of the little square commutes. □

5.4. **The free dioperad algebra.**

Definition 5.9. For Y a dioperad and V a vector space, let $\mathsf{Y} \circ \mathsf{V}$ be the Y-algebra given by

$$\begin{aligned}
\dashv :\ \mathsf{Y} \diamond (\mathsf{Y} \circ \mathsf{V}) &\xrightarrow{\cong} (\mathsf{Y} \diamond \mathsf{Y}) \circ \mathsf{V} \xrightarrow{\dashv \circ \mathrm{id}} \mathsf{Y} \circ \mathsf{V}. \\
\vdash :\ \mathsf{Y} \diamond (\mathsf{Y} \circ \mathsf{V}) &\xrightarrow{\cong} (\mathsf{Y} \diamond \mathsf{Y}) \circ \mathsf{V} \xrightarrow{\vdash \circ \mathrm{id}} \mathsf{Y} \circ \mathsf{V}.
\end{aligned}$$

It is clear that a dioperad axiom for Y in Lemma 4 implies the corresponding algebra axiom for $\mathsf{Y} \circ \mathsf{V}$. Similarly the bar unit of Y implies the bar unit axiom for $\mathsf{Y} \circ \mathsf{V}$. This defines a functor

$$\mathsf{Y} \circ (-) \;:\; \mathtt{Vect} \longrightarrow \mathsf{Y}\text{-}\mathtt{alg},$$

which is called the free functor.

We denote the Leibniz algebra $\mathrm{Der}(\mathsf{Y} \circ \mathsf{V})$ by the notation $\mathrm{Der}(gl_n, \mathsf{Y})$, where n is the dimension of V.

Remark. By definition, the composite map

$$\mathsf{Y} \circ \mathsf{V} \cong \mathsf{Y} \diamond (\mathsf{u} \circ \mathsf{V}) \longrightarrow \mathsf{Y} \diamond (\mathsf{Y} \circ \mathsf{V}) \xrightarrow{\dashv} \mathsf{Y} \circ \mathsf{V}$$

is the identity. This shows that the map $\xleftarrow{\dashv}$ is surjective for $\mathsf{Y} \circ \mathsf{V}$. In particular, Lemma 6 applies.

Proposition 4. *The free functor is the left adjoint to the forgetful functor*

$$\mathsf{Y}\text{-}\mathtt{alg} \longrightarrow \mathtt{Vect}.$$

In other words, $\mathsf{Y} \circ \mathsf{V}$ *is the free* Y*-algebra on* V*.*

One may phrase the above proposition by the following universal property.

Given a Y-algebra A and a linear map $F : \mathsf{V} \to \mathsf{A}$, there is a unique map of Y-algebras $\check{F} : \mathsf{Y} \circ \mathsf{V} \to \mathsf{A}$ which extends F.

It is defined as the composite map

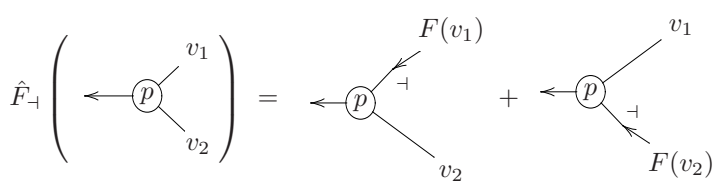
$$\check{F} : \ \mathsf{Y} \circ \mathsf{V} \xrightarrow{\mathrm{id} \circ F} \mathsf{Y} \circ \mathsf{A} \xrightarrow{\dashv} \mathsf{A}\,.$$

Note that the domain $\mathsf{Y} \circ \mathsf{A}$ forces the second map to be $\dashv$. The axioms $\dashv (\dashv, \dashv) = (\dashv) \dashv$ and $\vdash (\dashv, \dashv) = (\vdash) \dashv$ in Definition 5.7 applied to the Y-algebra A show that $\check{F}$ is a map of Y-algebras. The bar unit of Y shows that $\check{F}$ is unique.

Proposition 5. *A derivation on $\mathsf{Y} \circ \mathsf{V}$ is uniquely determined by its value on V. That is,*

$$\mathrm{Hom}(\mathsf{V}, \mathsf{Y} \circ \mathsf{V}) \xrightarrow{\cong} \mathrm{Der}(\mathsf{Y} \circ \mathsf{V}).$$

Proof. In analogy with the proof of Proposition 3, one can extend $F : \mathsf{V} \to \mathsf{Y} \circ \mathsf{V}$ to two maps $\hat{F}_\dashv, \hat{F}_\vdash : \mathsf{Y} \circ \mathsf{V} \to \mathsf{Y} \circ \mathsf{V}$ as follows. We illustrate it on an example.



And similarly $\hat{F}_\vdash$ is defined with $\vdash$ instead of $\dashv$. And by definition, $\hat{F}$ sends $\mathsf{Y}[0]$, the degree 0 part of $\mathsf{Y} \circ \mathsf{V}$, to 0. It is routine to check that $(\hat{F}_\dashv, \hat{F}_\vdash) \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{V})$. For example, the repeated use of the dioperad axiom

$$p \vdash (q \vdash r) = (p \vdash q) \vdash r = (p \dashv q) \vdash r$$

on Y yields the derivation axiom

$$\vdash \ (\ \tilde{\hat{F}}_\vdash \ ) \ = \ (\ \vdash \ ) \ \hat{F}_\vdash .$$

This defines the map, say L, in the proposition. The inverse map, say M, sends $(\xi_\dashv, \xi_\vdash)$ to

$$\xi_\vdash \mid_{\mathsf{V}} : \ \mathsf{V} \xrightarrow{\cong} u \circ \mathsf{V} \longrightarrow \mathsf{Y} \circ \mathsf{V} \xrightarrow{\xi_\vdash} \mathsf{Y} \circ \mathsf{V}\,.$$

One needs to check that L and M are inverses of each other. The bar unit of Y shows that $M \circ L = \mathrm{id}$. And $L \circ M = \mathrm{id}$ follows from the derivation axioms

$$\dashv \ (\ \tilde{\xi}_\vdash \ ) \ = \ (\ \dashv \ ) \ \xi_\dashv \quad \text{and} \quad \vdash \ (\ \tilde{\xi}_\vdash \ ) \ = \ (\ \dashv \ ) \ \xi_\vdash .$$

□

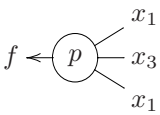
Definition 5.10. The Leibniz structure on $\mathrm{Hom}(\mathsf{V}, (\mathsf{Y} \circ \mathsf{V}))$ induced by the above isomorphism is given by

$$[F, G] = \hat{F}_\vdash \circ G - \hat{G}_\dashv \circ F. \tag{12}$$

One can describe this bracket pictorially by using the isomorphism

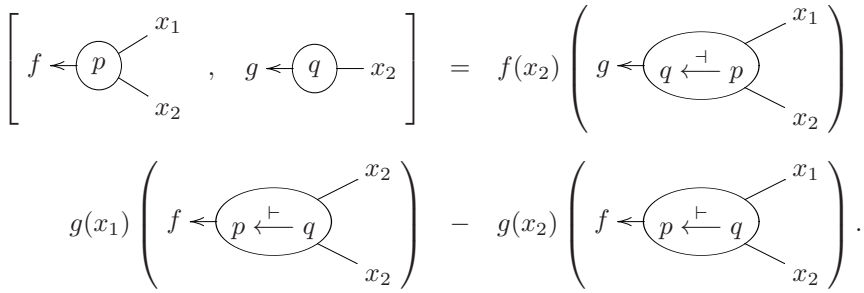
$$\mathrm{Hom}(\mathsf{V}, \mathsf{Y} \circ \mathsf{V}) \cong \mathsf{V}^* \otimes (\mathsf{Y} \circ \mathsf{V}).$$

An element of $\mathrm{Hom}(\mathsf{V}, \mathsf{Y} \circ \mathsf{V})$ is then an operad element p, with its inputs labelled by elements of V and its output by an element of V^*, as shown below.

$f \otimes (p \otimes \{x_1, x_3, x_1\}) \quad = \quad$ 

$\in \mathsf{V}^* \otimes (\mathsf{Y} \circ \mathsf{V}) \quad \text{with} \quad f \in \mathsf{V}^*.$

We illustrate the Leibniz bracket on $\mathsf{V}^* \otimes (\mathsf{Y} \circ \mathsf{V})$ by an example.



In the first term on the right, p is substituted into q and in the next two, q is substituted into each input of p. For each term, we pick a coefficient given by contracting an element of V^* with an element of V, along with the appropriate sign.

6. Dioperad geometry

In this section, we explain the geometry behind the dioperad constructions in Section 5, and how concepts about manifolds can be used to develop these ideas further. Further, we explain the basic setting for symplectic and orthogonal geometry for a reversible dioperad, by borrowing facts about symplectic manifolds.

6.1. **Dioperad manifolds.** We begin with an informal discussion. Let Y be a dioperad. For our intuition, we assume that there exists a category of Y-`manifolds` and an equivalence of categories

$$\mathsf{Y}\text{-}\mathtt{manifolds} \longleftrightarrow \mathsf{Y}\text{-}\mathtt{alg}.$$

We view a Y-algebra, say A, as the algebra of functions on a Y-manifold, say X. This allows us to interpret algebraic objects associated to A as familiar geometric notions of X. For example, one can view $\mathrm{Der}(\mathsf{A})$ as vector fields on X, denoted by $\mathfrak{X}(X)$.

To go a step further, one must define differential forms on X, denoted by $\Omega(\mathsf{A})$. Classically, these are functions on ΠTX, which is the total space of the odd tangent bundle to X. This suggests that we must also assume an equivalence of categories

$$\mathsf{Y}\text{-}\mathtt{supermanifolds} \longleftrightarrow \mathsf{Y}\text{-}\mathtt{superalg},$$

under which ΠTX and $\Omega(\mathsf{A})$ correspond to each other. The differential d on $\Omega(\mathsf{A})$ satisfies the Leibniz rule and hence is a superderivation. So one can view this as a vector field on ΠTX, that is, as an element of $\mathfrak{X}(\Pi TX)$.

The basic objects of interest are summarised in the table below.

Geometric objects	Algebraic objects	
X	$\mathsf{Y}\circ\mathsf{V}$	$\mathsf{Z}\circ\mathsf{V}$
$\mathfrak{X}(X)$	$\mathrm{Der}(\mathsf{Y}\circ\mathsf{V})$	$\mathrm{Der}(\mathsf{Z}\circ\mathsf{V})$
ΠTX	$\Omega(\mathsf{Y}\circ\mathsf{V})$	$\Omega(\mathsf{Z}\circ\mathsf{V})$
$\mathfrak{X}(\Pi TX)$	$\mathrm{Der}(\Omega(\mathsf{Y}\circ\mathsf{V}))$	$\mathrm{Der}(\Omega(\mathsf{Z}\circ\mathsf{V}))$

We repeat that the geometric objects in the first column do not exist; their role is to aid our intuition. If Y is reversible then there are two algebraic candidates for a given geometric notion and they are both useful. Since X corresponds to the free Y-algebra on V, we call it the standard Y-manifold on V. If V is a super vector space, as will be the case in orthogonal geometry, then we call X the standard Y-supermanifold on V.

6.2. **Dioperad supergeometry.** The material in Section 5 generalizes to superspaces, which we briefly record.

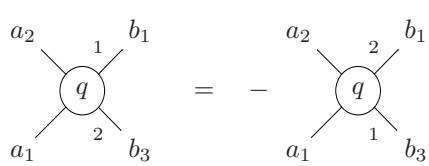
Definition 6.1. A super vector space A is a $\mathbb{Z}_2$-graded vector space $\mathsf{A}=\mathsf{A}_0\oplus\mathsf{A}_1$. We refer to A_0 and A_1 as the even and odd parts of A respectively. An even (resp. odd) map between A and B is a linear map between A and B which preserves (resp. switches) the grading. And $\mathrm{Hom}(\mathsf{A},\mathsf{B})$ is the direct sum of the spaces of even and odd maps. We write `SuperVect` for the category of super vector spaces.

Definition 6.2. For a super vector space A, let $\mathsf{Q}\circ\mathsf{A}$ be as in Definition 3.1, but where Σ_j acts on $\mathsf{A}^{\otimes j}$ by permuting the factors using the Koszul rule of signs. Namely, for a transposition $\pi=(i,i+1)\in\Sigma_j$, we have

$$\pi(\ldots\otimes a_i\otimes a_{i+1}\otimes\ldots)=(-1)^{|a_i||a_{i+1}|}\,(\ldots\otimes a_{i+1}\otimes a_i\otimes\ldots),$$

where $|\cdot|$ refers to the degree of the element.

Note that $\mathsf{Q}\circ\mathsf{A}$ is a super vector space. We denote an element of $\mathsf{Q}\circ\mathsf{A}$ by $q\otimes\{a_1,\ldots,a_j\}\otimes[b_1,\ldots,b_k]$. The a_i's and b_i's belong to the even and odd part of A respectively. The square brackets indicate that permuting the b_i's incurs a sign, namely the sign of the permutation. For a picture, we draw



$\in\mathsf{Q}\circ\mathsf{A}\,.$

The first picture stands for $q\otimes\{a_1,a_2\}\otimes[b_1,b_3]$.

Proceeding along these lines, for a dioperad Y, we define Y-`superalg`, the category of Y-superalgebras. Further the space of superderivations $\mathrm{Der}(\mathsf{A})$ of a Y-superalgebra A is a Leibniz superalgebra. Propositions 4 and 5 continue to hold in the super setting. Namely, $\mathsf{Y}\circ\mathsf{W}$ is the free Y-superalgebra on the super vector space W, and $\mathrm{Hom}(\mathsf{W},\mathsf{Y}\circ\mathsf{W})\cong\mathrm{Der}(\mathsf{Y}\circ\mathsf{W})$.

Let W be a supe vector space of dimension $(k|l)$. In what follows, we denote by X the standard Y-supermanifold on W, with $x_1,\ldots,x_k$ and $\theta_1,\ldots,\theta_l$ as the even and odd coordinates respectively.

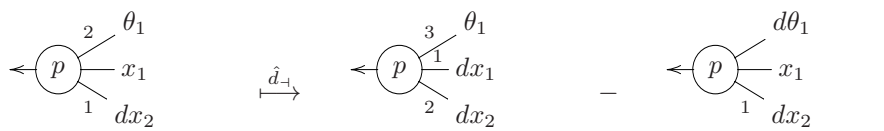
Definition 6.3. We now define one notion of differential forms on X. Let

$$\Omega(\mathsf{Y}\circ\mathsf{W})\;:=\;\mathsf{Y}\circ(\mathsf{W}\oplus\Pi\,\mathsf{W})\;=\;\text{The free }\mathsf{Y}\text{-superalgebra on }\mathsf{W}\oplus\Pi\,\mathsf{W},$$

where Π is the functor on `SuperVect` that switches parity. Let

$$d \in \mathrm{Hom}(\mathsf{W} \oplus \Pi\,\mathsf{W}, \Omega(\mathsf{Y} \circ \mathsf{W}))$$

send W isomorphically onto $\Pi\,\mathsf{W} \cong (u[1] \otimes \Pi\,\mathsf{W})$ and send $\Pi\,\mathsf{W}$ to 0. Then d defines an odd superderivation $(\hat{d}_{\dashv}, \hat{d}_{\vdash})$ on $\Omega(\mathsf{Y} \circ \mathsf{W})$. For simplicity, we denote this again by d. Since u is a unit only on the bar side, $\hat{d}_{\dashv}$ and $\hat{d}_{\vdash}$ are different operators. It follows from the definition that $\hat{d}_{\dashv}^2 = 0$. To give an illustration of how $\hat{d}_{\dashv}$ works,



$p \otimes \{x_1\} \otimes [dx_2, \theta_1]$ $\qquad$ $p \otimes [dx_1, dx_2, \theta_1]$ $\qquad$ $p \otimes \{x_1, d\theta_1\} \otimes [dx_2]$

Observe that for $\hat{d}_{\vdash}$, the term $p \xleftarrow{\vdash} u$ would enter the picture.

Apart from the $\mathbb{Z}_2$-grading, $\Omega(\mathsf{Y} \circ \mathsf{W})$ has a $\mathbb{Z}$-grading given by the number of occurrences of elements of $\Pi\,\mathsf{W}$. We write $\Omega^i(\mathsf{Y} \circ \mathsf{W})$ for the ith graded part. Note that $\Omega^0(\mathsf{Y} \circ \mathsf{W}) = \mathsf{Y} \circ \mathsf{W}$. Thus we obtain a chain complex $(\Omega^*(\mathsf{Y} \circ \mathsf{W}), \hat{d}_{\dashv})$, whose homology is computed in Proposition 6.

Remark. One can also define the space $\Omega(\mathsf{Y} \circ \mathsf{W})$ by an universal property as the free differential Y-superalgebra on W.

Definition 6.4. Let $\xi \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{W})$ be a vector field on X. It gives rise to two vector fields, namely L_ξ and i_ξ, on ΠTX as follows. We define the Lie derivative $L_\xi \in \mathrm{Der}(\Omega(\mathsf{Y} \circ \mathsf{W}))$ and the contraction $i_\xi \in \mathrm{Der}(\Omega(\mathsf{Y} \circ \mathsf{W}))$ by specifying them on $\mathsf{W} \oplus \Pi\,\mathsf{W}$ as below.

$$L_\xi(w) = (-1)^{|\xi|}\xi(w),\ \ L_\xi(dw) = \hat{d}_{\dashv}(\xi(w)) \ \text{ and } \ i_\xi(w) = 0,\ \ i_\xi(dw) = (-1)^{|\xi|}\xi(w),$$

for every $w \in \mathsf{W}$ and where $|\xi|$ denotes the degree of ξ. The operators L_ξ and i_ξ have degrees $|\xi|$ and $|\xi| + 1$ respectively.

Lemma 7. *For vector fields $\xi, \eta \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{W})$, the following Leibniz super commutation relations hold.*

$$[i_\xi, d] = L_\xi, \quad [i_\xi, i_\eta] = 0, \quad [L_\xi, i_\eta] = i_{[\xi,\eta]}, \quad [L_\xi, L_\eta] = L_{[\xi,\eta]}.$$

Proof. To prove this lemma, it is better to regard the super vector fields as elements of $\mathrm{Hom}(\mathsf{W} \oplus \Pi\,\mathsf{W}, \Omega(\mathsf{Y} \circ \mathsf{W}))$. From the super-version of Equation (12), we have

$$[i_\xi, d] \ = \ (\hat{i_\xi})_{\vdash} \circ d - (-1)^{|\xi|+1}\ \hat{d}_{\dashv} \circ i_\xi$$

From Definition 6.4, the formula above, and $\hat{d}_{\dashv}^2 = 0$, we obtain

$$[i_\xi, d](w) = (-1)^{|\xi|}\xi(w) \quad \text{and} \quad [i_\xi, d](dw) = \hat{d}_{\dashv}(\xi(w)),$$

which is same as $L_\xi(w)$ and $L_\xi(dw)$ respectively. This proves the first formula. The remaining proofs are similar. □

Proposition 6. *We have*

$$H^i(\Omega^*(\mathsf{Y} \circ \mathsf{W}), \hat{d}_{\dashv}) = \begin{cases} 0 & \text{if } i > 0, \\ \mathsf{Y}[0] & \text{if } i = 0. \end{cases}$$

Proof. Consider the linear Euler vector field $e \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{W})$, defined by $e(w) = w$ for $w \in \mathsf{W}$. Then $L_e(w) = w$ and $L_e(dw) = dw$ and hence L_e is the Euler vector field on ΠTX. With respect to the $\mathbb{Z}$-grading on $\Omega(\mathsf{Y} \circ \mathsf{W})$, the operators $(\hat{L}_e)_\dashv$, $(\hat{\imath}_e)_\dashv$ and $\hat{d}_\dashv$ have degrees 0, -1 and 1 respectively. It is clear that for degree greater than 0, the operator $(\hat{L}_e)_\dashv$ is invertible, and induces an isomorphism on cohomology. On the other hand, one half of Cartan's formula (the first formula in Lemma 7), namely,

$$[(\hat{\imath}_\xi)_\dashv, \hat{d}_\dashv] \;=\; (\hat{L}_\xi)_\dashv$$

for $\xi = e$ shows that it induces the zero map on cohomology. So we conclude that the chain complex $(\Omega^*(\mathsf{Y} \circ \mathsf{W}), \hat{d}_\dashv)$ is exact for degree greater than 0. The computation for degree 0 follows from the fact that the kernel of $(\hat{L}_e)_\dashv$ is precisely $\mathsf{Y}[0]$. □

6.3. **The mating functor for algebras.** Let Y be a reversible dioperad and $\mathsf{Z} = \mathsf{YY}$ be its mated species.

Definition 6.5. We define the mating functor

$$\mathsf{Y}\text{-}\mathtt{alg} \longrightarrow \mathtt{Vect},$$

that maps a Y-algebra A to the vector space

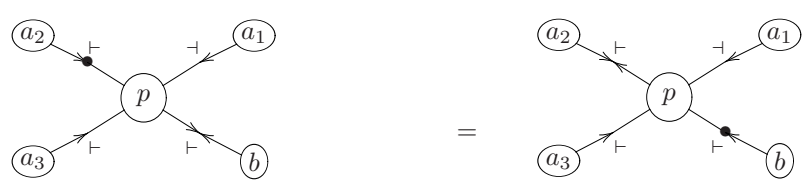
$$\mathsf{A} \otimes \mathsf{A} \otimes k\{\dashv, \vdash\},$$

subject to the two relations below. We denote $a \otimes b \otimes \dashv$ by the notation $a \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b$ and $a \otimes b \otimes \vdash$ by the notation $a \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b$.

(R1) $$a \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b \;=\; b \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow a$$

(R2) $$\begin{aligned} &(\{a_1, \ldots, a_i, \ldots, a_j\} \overset{\dashv}{\Longrightarrow} p) \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b \\ &\quad = \quad (\{a_1, \ldots, a_i, \ldots, a_j\} \overset{\vdash}{\Longrightarrow} p) \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b \\ &\quad = \quad a_i \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (\mathbf{r}(p) \overset{\vdash}{\Longleftarrow} \{a_1, \ldots, b, \ldots, a_j\}). \\ &(\{a_1, \ldots, a_i, \ldots, a_j\} \overset{\vdash}{\Longrightarrow} p) \xrightarrow{\dashv}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b \\ &\quad = \quad a_i \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (\mathbf{r}(p) \overset{\dashv}{\Longleftarrow} \{a_1, \ldots, b, \ldots, a_j\}). \end{aligned}$$

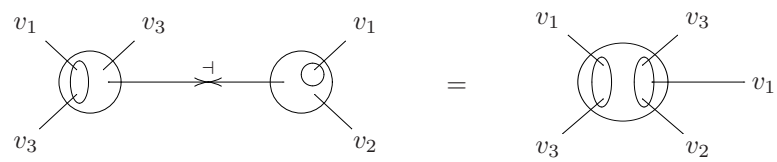
The notation used is as in Definition 5.7. As a pictorial illustration of the first part of relation $(R2)$,



$(\{a_1, a_2, a_3\} \overset{\vdash}{\Longrightarrow} p) \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow b$ $\qquad$ $a_2 \xrightarrow{\vdash}\!\!\!\!\!\!\times\!\!\!\!\!\!\leftarrow (\mathbf{r}(p) \overset{\vdash}{\Longleftarrow} \{a_1, b, a_3\})$

We denote the image of the Y-algebra A under the mating functor by the notation AA and call it the mated vector space of A.

Exercise 3. For a vector space V, show that the mating functor maps the free Y-algebra $\mathsf{Y}\circ\mathsf{V}$ to the vector space $\mathsf{YY}\circ\mathsf{V}$. As an example,



which shows the mating for the free algebra of the dioperad s.

We regard $v \in \mathsf{V}$ as an element of $\mathsf{Y}\circ\mathsf{V}$ via $\mathsf{V} \cong \mathsf{u}\circ\mathsf{V} \to \mathsf{Y}\circ\mathsf{V}$. Show that for $v_1, v_2 \in \mathsf{V}$, we have

$$v_1 \overset{\vdash}{\rightarrow\!\!\!\leftarrow} v_2 \quad = \quad v_1 \overset{\dashv}{\rightarrow\!\!\!\leftarrow} v_2.$$

In this case, we drop $\vdash$ and $\dashv$ from the notation and denote the above unambiguously by $v_1 \rightarrow\!\!\!\leftarrow v_2$.

Definition 6.6. Let A be a Y-algebra and AA be its mated vector space. Further let $(\xi_\vdash, \xi_\dashv) \in \mathrm{Der}(\mathsf{A})$. This defines a pair of operators $\xi_\vdash, \xi_\dashv : \mathsf{AA} \to \mathsf{AA}$ as follows.

$$\begin{aligned}
\xi_\dashv(a \overset{\vdash}{\rightarrow\!\!\!\leftarrow} b) &= \xi_\dashv(a) \overset{\vdash}{\rightarrow\!\!\!\leftarrow} b \;+\; a \overset{\vdash}{\rightarrow\!\!\!\leftarrow} \xi_\dashv(b)\\
\xi_\vdash(a \overset{\vdash}{\rightarrow\!\!\!\leftarrow} b) &= \xi_\vdash(a) \overset{\dashv}{\rightarrow\!\!\!\leftarrow} b \;+\; a \overset{\vdash}{\rightarrow\!\!\!\leftarrow} \xi_\vdash(b)
\end{aligned}$$

One can check that they respect the relations ($R1$) and ($R2$). Let $\mathrm{Der}(\mathsf{AA})$ consists of pairs $(\xi_\vdash, \xi_\dashv)$ of such operators. Then $\mathrm{Der}(\mathsf{AA})$ is a Leibniz algebra with bracket given by Equation (11) and the surjection $\mathrm{Der}(\mathsf{A}) \to \mathrm{Der}(\mathsf{AA})$ is a map of Leibniz algebras.

Exercise 4. Define the super-version of the mating functor

$$\texttt{Y-superalg} \longrightarrow \texttt{Vect},$$

and show that it sends $\mathsf{Y}\circ\mathsf{W}$ to $\mathsf{Z}\circ\mathsf{W}$. Also generalize the second claim in Exercise 3. Further show that Definition 6.6 can be extended to superalgebras. In particular, this defines a map $\mathrm{Der}(\mathsf{Y}\circ\mathsf{W}) \to \mathrm{Der}(\mathsf{Z}\circ\mathsf{W})$ of Leibniz superalgebras.

Definition 6.7. We now define the second notion of differential forms on X. Let

$$\Omega(\mathsf{Z}\circ\mathsf{W}) \;:=\; \mathsf{Z}\circ(\mathsf{W}\oplus\Pi\,\mathsf{W}).$$

Applying the above exercise with W replaced by $\mathsf{W}\oplus\Pi\,\mathsf{W}$, we observe that the mating functor sends $\Omega(\mathsf{Y}\circ\mathsf{W})$ to $\Omega(\mathsf{Z}\circ\mathsf{W})$. Further we have a map $\mathrm{Der}(\Omega(\mathsf{Y}\circ\mathsf{W})) \to \mathrm{Der}(\Omega(\mathsf{Z}\circ\mathsf{W}))$ of Leibniz superalgebras. For $\xi \in \mathrm{Der}(\mathsf{Y}\circ\mathsf{W})$, this yields $d, L_\xi, i_\xi \in \mathrm{Der}(\Omega(\mathsf{Z}\circ\mathsf{W}))$, which satisfy the commutation relations of Lemma 7. By analogous reasoning, we obtain a chain complex $(\Omega^*(\mathsf{Z}\circ\mathsf{W}), \hat{d}_\dashv)$, whose degree zero part is $\mathsf{Z}\circ\mathsf{W}$ and whose homology is given by

$$H^i(\Omega^*(\mathsf{Z}\circ\mathsf{W}), \hat{d}_\dashv) = \begin{cases} 0 & \text{if } i > 0, \\ \mathsf{Z}[0] & \text{if } i = 0. \end{cases}$$

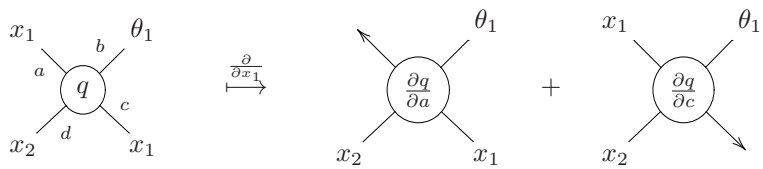
The picture illustration for the map $\hat{d}_\dashv$ is as shown in Definition 6.3, except that there is no output arrow.

6.4. **The partial derivative for algebras.** We now define and state the dioperad analogues of some basic facts about partial derivatives on manifolds. The justification for the claims made is given by Proposition 1.

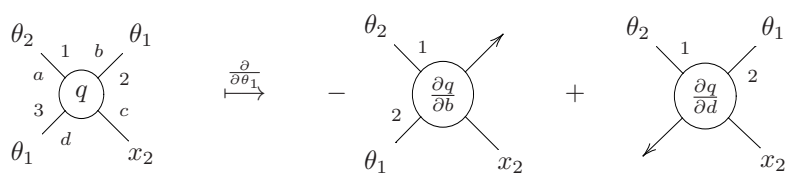
Definition 6.8. The spaces $\mathsf{Z} \circ \mathsf{W}$ and $\mathsf{Y} \circ \mathsf{W}$ can both be interpreted as functions on X. For $1 \leq i \leq k$ and $1 \leq j \leq l$, we define partial derivatives

$$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \theta_j} \ : \ \mathsf{Z} \circ \mathsf{W} \to \mathsf{Y} \circ \mathsf{W}$$

as follows.



Namely, to define $\frac{\partial q}{\partial x_1}$, we cut the inputs of q attached to x_1, one at a time. For clarity, we have also indicated the labels on the inputs of q.



Namely, to define $\frac{\partial q}{\partial \theta_1}$, we cut the inputs attached to θ_1, one at a time, first reordering the numbers so that the input being cut has number 1 and then shifting down the numbers of the remaining inputs.

Lemma 8. *A one form on X, that is, an element of $\Omega^1(\mathsf{Z} \circ \mathsf{W})$, can be uniquely written as*

$$\sum_i dx_i \overset{\vdash}{\rightarrow\!\!\leftarrow} a_i \ + \ \sum_j d\theta_j \overset{\vdash}{\rightarrow\!\!\leftarrow} b_j.$$

Further for a function on X, that is, an element $H \in \mathsf{Z} \circ \mathsf{W}$, we have

$$\hat{d}_{\dashv}(H) \quad = \quad \sum_i dx_i \overset{\vdash}{\rightarrow\!\!\leftarrow} \frac{\partial H}{\partial x_i} \ + \ \sum_j d\theta_j \overset{\vdash}{\rightarrow\!\!\leftarrow} \frac{\partial H}{\partial \theta_j}.$$

Note that the bar is always on the side of the diferentials.

Exercise 5. For the subset dioperad s, make the above ideas explicit.

6.5. **Symplectic dioperad geometry.** We specialise to $\mathsf{W} = \mathsf{V}_{2n}$, a super vector space of dimension $(2n|0)$ with basis $p_1, \ldots, p_n, q_1, \ldots, q_n$. There are no odd coordinates. Using the even number of even coordinates, we define the symplectic form

$$\omega \ := \ \sum_{i=1}^{n} dp_i \rightarrow\!\!\leftarrow dq_i \quad \in \ \Omega^2(\mathsf{Z} \circ \mathsf{V}_{2n})$$

This defines a symplectic dioperad manifold (X, ω). The unambiguity of the notation follows from Exercises 3 and 4. Observe that the super-version of relation $(R1)$ implies that

$$dp_i \rightarrow\!\!\leftarrow dq_i \quad = \quad -\,(dq_i \rightarrow\!\!\leftarrow dp_i).$$

Hence the order of dp_i and dq_i is important.

Definition 6.9. Let now define the Leibniz algebra of symplectic, or equivalently, Hamiltonian vector fields on (X, ω). Let

$$\mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n}, \omega) \ := \ \{\xi \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n}) \mid (\hat{L_\xi})_{\vdash}\ \omega = 0\}.$$

We note that the bracket on $\mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n})$ restricts to the above subspace. This follows from the following part of Lemma 7.

$$(\widehat{L_{[\xi,\eta]}})_{\dashv} \ = \ (\hat{L_\xi})_{\vdash} \circ (\hat{L_\eta})_{\vdash} - (\hat{L_\eta})_{\dashv} \circ (\hat{L_\xi})_{\vdash}.$$

For the expression on the right, we also made use of Lemma 6.

We also denote the Leibniz algebra $\mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n}, \omega)$ by the notation $\mathrm{Der}(sp_{2n}, \mathsf{Y})$.

Definition 6.10. We interpret the space $\mathsf{Z} \circ \mathsf{V}_{2n}$ as Hamiltonian functions on the symplectic dioperad manifold (X, ω). We define the Poisson bracket

$$\{\ ,\ \} : \mathsf{Z} \circ \mathsf{V}_{2n} \otimes \mathsf{Z} \circ \mathsf{V}_{2n} \to \mathsf{Z} \circ \mathsf{V}_{2n}$$

using the formula

$$\{F, H\} \ := \ \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \ \overset{\dashv}{\rightarrowtail\!\!\!\!\!\leftarrowtail} \ \frac{\partial H}{\partial q_i} \ - \ \frac{\partial F}{\partial q_i} \ \overset{\dashv}{\rightarrowtail\!\!\!\!\!\leftarrowtail} \ \frac{\partial H}{\partial p_i}$$

This defines a Leibniz structure on $\mathsf{Z} \circ \mathsf{V}_{2n}$.

Proposition 7. *There is a split short exact sequence of Leibniz algebras*

$$0 \to \mathsf{Z}[0] \hookrightarrow \mathsf{Z} \circ \mathsf{V}_{2n} \twoheadrightarrow \mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n}, \omega) \to 0.$$

With our conventions below, the second map is an anti-Leibniz map.

We now outline the proof of the above proposition, which imitates the standard proof in classical symplectic geometry.

Lemma 9. *The map* $\mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n}) \to \Omega^1(\mathsf{Z} \circ \mathsf{V}_{2n})$ *which sends* ξ *to* $(\hat{i_\xi})_{\vdash}\ \omega$ *is an isomorphism.*

Proof. We have the chain of equalities

$$\begin{aligned} (\hat{i_\xi})_{\vdash}\ \omega \ &= \ \textstyle\sum_{i=1}^{n} (\hat{i_\xi})_{\vdash}\ (dp_i \ \rightarrowtail\!\!\!\!\!\leftarrowtail \ dq_i) \\ &= \ \textstyle\sum_{i=1}^{n} dp_i \ \overset{\vdash}{\rightarrowtail\!\!\!\!\!\leftarrowtail} \ \xi(q_i) \ + \ dq_i \ \overset{\vdash}{\rightarrowtail\!\!\!\!\!\leftarrowtail} \ \xi(p_i). \qquad \text{(Definition 6.6)} \end{aligned}$$

This shows that $(\hat{i_\xi})_{\vdash}\ \omega$ is determined by $\xi(p_i)$ and $\xi(q_i)$, which also determine ξ. The result now follows from Lemma 8. □

It follows from Cartan's formula (first formula in Lemma 7) that

$$\xi \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_{2n}, \omega) \quad \Longleftrightarrow \quad \hat{d}_{\dashv}((\hat{i_\xi})_{\vdash}\ \omega) = 0.$$

Hence under the isomorphism of Lemma 9, symplectic vector fields correspond to closed 1 forms. Define Hamiltonian vector fields to be the ones that correspond to exact 1 forms. However, Definition 6.7 says that $H^1(\Omega^*(\mathsf{Z} \circ \mathsf{V}_{2n}), \hat{d}_{\dashv}) = 0$. Hence the Hamiltonian and symplectic vector fields coincide in this case. This defines the short exact sequence in Proposition 7. For the second map, $H \mapsto \xi_H$ is defined by the equation

$$\hat{d}_{\dashv}(H) \ = \ (\widehat{i_{\xi_H}})_{\vdash}\ \omega.$$

Writing both one forms in the unique expression of Lemma 8, we derive the Hamiltonian equations

$$\xi_H(p_i) = \frac{\partial H}{\partial q_i} \quad \text{and} \quad \xi_H(q_i) = -\frac{\partial H}{\partial p_i}. \tag{13}$$

It remains to check that the second map is an anti-Leibniz morphism. This follows from the following formulas

$$(\widehat{i_{[\xi_F,\xi_H]}})_{\vdash}\ \omega \ = \ \hat{d}_{\dashv}((\widehat{i_{\xi_F}})_{\vdash}(\hat{d}_{\dashv}(H))) \qquad \text{and} \qquad (\widehat{i_{\xi_F}})_{\vdash}(\hat{d}_{\dashv}(H)) \ = \ -\{F,H\},$$

which are left as an exercise to the reader.

6.6. **Orthogonal dioperad geometry.** We specialise to $\mathsf{W} = \mathsf{V}_n^-$, a super vector space of dimension $(0|n)$ with basis $\theta_1, \ldots, \theta_n$. There are no even coordinates. We define the symmetric two tensor

$$\rho \ := \ \sum_{i=1}^{n} d\theta_i \ \rightarrowtail\!\!\!\!\leftarrowtail \ d\theta_i \quad \in \ \Omega^2(\mathsf{Z} \circ \mathsf{V}_n^-).$$

This defines an orthogonal dioperad manifold (X, ρ). We now proceed as in the symplectic case; the interested reader may fill in the details.

Definition 6.11. Let now define the Leibniz superalgebra of orthogonal vector fields on (X, ρ). Let

$$\mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_n^-, \rho) \ := \ \{\xi \in \mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_n^-) \mid (\hat{L_\xi})_{\vdash}\ \rho = 0\}.$$

We also denote this Leibniz superalgebra by the notation $\mathrm{Der}(o_n, \mathsf{Y})$.

Definition 6.12. We interpret the space $\mathsf{Z} \circ \mathsf{V}_n^-$ as functions on the orthogonal dioperad manifold (X, ρ). We define the Poisson bracket

$$\{\ ,\ \} : \mathsf{Z} \circ \mathsf{V}_n^- \otimes \mathsf{Z} \circ \mathsf{V}_n^- \to \mathsf{Z} \circ \mathsf{V}_n^-$$

using the formula

$$\{F, H\} \ := \ (-1)^{|F|} \sum_{i=1}^{n} \frac{\partial F}{\partial \theta_i} \ \overset{\dashv}{\rightarrowtail\!\!\!\!\leftarrowtail} \ \frac{\partial H}{\partial \theta_i}.$$

This defines a Leibniz superalgebra structure on $\mathsf{Z} \circ \mathsf{V}_n^-$.

Proposition 8. *There is a split short exact sequence of Leibniz superalgebras*

$$0 \to \mathsf{Z}[0] \hookrightarrow \mathsf{Z} \circ \mathsf{V}_n^- \twoheadrightarrow \mathrm{Der}(\mathsf{Y} \circ \mathsf{V}_n^-, \rho) \to 0.$$

The second map sends H to ξ_H, where $\xi_H(\theta_i) = \frac{\partial H}{\partial \theta_i}$.

7. Graph homology

In this section, we give the definitions of the graph complexes that occur in Conjecture 2.

Definition 7.1. A graph is a 1 dimensional CW complex. For a graph Γ, we denote the set of vertices by $V(\Gamma)$, the set of edges by $E(\Gamma)$, the set of ends of an edge e by $V(e)$ and the set of edges incident at a vertex v by $E(v)$.

Definition 7.2. For a set S, let $\mathbb{K}\,\mathsf{S}$ be the vector space over $\mathbb{K}$ which has the elements of S as a basis. For a vector space W of dimension n, we use the notation $\det W = \Lambda^n W$.

7.1. **The graph complex $\hat{\mathcal{C}}(gl, \mathsf{Y})$.** Let Y be a dioperad. A Y-graph is a directed graph Γ such that for every vertex v, there is exactly one outgoing edge and a Y-structure is specified on the set of incoming edges at v. We will use the letter Γ to denote a Y-graph as well as its underlying graph.

Definition 7.3. Let Γ be a Y-graph and L be the set of linear orders on $V(\Gamma)$. A labelling l of Γ is an element of $\mathbb{K}L$. We say that (Γ, l) is a labelled Y-graph.

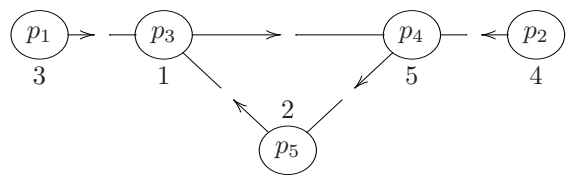


FIGURE 1. A labelled Y-graph.

Figure 1 shows a labelled Y-graph with 5 vertices and 5 edges. The edges are drawn broken to emphasize that the graph is made of 5 dioperad elements with $p_1, p_2 \in \mathsf{Y}[0]$, $p_3, p_4 \in \mathsf{Y}[2]$ and $p_5 \in \mathsf{Y}[1]$. The numbers written close to the vertices show the linear order on $V(\Gamma)$.

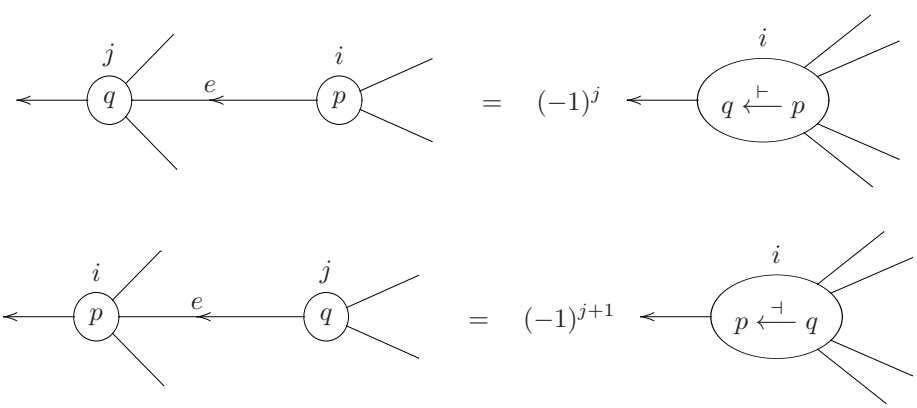
Definition 7.4. The kth chain group of the chain complex $\hat{\mathcal{C}}(gl, \mathsf{Y})$, which we denote $\hat{\mathcal{C}}_k(gl, \mathsf{Y})$, is the vector space over $\mathbb{K}$ generated by all labelled connected Y-graphs (Γ, l) with k vertices, upto automorphism, and vertex linearity and the condition

$$(\Gamma, k_1 l_1 + k_2 l_2) \;=\; k_1(\Gamma, l_1) + k_2(\Gamma, l_2), \quad \text{for} \quad k_1, k_2 \in \mathbb{K}. \tag{14}$$

The boundary map $\partial_E : \hat{\mathcal{C}}_k(gl, \mathsf{Y}) \to \hat{\mathcal{C}}_{k-1}(gl, \mathsf{Y})$ is defined using edge contractions. We do not contract loops. More precisely, we have

$$\partial_E(\Gamma, l) = \sum_{e \in E(\Gamma)} (\Gamma/e, l/e), \tag{15}$$

where Γ/e and l/e are defined as below.



Let p and q be Y-structures and i and j be the labels of the vertices connecting the edge e with $i < j$. Then Γ/e is the graph Γ with the edge e contracted, the

Y-structure on the resulting vertex being $p \overset{\dashv}{\longrightarrow} q$ if e points from i to j, and $p \overset{\dashv}{\longleftarrow} q$ if e points from j to i. To get l/e, we label the resulting vertex i, decrease the labels greater than j by one, and multiply this labelling by $(-1)^j$ if e points from i to j, and by $(-1)^{j+1}$ if it points from j to i.

The associativity property of dioperad substitution and the choice of sign imply that $\partial_E^2 = 0$. This defines the chain complex $\hat{\mathcal{C}}(gl, \mathsf{Y}) = (\hat{\mathcal{C}}_*(gl, \mathsf{Y}), \partial_E)$.

7.2. **The graph complexes $\hat{\mathcal{C}}(sp, \mathsf{Y})$ and $\hat{\mathcal{C}}(o, \mathsf{Y})$.** Let Z be the mated species of a reversible dioperad Y. A Z-graph is a graph Γ such that for every vertex v, a Z-structure is specified on the set of edges incident at v.

Definition 7.5. We define two different notions of labelling for a Z-graph Γ.

A labelling l of Γ is an element of $\mathbb{K}L$, where L is the set of linear orders on $V(\Gamma)$. We say that (Γ, l) is a labelled Z-graph.

An odd labelling l^- of a Z-graph Γ is an element of $\mathbb{K}L \otimes \bigotimes_{v \in V(\Gamma)} \det \mathbb{K}E(v)$. We say that (Γ, l^-) is an odd labelled Z-graph. A way to represent an odd labelling is to order the order the edges incident on every fixed vertex. An odd permutation of the labels on the edges incident to a fixed vertex reverses the labelling.

Definition 7.6. The kth chain group of $\hat{\mathcal{C}}(sp, \mathsf{P})$, which we denote $\hat{\mathcal{C}}_k(sp, \mathsf{P})$, is the vector space over $\mathbb{K}$ generated by all connected labelled Z-graphs (Γ, l) with k vertices, upto automorphism, and vertex linearity and Equation (14). The boundary map $\partial_E : \hat{\mathcal{C}}_k(sp, \mathsf{P}) \to \hat{\mathcal{C}}_{k-1}(sp, \mathsf{P})$ is defined using Equation (15), where l/e is as before and Γ/e is the graph Γ with the edge e contracted, the Y-structure on the resulting vertex being $p \overset{\dashv}{\rightarrowtail\!\!\!\!\leftarrowtail} q$. Note that the bar is on the side of the vertex with the larger label.

Definition 7.7. The chain complex $\hat{\mathcal{C}}(o, \mathsf{P})$ is defined similarly to $\hat{\mathcal{C}}(sp, \mathsf{P})$, using connected odd labelled Z-graphs. The induced labelling l^-/e is obtained the following way: let v_1 and v_2 be the ends of the edge e. Choose a representative of l where the edge e has label 1 for both v_1 and v_2; give the new vertex v arising from the contraction of e the label i, decrease the labels greater than j by one, renumber the labels on $E(v)$ by preserving the relative order on the labels of $E(v_1)$ and $E(v_2)$ and shifting up the labels on $E(v_2)$. Finally multiply this labelling by the sign $(-1)^{j+|v_1|}$ if the degree of v_2 is even and by $(-1)^{i+k+1+|v_1|}$ if the degree of v_2 is odd, where k is the number of vertices of odd degree with label between i and j.

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