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# PROJECTIONS, SHELLINGS AND DUALITY

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Abstract. Projection maps which appear in the theory of buildings and oriented matroids are closely related to the notion of shellability. This was first observed by Björner [9]. In this paper, we give an axiomatic treatment of either concept and show their equivalence. We also axiomatize duality in this setting. As applications of these ideas, we prove a duality theorem on buildings and give a geometric interpretation of the flag $h$ vector. The former may be regarded as a $q$-analogue of the Dehn-Sommerville equations. We also briefly discuss the connection with the random walks introduced by Bidigare, Hanlon and Rockmore [5].

## Contents



## 1. Introduction

Projection maps appeared independently in the theory of buildings developed by Tits [33] and in the theory of oriented matroids [10]. Given the importance of these maps in either theory, it is reasonable to try to formulate them axiomatically. The close connection between these maps and the notion of shellability was first observed by Björner [9]. In this paper, we study projection maps axiomatically keeping the viewpoint of [9]. This leads us to the notion of many "compatible" shellings rather than a single shelling. To complete the picture, we also consider restriction maps that are useful to keep track of a shelling. For ideas closely related to this paper, see [27, 16, 21]. The main result of this paper (Theorem 1) may be

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informally stated as

$$\text{Projection maps} \Longleftrightarrow \text{Compatible shellings.}$$

Thus far, people seem to be interested in the question "Is a complex shellable?" The next question that should be asked is "How do we study the space of all shellings of a complex?" As explained in the previous paragraph, projection maps give us an approach to this hard problem. We mention that though compatible shellings form a fairly large class of shellings, they are far from giving all possible shellings of a complex.

For a labeled shellable complex $\Delta$, Björner [9, Theorem 1.6] gave a simple geometric interpretation of the flag $h$ vector. In our case, $\Delta$ has many "compatible" shellings rather than a single shelling. This allows us to define local flag $h$ vectors of $\Delta$. And the average of the local vectors gives us the usual vector. Also the local vectors have an even simpler geometric interpretation. Further the local vectors are all equal to the usual vector when the projection maps satisfy some commutativity relations. These relations are related to the uniformity of the stationary distribution of certain random walks introduced by Bidigare, Hanlon and Rockmore [5]. In this context, we generalize a result on uniform stationary distributions obtained in [6].

Next we formalize the notion of duality by adding an "opposite" axiom to our axiomatic setup. It is strong enough to imply the Dehn-Sommerville equations. And as one expects, this axiom can hold only for a simplicial complex homotopy equivalent to a sphere. From the viewpoint of shellings, the relevant concept is that of shelling reversal, i.e., when is the reverse of a shelling again a shelling?

The Solomon-Tits theorem says that a (thick) spherical building $\Delta$ has the homotopy type of a (non-trivial) wedge of spheres. Hence by the observation in the previous paragraph, $\Delta$ cannot satisfy the opposite axiom. However $\Delta$ has a remarkable duality which can be expressed in terms of the flag $h$ vector (Theorem 6). This may be regarded as a $q$-analogue of the Dehn-Sommerville equations.

**Organization of the paper.** The next two sections provide basic definitions and motivating examples. The axiomatic theory involving projection maps, shellings and restriction maps is presented in Section 4 and later in a more intuitive metric setup in Section 6. In Section 7 we explain the connection with the flag $h$ vector. The connection with random walks is explained in Section 8. In the next two sections, we study the notion of duality first for thin complexes and then for buildings. In the final section, we outline some problems for further study.

## 2. Projection maps

In this section, we give four examples of the mini-theory that we will present in Section 4. We will return to these examples again in Section 6, where we will show that they are indeed examples of our theory. Terms like gallery connected, convex, etc. that we freely use here are also explained in that section. For now, we present the examples from the viewpoint of projection maps. They fit into the framework of LRBs (non-associative in general) as indicated in Table 1.

The question marks say that a building-like analogue for the more general case of hyperplane arrangements is unknown. Though LRBs in general are not examples

Table 1. Examples of the theory.

| LRB | Non-associative LRB |
|---|---|
| Hyperplane arrangements | ??? |
| Reflection arrangements | Buildings |
| Distributive lattices | Modular lattices |

of our theory they come quite close as we will see in Section 5; and hence can potentially give more examples. Keeping this in mind, we start with a brief review of LRBs. Our exposition in Section 2.1 is mainly taken from [14].

2.1. **Left regular bands.** Let $S$ be a semigroup (finite, with identity). A *left-regular band*, or LRB, is a semigroup $S$ that satisfies the identities

$$x^2 = x \quad \text{and} \quad xyx = xy \tag{D}$$

for all $x, y \in S$. We call (D) the "deletion property", because it admits the following restatement: Whenever we have a product $x_1 x_2 \cdots x_n$ in $S$, we can delete any factor that has occurred earlier without changing the value of the product.

Alternatively, one can say that $S$ is a *LRB* if there are a lattice $L$ and a surjection $\operatorname{supp}\colon S \twoheadrightarrow L$ satisfying

$$\operatorname{supp} xy = \operatorname{supp} x \vee \operatorname{supp} y \tag{1}$$

and

$$xy = x \quad \text{if } \operatorname{supp} y \leq \operatorname{supp} x. \tag{2}$$

Here $\vee$ denotes the join operation (least upper bound) in $L$. A good reference for LRBs is [14]. It explains the equivalence of the above two definitions and also contains plenty of examples. More information about LRBs can be found in [17, 24, 25]. Early references to the identity $xyx = xy$ are [19, 29].

We can also define a partial order on a LRB by setting

$$x \leq y \iff xy = y. \tag{3}$$

We now switch to a slightly different notation which we will be using for the most part. We denote a LRB by $\mathcal{F}$ and call elements of $\mathcal{F}$ faces. A face $C$ is a *chamber* if $CF = C$ for all $F \in \mathcal{F}$, or equivalently, if it is maximal in the partial order on $\mathcal{F}$ specified by equation (3). The set of chambers, which we denote $\mathcal{C}$, is an ideal in $\mathcal{F}$. This gives us a map $\mathcal{F} \times \mathcal{C} \to \mathcal{C}$ that maps $(F, C)$ to $FC$. We call $FC$ the *projection* of $C$ on $F$. In this paper, we are mainly interested in these projection maps rather than the full product in $\mathcal{F}$.

*Example* 1. Hyperplane arrangements: The motivating example of a LRB is the poset of regions of a central hyperplane arrangement (or more generally of an oriented matroid). A good reference for this example is [14, Appendix A]. More details can be found in [5, 6, 10, 13, 15, 23, 34]. Briefly, a finite set of linear hyperplanes (i.e. hyperplanes passing through the origin) in a real vector space $V$ divides $V$ into regions called *chambers*. These are polyhedral sets, which have *faces*. The totality $\mathcal{F}$ of all the faces is a poset under the face relation. Less obviously, $\mathcal{F}$ admits a product, making it a LRB. The lattice $L$ is the intersection lattice of the arrangement (or more generally the underlying matroid).

The product can be described combinatorially by encoding faces using sign sequences. In fact one way to axiomatize an oriented matroid is in terms of sign sequences and this product. The deletion property $(D)$ for $\mathcal{F}$ can be checked directly. The product in $\mathcal{F}$ can also be described geometrically. For instance, the projection $FC$ is the chamber closest to $C$ having $F$ as a face.

The case that is directly relevant to our theory is that of *simplicial* hyperplane arrangements, i.e., the chambers are simplicial cones and hence $\mathcal{F}$ is a simplicial complex. As an interesting example, we mention Coxeter complexes which arise from reflection arrangements.

Now let $\mathcal{D} \subseteq \mathcal{C}$ be a convex set of chambers and let $\mathcal{G}$ be the set of faces of all the chambers in $\mathcal{D}$. Then using the geometric description of the product in $\mathcal{F}$, for instance, one can check that $\mathcal{G}$ is also a LRB. We now give a concrete example of this type.

*Example* 2. Distributive lattices: A good reference for this example is [14, Section 4]. We will soon see that it generalizes to the case of modular lattices (Example 4). We start with the basic example of the Boolean lattice $\mathcal{B}_n$ of rank $n$ consisting of all subsets of an $n$-set ordered under inclusion. Its flag (order) complex $\Delta(\mathcal{B}_n)$ is the Coxeter complex of type $A_{n-1}$ and corresponds to the braid arrangement. Hence the set of faces of $\Delta(\mathcal{B}_n)$ is a LRB. Moreover the product can be described entirely using the lattice structure (meets and joins) of $\mathcal{B}_n$.

More generally, the set of faces $\mathcal{F}$ of the flag (order) complex $\Delta(M)$ of any distributive lattice $M$ is also a LRB. A simple way to verify this is to appeal to the well-known fact that $M$ can be embedded as a sublattice of the Boolean lattice. More geometrically, Abels [1, Proposition 2.5] has described a way of constructing an embedding which makes the set of chambers in $\Delta(M)$ (i.e., the maximal chains in $M$) correspond to a convex set of chambers in $\Delta(\mathcal{B}_n)$, the complex of the braid arrangement.

2.2. **Non-associative LRBs.** As one may expect, these are the same as LRBs except that we no longer require associativity. However, we do require that $xyx$ be well-defined for all $x, y$; that is, $xyx = (xy)x = x(yx)$. With this restriction, the deletion property makes sense. In the non-associative setting, the second definition of LRBs involving a lattice $L$ does not make much sense and we omit it. Also transitivity of the relation $\leq$ defined by equation (3) is no longer automatic. So we have two choices; either to take transitive closure of the relation or simply to impose transitivity as an additional condition. In the two examples that we consider, transitivity of the relation $\leq$ is in fact automatic and hence we do not pursue this issue further.

To avoid confusion later, we mention that the term LRB always means an associative LRB. Whenever we want to include the non-associative case, we will say so explicitly.

*Example* 3. Buildings: A good reference for this example is [33]; also see [13, 28]. Some of the terminology used here is explained at the beginning of Section 6. Also the building of type $A_{n-1}$ is described briefly in Example 5 in Section 10. Let $W$ be a Coxeter group and $\Sigma(W)$ its Coxeter complex. Roughly a building $\Delta$ of type $W$ is a union of subcomplexes $\Sigma$ (called *apartments*) which fit together nicely. Each apartment $\Sigma$ is isomorphic to $\Sigma(W)$. For any two simplices in $\Delta$, there is an apartment $\Sigma$ containing both of them. As a simplicial complex, $\Delta$ is pure, labeled and gallery connected. Also each apartment is convex. There is a $W$-valued distance function $\delta : \mathcal{C} \times \mathcal{C} \to W$ that generalizes the gallery distance. Furthermore for any apartment $\Sigma$ and chamber $C \in \mathcal{C}$, there is a retraction $\rho = \rho_{\Sigma,C} : \Delta \to \Sigma$ satisfying $\delta(C, \rho(D)) = \delta(C, D)$ for any $D \in \mathcal{C}$.

We denote the set of faces by $\mathcal{F}$. For $F, G \in \mathcal{F}$, we choose an apartment $\Sigma$ containing $F$ and $G$ and define $FG$ to be their product in $\Sigma$. Since $\Sigma$ is a Coxeter complex, we know how to do this (Example 1). Furthermore it can be shown that the product does not depend on the choice of $\Sigma$. Hence this defines a product on the set of faces $\mathcal{F}$ of $\Delta$. And it is compatible with the retraction $\rho = \rho_{\Sigma,C}$, namely, $\rho(F)C = \rho(FC)$ for any face $F$ of $\Delta$.

*Example* 4. Modular lattices: A good reference for this example is [1] where you can find proofs of all the facts that we state here. The flag (order) complex $\Delta(M)$ of a modular lattice $M$ is a labeled simplicial complex. A face of $\Delta$ is a chain in $M$. As usual, we denote the set of faces by $\mathcal{F}$. The chambers $\mathcal{C}$ of $\Delta$ are the maximal chains in $M$. We define the map $\mathcal{F} \times \mathcal{C} \to \mathcal{C}$ as follows. For $\mathcal{F} \in \mathcal{F}$ and $C \in \mathcal{C}$, define $FC$ to be the unique chamber containing $F$ that is contained in the sublattice generated by $F$ and $C$. More generally, for $F, G \in \mathcal{F}$, the face $FG$ is the chain in $M$ obtained by refining the chain $F$ by the chain $G$, using meets and joins as in a Jordan–Hölder product.

An interesting subclass of modular lattices is that of distributive lattices that we discussed earlier. Another interesting fact is that in a modular lattice, the sublattice generated by any two chains is distributive; see [7, pg 66]. Hence modular lattices may be regarded as generalizations of buildings of type $A_{n-1}$, with the distributive lattices playing the role of apartments (and their convex subsets).

It may be possible to generalize this example in various directions. For example, one may consider more general lattices like supersolvable lattices [31]. It is also a challenge to find analogues of modular lattices that generalize buildings of types other than $A_{n-1}$.

*Remark.* For buildings, while there is always an apartment containing two given chambers, there may not be an apartment containing three given chambers. Similarly for modular lattices, the sublattice generated by three chains may not be distributive. This is the basic reason why the product in buildings and modular lattices is not associative. We do not know of any algebraic tools relevant to the study of non-associative LRBs.

*Remark.* All four examples that we discussed share some common geometric properties. The LRB $\mathcal{F}$ (non-associative in the last two examples) is a simplicial (or polyhedral) complex. (Later in the paper, we will call such an $\mathcal{F}$, a *simplicial LRB.*) Furthermore $\mathcal{F}$ is pure and gallery connected. Geometrically the projection $FC$ is the chamber closest to $C$ having $F$ as a face. Also $FG = \cap_{G \leq D, D \in \mathcal{C}} FD$ for $F, G \in \mathcal{F}$.

## 3. Shellings and restriction maps

Let $\Delta$ be a finite pure $d$-dimensional simplicial complex. The term *pure* means that all maximal simplices have the same dimension. We will call the maximal simplices chambers. Let $\mathcal{F}$ be the set of all faces of $\Delta$ and let $\mathcal{C}$ be the set of chambers of $\Delta$. Also let $\Delta_{\geq F}$ be the simplicial subcomplex of $\Delta$ consisting of all faces of the faces that contain $F$. Similarly $\mathcal{C}_{\geq F}$ stands for the set of chambers of

$\Delta_{\geq F}$. Let the partial order $\leq$ denote the face relation; that is, $F \leq G$ if $F$ is a face of $G$.

We say that $\Delta$ is *shellable* if there is a linear order $\leq_S$ on the set of chambers $\mathcal{C}$ of $\Delta$ such that for every $D \in \mathcal{C}$ except the first in the linear order, we have $D \cap (\cup_{E <_S D} E)$ is pure $(d-1)$-dimensional; that is, the intersection of $D$ with the chambers that came before it in the linear order $\leq_S$ is a non-empty union of certain facets of $D$. Less formally, a shelling gives a systematic way to build $\Delta$ by adding one chamber at a time. And the subcomplex obtained at each stage (and in particular the entire complex $\Delta$) is gallery connected.


Figure 1. The restriction map at work.

To a shelling $\leq_S$, we can associate a *restriction map* $R : \mathcal{C} \to \mathcal{F}$ with $R(D)$ defined to be the face of $D$ spanned by those vertices $v$ of $D$ for which $D \setminus v \leq D \cap (\cup_{E <_S D} E)$. Here $D \setminus v$ is the facet of $D$ that does not contain the vertex $v$. Figure 1 shows two cases that can arise in the rank 3 case. The facets of $D$ that are shown by dark lines indicate the intersection of $D$ with the chambers that came before it in the shelling order. Observe that $R(D) \leq F \leq D \Leftrightarrow F$ shows up for the first time when we adjoin $D$. This shows that $\mathcal{F}$ can be expressed as a disjoint union

$$\mathcal{F} = \sqcup_{D \in \mathcal{C}} \{F \in \mathcal{F} \mid R(D) \leq F \leq D\}. \tag{4}$$

Compare this statement with the restriction axiom $(R2)$ in Section 4.1. Also note that $R(D) = \emptyset$, the empty face of $\Delta$, if and only if $D$ is first in the linear order $\leq_S$. To summarize, the restriction map $R$ gives us the local data at every chamber associated with the shelling $\leq_S$. It does not have enough information however to allow us to reconstruct $\leq_S$. This is because the essence of a shelling is really a partial order. Here is what we can do. We can define a partial order $\leq_C$ on $\mathcal{C}$ as the transitive closure of the relation:

$$E \leq_C D \text{ if } R(E) \leq D. \tag{5}$$

Observe that the chamber $C$ by which we indexed our partial order is the one that occurred first in $\leq_S$. The motivation for this notation will be more clear when we study the shelling axioms in Section 4.1. One can show directly that any linear extension of $\leq_C$ is a shelling of $\Delta$ (compare this statement with the shelling axiom $(S2)$) and the shelling $\leq_S$ that we started with is one of them. We can also define a more refined partial order $\leq_C^r$ on $\mathcal{C}$ as the transitive closure of the relation:

$$E \leq_C^r D \text{ if } R(E) \leq D \text{ and } E \text{ is adjacent to } D. \tag{6}$$

We will show (Lemma 2) that for a thin complex, the partial orders $\leq_C$ and $\leq_C^r$ on $\mathcal{C}$ are identical. The term *thin* means that every facet is contained in exactly 2 chambers.

Now suppose instead that we start not with a shelling order $\leq_S$ but only with a map $R : \mathcal{C} \to \mathcal{F}$ that satisfies equation (4). Then can we say that $R$ is the restriction map of a shelling? We know only a partial answer. The first thing to do is to define a relation $\leq_C$ using equation (5). (The chamber $C$ is the unique chamber satisfying $R(C) = \emptyset$.) If the relation $\leq_C$ happens to be a partial order then it follows that any linear extension of $\leq_C$ is a shelling of $\Delta$ and the restriction map of any of these is $R$.

The above statements will be justified in the course of proving Theorem 1. For future use, we record three useful results about shellable complexes.

**Lemma 1.** *Let $F$ be any face of $\Delta$ and let $\leq_S$ be a shelling of $\Delta$. Then this linear order when restricted to $\mathcal{C}_{\geq F}$ is a shelling of $\Delta_{\geq F}$. In other words, the link $\mathrm{lk}(F, \Delta)$ of any face $F$ in a shellable complex $\Delta$ is again shellable with the induced order.*

*Proof.* Let $\leq_S$ be a shelling of $\Delta$ and let $\leq'_S$ be its restriction to $\mathcal{C}_{\geq F}$. By the definition of a shelling we have

$$D \cap (\cup_{E <_S D} E) = \cup_{G \in \mathcal{F}_D} G, \tag{*}$$

where $\mathcal{F}_D$ is a subset of the set of facets of $D$. Let $\mathcal{F}'_D$ be the subset of $\mathcal{F}_D$ consisting of those facets that contain $F$. The lemma follows from the following claim.

Claim: For $D \geq F$, we have $D \cap (\cup_{E <'_S D} E) = \cup_{G \in \mathcal{F}'_D} G$.
Note that if $D$ is not the first element in $\leq'_S$ then $F \subseteq LHS$. In that case the set $\mathcal{F}'_D$ will be non-empty as required.
Proof of the claim. ($\supseteq$) Let $G \in \mathcal{F}'_D$. By equation $(*)$, there is a $E \geq G$ such that $E <_S D$. Since $G \geq F$, it follows that $E \geq F$ and hence $E <'_S D$.
($\subseteq$) Let $F' \subseteq LHS$. We may assume that $F' \geq F$. Since otherwise we may replace $F'$ by the face spanned by $F$ and $F'$ which still belongs to the LHS. Applying $(*)$, there is a $G \in \mathcal{F}_D$ such that $G \geq F' \geq F$. So $G \in \mathcal{F}'_D$. Hence $F' \subseteq RHS$. $\square$

**Lemma 2.** *Let $\Delta$ be a thin shellable complex with shelling $\leq_S$ and restriction map $R$. Let $\leq_C$ and $\leq^r_C$ be the partial orders on $\mathcal{C}$ defined by equations (5) and (6) respectively. Then the partial orders $\leq_C$ and $\leq^r_C$ on $\mathcal{C}$ are identical.*

*Proof.* We only need to show that $E \leq_C D \Rightarrow E \leq^r_C D$, or equivalently, $R(E) \leq D$ implies $E \leq^r_C D$. We do this by constructing a gallery from $E$ to $D$ such that for consecutive chambers $E'$ and $E''$ in the gallery, we have $R(E') \leq E''$.

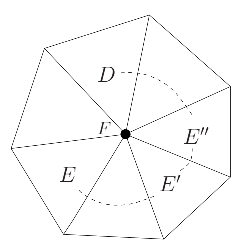



FIGURE 2. A gallery from $E$ to $D$ in $\Delta_{\geq F}$ where $F = R(E)$.

By Lemma 1, the shelling $\leq_S$ restricts to a shelling $\leq'_S$ of $\Delta_{\geq R(E)}$. Hence $\Delta_{\geq R(E)}$ is gallery connected at every stage of the shelling $\leq'_S$. And $E$ is the starting chamber of the shelling. This allows us to choose a gallery from $E$ to $D$ in $\Delta_{\geq R(E)}$ consistent with the shelling $\leq'_S$. This automatically implies consistency with the original shelling $\leq_S$. Let $E'$ and $E''$ be two consecutive chambers in the gallery such that $E' <_S E''$. Since $\Delta$ is thin, the common facet of $E'$ and $E''$ appears for the first time in the shelling $<_S$ when we adjoin $E'$. Hence we get $R(E') \leq E''$. This is exactly what we wanted to show. Hence the partial orders $\leq_C$ and $\leq_C^r$ are identical.

□

**Proposition 1.** [9, Theorem 1.3] *A finite $d$ dimensional shellable complex $\Delta$ is homotopy equivalent to a wedge of $d$-spheres. The number of these spheres is given by $|\{D \mid R(D) = D\}|$, where $R$ is the restriction map associated with any shelling of $\Delta$.*

Note that if $|\{D \mid R(D) = D\}| = 1$ then the unique chamber $D$ for which $R(D) = D$ is the one that gets shelled in the end.

## 4. Axioms

In this section we provide an axiomatic setup that relates projection maps, shellings and restriction maps. Motivation for some of the axioms was given in the previous two sections. The axioms are somewhat abstract and you may want to simply glance at them now. They are split into three categories. The first two axioms in each of the three categories really deal with a fixed chamber $C \in \mathcal{C}$. As we vary $C \in \mathcal{C}$, it is natural to impose some *compatibility* condition. This is the content of the third (compatibility) axiom. It is easiest to swallow in the shelling case. We do not know of any way to make it more palatable in the other two cases.

The main result of this section (Theorem 1) says that for a pure simplicial complex, the different sets of axioms are equivalent. The way to pass from one set of axioms to another is explained immediately after the statements of all the axioms. The examples mentioned in Section 2 are discussed from this axiomatic viewpoint in Section 6. It is a good idea to read that section before reading the proof of Theorem 1. Apart from serving as motivation, it will enable you to understand the geometric content of every step in the proof.

Let $\Delta$ be a finite pure simplicial complex. Let $\mathcal{F}$ be the set of all faces of $\Delta$ and let $\mathcal{C}$ be the set of chambers of $\Delta$. We always consider the empty face $\emptyset$ to be a face of $\Delta$. Let the partial order $\leq$ denote the face relation; that is, $F \leq G$ if $F$ is a face of $G$. Also let $\lessdot$ stand for the cover relation; that is, $G \lessdot D$ if $G$ is a codimension 1 face of $D$. If $D$ is a chamber and $G \lessdot D$ then $G$ is a facet of $\Delta$.

4.1. **The axioms.** Before we state the projection axioms, we need a definition. Given a map $\mathcal{F} \times \mathcal{C} \to \mathcal{C}$, we say that $E$ is *weakly $C$ adjacent* to $D$ if they have a common face $F$ such that $FC = D$. We may write this as $D \overset{F}{-} E$.

Note that this is not a symmetric relation. Also we do not require $F$ to be a facet. In fact, $F$ could also be the empty face. This explains the term "weakly adjacent". A *weak $C$ gallery* is a sequence of chambers $C_1, \ldots, C_n$ such that $C_{i+1}$ is weakly $C$ adjacent to $C_i$ for $1 \leq i \leq n-1$. We write this as $C_1 - \ldots - C_n$. If more precision is required then we write $C_1 \overset{F_1}{-} C_2 \overset{F_2}{-} \ldots \overset{F_{n-1}}{-} C_n = D$, where $F_1, \ldots, F_{n-1}$ is a sequence of faces such that, $F_i C = C_i$ and $F_i \leq C_i, C_{i+1}$. It will be seen from axiom $(P1)(ii)$ that $\emptyset C = C$. Hence $C$ can always be tagged on as the first element of any weak $C$ gallery.

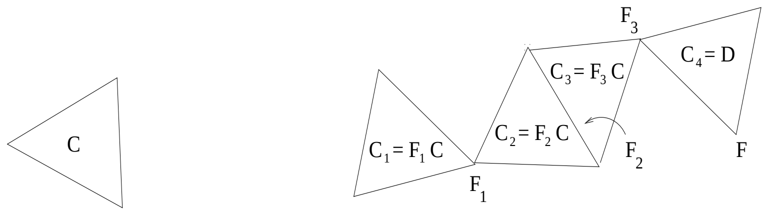



FIGURE 3. A weak $C$ gallery $C_1 - C_2 - C_3 - C_4$.

**Projection axioms** $(P)$**.** For every $F \in \mathcal{F}$, there is a projection map $\mathcal{C} \to \mathcal{C}$ which we write $C \mapsto FC$ that satisfies

- (P1) (i) If $FC = D$ then $F \le D$.
  (ii) If $F \le C$ then $FC = C$.
  (iii) If $FC = D$ and $F \le G \le D$ then $GC = D$.
- (P2) If $F \le D$ and $GC = D$ for all $F \le G \lessdot D$ then $FC = D$.
- (P2′) If $F_1C = F_2C = D$ then there exists a face $F \le F_1, F_2$ such that $FC = D$.
- (P3) If $FC = D$ and $C_1 - C_2 - \ldots - C_n = D$ is any weak $C$ gallery from $C_1$ to $D$ then $FC_1 = D$.

Axiom $(P1)(ii)$ implies that $F = \emptyset$ acts as the identity on $\mathcal{C}$. Also note that it is a special case of axiom $(P1)(iii)$ obtained by setting $F = \emptyset$ and $C = D$. For axiom $(P2)$, as a special case when $F = \emptyset$, we get: If $GC = D$ for all $G \lessdot D$ then $C = D$. As another special case, when $F = D \in \mathcal{C}$, the second condition holds vacuously and we get $DC = D$. In Figure 3, we have illustrated a weak $C$ gallery. If the face $F$ is such that $FC = D$ then axiom $(P3)$ says that $FC_1 = D$.

The axiom $(P2')$ is motivated by the theory of descent sets (see the discussion at the end of Section 6.2). In the course of proving Theorem 1, we will never use axiom $(P2')$. The justification is the following simple lemma, whose proof we omit.

**Lemma 3.** *Let $\Delta$ be a simplicial complex with a map $\mathcal{F} \times \mathcal{C} \to \mathcal{C}$. Then $\Delta$ satisfies $(P1)$ and $(P2)$ $\Leftrightarrow$ $\Delta$ satisfies $(P1)$ and $(P2')$.*

**Restriction axioms** $(R)$**.** For every $C \in \mathcal{C}$, there is a restriction map $R_C : \mathcal{C} \to \mathcal{F}$ that satisfies

- (R1) For any $C, D \in \mathcal{C}$, we have $R_C(C) = \emptyset$ and $R_C(D) \le D$.
- (R2) For any $C \in \mathcal{C}$, we have $\mathcal{F} = \sqcup_{D \in \mathcal{C}} \{F \in \mathcal{F} \mid R_C(D) \le F \le D\}$.
- (R3) If $R_C(C_1) \le C_2, R_C(C_2) \le C_3, \ldots, R_C(C_{n-1}) \le C_n = D$ then $R_{C_1}(D) \le R_C(D) \le D$.

The symbol $\sqcup$ in axiom $(R2)$ stands for disjoint union. Also for simplicity of notation and suggestiveness, we will write the set of inequalities in the "if part" of axiom $(R3)$ as $C_1 \xrightarrow{R_C(C_1)} C_2 \xrightarrow{R_C(C_2)} \ldots \xrightarrow{R_C(C_{n-1})} C_n$. We will refer to this diagram as a sequence of $C$ inequalities. In proving the equivalence of the axioms, we will see that a sequence of $C$ inequalities is indeed a weak $C$ gallery (and extremal in a certain sense). The ambiguity in notation will then disappear.

**Shelling axioms** $(S)$**.** For every $C \in \mathcal{C}$, there is a partial order $\le_C$ on $\mathcal{C}$ that satisfies

- (S1) For any $F \in \mathcal{F}$, the partial order $\le_C$ restricted to $\mathcal{C}_{\ge F}$ has a unique minimal element. For the empty face $F = \emptyset$, this unique minimal element is $C$ itself.

(S2) Every linear extension of $\leq_C$ is a shelling of $\Delta$.
(S2′) There exists a linear extension of $\leq_C$ that is a shelling of $\Delta$.
(S3) The partial orders are compatible in the sense that if $D \leq_C D_1 \leq_C D_2$ then $D_1 \leq_D D_2$.

Axiom $(S2)$ clearly implies axiom $(S2')$. In the course of proving Theorem 1, we will see that we can drop axiom $(S2)$ altogether and replace it by the weaker axiom $(S2')$. Hence the two axioms can be used interchangeably.

4.2. **Connection between the axioms.** We first explain the basic idea. Projection maps on $\Delta$ give rise to many shellings of $\Delta$. In fact we get a partial order (shelling) $\leq_C$ for every $C \in \mathcal{C}$ as follows. We say $E \leq_C D$ if there is a common face $F$ of the chambers $D$ and $E$ such that $FC = E$. This is illustrated in Figure 4. To be technically correct, the partial order $\leq_C$ is the transitive closure of the above relation.

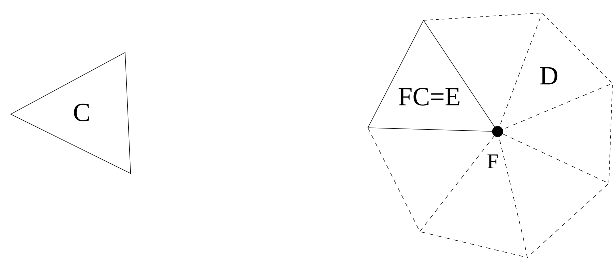



Figure 4. $E$ occurs before $D$ in the shelling associated to $C$.

To put it in words, among all chambers that contain $F$, the chamber that is smallest in the partial order $\leq_C$ is the chamber $FC$. And this is true for every $F \in \mathcal{F}$. The restriction map $R_C$ associated to $\leq_C$ can be defined as follows. The face $R_C(D)$ is the smallest face $F$ of $D$ such that $FC = D$.

We now record the above idea a little more formally. It is this formal connection between the axioms that we will use to prove Theorem 1. Going from

$(P)$ to $(R)$: For every $C \in \mathcal{C}$, we define a map $R_C : \mathcal{C} \to \mathcal{F}$. For $D \in \mathcal{C}$, we let $R_C(D)$ be the face spanned by those vertices $v$ of $D$ which satisfy $(D \setminus v)C \neq D$.

$(R)$ to $(S)$: For every $C \in \mathcal{C}$, we define a partial order $\leq_C$ to be the transitive closure of the relation: $E \leq_C D$ if $R_C(E) \leq D$. (Compare with equation (5).)

$(S)$ to $(P)$: For $F \in \mathcal{F}$ and $C \in \mathcal{C}$, define $FC$ to be the unique minimal element in $\mathcal{C}_{\geq F}$ with respect to the partial order $\leq_C$. Here we used axiom $(S1)$.

4.3. **A shellable complex with a transitive group action.** In Section 2 we discussed some examples from the viewpoint of projection maps. Now we consider an example that is more natural from the point of view of shellability. It remains to be seen whether this approach can be formalized to get more examples.

Let $\Delta$ be a simplicial complex of rank 2 that triangulates a circle; see Figure 5. For every $C \in \mathcal{C}$, we define the partial order $\leq_C$ to be the total order on $\mathcal{C}$ that shells $\Delta$ in the clockwise direction starting at $C$. The shelling axioms are easily checked. Note that there is a transitive action of $\mathbb{Z}/n\mathbb{Z}$ on $\Delta$, where $n$ is the number

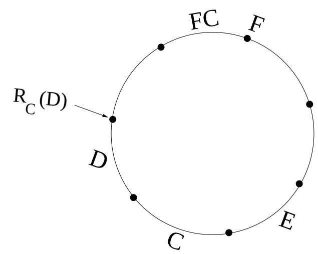



FIGURE 5. A non-metrical example.

of edges in $\Delta$. And this action is in some sense compatible with the partial orders $\leq_C$.

Using the connections between the three sets of axioms sketched above, we now describe the restriction and projection maps. Note that $R_C(C) = \emptyset$ and $R_C(E) = E$, where $E$ is the chamber adjacent to $C$ in the anticlockwise direction. For any other chamber $D$, $R_C(D)$ is the vertex of $D$ that is further from $C$ in the clockwise direction. For the projection maps, the only non-trivial case is when $F$ is a vertex. If $F \leq C$ then $FC = C$. If not then we define $FC$ to be the projection of $C$ on $F$ in the clockwise direction. Note that $FC$ is not necessarily the closest chamber to $C$ that contains $F$. In this sense $\Delta$ is a non-metrical example. It is a good exercise to directly check the projection and restriction axioms.

*Remark.* In the examples that we gave in Section 2, we always had a map $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$. In our axiomatic setting, we may define such a map by using the projection maps $\mathcal{C} \to \mathcal{C}$. For example, we can set $FG = \cap_{G \leq D, D \in \mathcal{C}} FD$ for $F, G \in \mathcal{F}$. However it is not clear what this would imply. For instance, we may ask whether we always get a LRB (non-associative included). In the example above, if we try to extend the projection maps to a product $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$, then $\Delta$ is at best a non-associative LRB.

4.4. **Main result.** We now prove the main result of this paper.

**Theorem 1.** *Let $\Delta$ be a finite pure simplicial complex. Then $\Delta$ satisfies $(P)$ $\Leftrightarrow$ $\Delta$ satisfies $(R)$ $\Leftrightarrow$ $\Delta$ satisfies $(S)$.*

*Proof.* We show $(P) \Rightarrow (R) \Rightarrow (S) \Rightarrow (P)$ using the connections between the axioms that we have already outlined. The proof is fairly routine. There is no particular reason why we choose this circle of implications. For instance, as an exercise, you may try to show $(P) \Rightarrow (S)$ directly.

$(P) \Rightarrow (R)$**.** We verify axioms $(R1)$, $(R2)$ and $(R3)$.

$(R1)$. This is immediate from the definition of $R_C$ and axiom $(P1)(ii)$.

$(R2)$. Let $G$ be any facet of $D$. Then by the definition of $R_C$, we have $GC = D \Leftrightarrow R_C(D) \leq G$. Now by axiom $(P2)$, we get $R_C(D)C = D$. In fact $R_C(D)$ is the unique smallest face of $D$ with this property. This follows from axiom $(P1)(iii)$ and the above "if and only of" statement. Hence $\{F \in \mathcal{F} \mid R_C(D) \leq F \leq D\} = \{F \in \mathcal{F} \mid FC = D\}$. Axiom $(R2)$ is immediate from this description.

$(R3)$. A sequence of $C$ inequalities $C_1 \xrightarrow{R_C(C_1)} C_2 \xrightarrow{R_C(C_2)} \dots \xrightarrow{R_C(C_{n-1})} C_n = D$ is in fact a weak $C$ gallery. This is due to the fact that $R_C(C_i)C = C_i$. Also $R_C(D)$ is such that $R_C(D)C = D$. Hence applying axiom $(P3)$ to the above weak $C$ gallery with $F = R_C(D)$, we get $R_C(D)C_1 = D$. Since $R_{C_1}(D)$ is the smallest face $F$ such that $FC_1 = D$, we get $R_{C_1}(D) \leq R_C(D) \leq D$.

$(R) \Rightarrow (S)$**.** We first show that $\leq_C$ is a partial order. For this we will use all three restriction axioms. The only thing to check is antisymmetry. Expanding out the definition of $\leq_C$ that we have given, $C_1 \leq_C C_n$ if we have a sequence of $C$ inequalities $C_1 \xrightarrow{R_C(C_1)} C_2 \xrightarrow{R_C(C_2)} \dots \xrightarrow{R_C(C_{n-1})} C_n$. To check antisymmetry, we show that a (non-trivial) sequence as above cannot close in on itself. Suppose not, then we have a minimal circular sequence,

$$C_1 \xrightarrow{R_C(C_1)} C_2 \xrightarrow{R_C(C_2)} \dots \xrightarrow{R_C(C_{n-1})} C_n \xrightarrow{R_C(C_n)} C_{n+1} = C_1,$$

which is shown schematically in Figure 6. Then by axiom $(R3)$, we obtain $R_{C_1}(C_n) \leq R_C(C_n) \leq C_n$. However by axiom $(R1)$ we also have $\emptyset = R_{C_1}(C_1) \leq R_C(C_n) \leq C_1$. Since $C_1 \neq C_n$, this contradicts axiom $(R2)$. This shows that $\leq_C$ is a partial order.

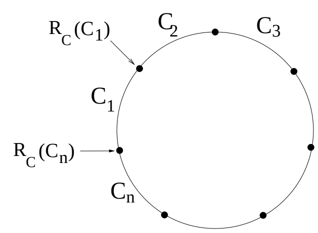



FIGURE 6. A situation that can never occur.

*Remark.* In terms of projection maps, $R_C(C_n)C_1$ is forced to be both $C_1$ and $C_n$, which shows why the situation in Figure 6 never occurs.

Now we verify axioms $(S1)$, $(S2)$ and $(S3)$.

$(S1)$. By axiom $(R2)$, for any $F \in \mathcal{F}$, there is a unique chamber $D$ such that $R_C(D) \leq F \leq D$. For any other chamber $E$ containing $F$, we have $R_C(D) \leq E$. Hence by definition of $\leq_C$, we have $D \leq_C E$. This shows that $D$ is the unique minimal element for $\leq_C$ when restricted to $\mathcal{C}_{\geq F}$.

$(S2)$. Let $\leq_S$ be any linear extension of $\leq_C$. Since $R_C(C) = \emptyset$, we know that $C$ is the unique smallest element in the partial order $\leq_C$. To show that $\leq_S$ is a shelling, it is enough to prove the following.
Claim. For any $D \in \mathcal{C}$ such that $D \neq C$, we have $D \cap (\cup_{E <_S D} E) = \cup_{v \in R_C(D)} D \setminus v$. Note that the RHS is a non-empty union of certain facets of $D$ as required in the shelling condition.
Proof of the claim.($\supseteq$) Since $R_C(D) \nsubseteq D \setminus v$ for $v \in R_C(D)$, we see by axiom $(R2)$ that $R_C(E) \leq D \setminus v \leq E$ for some $E \neq D$. Now since $R_C(E) \leq D$, we have $E <_S D$. Thus $E$ is a chamber such that $D \setminus v \leq E$ and $E <_S D$ as required.
($\subseteq$) We show that if $F$ is a face of $D$ such that $F \nsubseteq RHS$ then $F \nsubseteq LHS$. If $F \nsubseteq RHS$ then $R_C(D) \leq F$. Hence if $F \leq E$ for any $E \in \mathcal{C}$ then $R_C(D) \leq E$, that is, $D \leq_S E$. Hence $F \nsubseteq LHS$.

$(S3)$. If $D \leq_C D_1 \leq_C D_2$ then we have a sequence of $C$ inequalities $D \xrightarrow{R_C(D)} \dots - D_1 \xrightarrow{R_C(D_1)} \dots - D_2$. Then axiom $(R3)$ implies that $D_1 \xrightarrow{R_D(D_1)} \dots - D_2$ is a sequence of $D$ inequalities. Note that we have replaced $C$ by $D$. So $D_1 \leq_D D_2$.

$(S) \Rightarrow (P)$**.** Now we verify axioms $(P1)$, $(P2)$ and $(P3)$. For this we will use the weaker axiom $(S2')$ instead of $(S2)$.

$(P1)$. (i) This follows directly from the definition of the product.
(iii) Here we assume that $FC = D$ and $F \leq G \leq D$. The first assumption says that $D$ is the unique minimal element in the partial order $\leq_C$ restricted to $\mathcal{C}_{\geq F}$. Since $F \leq G \leq D$, it follows that $D$ is also the unique minimal element in the partial order $\leq_C$ restricted to $\mathcal{C}_{\geq G}$. Hence $GC = D$. As mentioned before, axiom $(P1)(ii)$ is a special case of $(P1)(iii)$.

$(P2)$. Using axiom $(S2')$, we choose a linear extension $\leq_S$ of $\leq_C$ which is a shelling of $\Delta$. By Lemma 1, this linear order when restricted to $\mathcal{C}_{\geq F}$ gives a shelling of $\Delta_{\geq F}$, and similarly of $\Delta_{\geq G}$ for all $F \leq G \lessdot D$. Our assumption $GC = D$ for all $F \leq G \lessdot D$ says that for all such $G$, the shelling order restricted to $\mathcal{C}_{\geq G}$ has $D$ as the first element. If we assume that $FC \neq D$ then $D$ is not the first element in the shelling order restricted to $\mathcal{C}_{\geq F}$. Hence $D \cap (\cup_{E <_S D, E \in \mathcal{C}_{\geq F}} E)$ is a non-empty union of certain facets of $D$. This implies that there is a facet $G \lessdot D$ for which the shelling of $\Delta_{\geq G}$ did not start with $D$. This is a contradiction. Therefore $FC = D$.

$(P3)$. If $D$ and $E$ are weakly $C$ adjacent then $D \leq_C E$. Hence if $C_1 - C_2 - \ldots - C_n = D$ is a weak $C$ gallery then $C_1 \leq_C D$. Let $F$ be such that $FC = D$. Now if $E \in \mathcal{C}_{\geq F}$ then $D \leq_C E$. Therefore $C_1 \leq_C D \leq_C E$. Applying axiom $(S3)$, we get $D \leq_{C_1} E$ for all $E \in \mathcal{C}_{\geq F}$. Hence $FC_1 = D$.

□

From now on, if $\Delta$ is a pure simplicial complex that satisfies $(P)$, or equivalently, $(R)$ or $(S)$, then we will simply say that $\Delta$ satisfies our axioms.

## 5. An almost example

The purpose of this section is to identify a potential source of examples that satisfy our axioms. In Section 4 the poset $\mathcal{F}$ was the face lattice of a simplicial complex $\Delta$. It is natural to consider a more general situation, where instead of $\Delta$, we just have a poset $\mathcal{F}$ with the maximal elements playing the role of $\mathcal{C}$. The projection axioms still make sense though the role they now play is not exactly clear. An interesting case to consider is when $\mathcal{F}$ is a LRB. We will show that a LRB satisfies axioms $(P1)$ and $(P3)$. However it can easily violate axiom $(P2)$. We first discuss this situation. We will see that axiom $(P2')$ is better suited for posets than axiom $(P2)$. See also the remark at the end of Section 6.2.

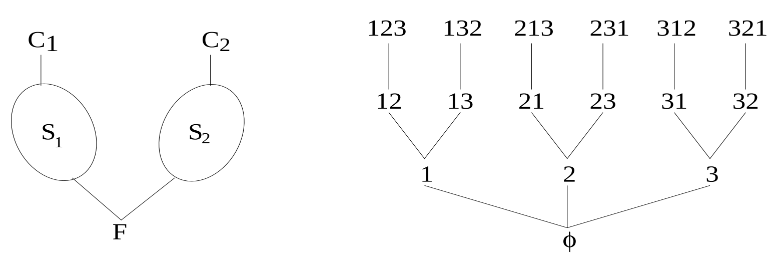



Figure 7. LRBs that violate axiom $(P2)$.

5.1. **On why a LRB violates axiom** $(P2)$**.** To see what goes wrong, consider the following example. Let $S_1$ and $S_2$ be any two LRBs. Consider the LRB $S$ shown in Figure 7 (on the left) with $F_1F_2 = C_1$ and $F_2F_1 = C_2$ for $F_1 \in S_1$ and $F_2 \in S_2$. We first check that $S$ is indeed a LRB. Then by definition $GC_1 = C_2$ for $F < G \lessdot C_2$. However $FC_1 = C_1 \neq C_2$. Hence $S$ violates axiom $(P2)$. We also see that $S$ violates axiom $(P2')$.

To give a concrete example that illustrates the same problem, consider the *free* LRB with identity on $n$ generators, denoted $F_n$. The elements of $F_n$ are sequences $x = (x_1, \ldots, x_l)$ of distinct elements of the set $[n] = \{1, \ldots, n\}$, $0 \leq l \leq n$. We multiply two such sequences by

$$(x_1, \ldots, x_l)(y_1, \ldots, y_m) = (x_1, \ldots, x_l, y_1, \ldots, y_m)\widehat{\ },$$

where the hat means "delete any element that has occurred earlier". For example,

$$(2\,1)(3\,5\,4\,1\,6) = (2\,1\,3\,5\,4\,6).$$

Figure 7 (on the right) shows the Hasse diagram of $F_3$, the free LRB on 3 generators. Check that $F_n$ violates axiom $(P2)$. Also note that every chamber has only one facet. However note that $F_n$ does satisfy axiom $(P2')$.

These examples show that a LRB is not a geometric object in any sense of the term. A natural question that comes up is whether a simplicial LRB always satisfies axiom $(P2)$. We do not know the answer. For the LRBs that we considered in Section 2, the answer is yes. In other words, the LRB associated to any hyperplane arrangement satisfies axiom $(P2)$ (and hence the same holds for flag (order) complex $\Delta(M)$ of a distributive lattice $M$). It is an easy and illuminating exercise to check this directly using the definition of the product involving sign sequences.

5.2. **A LRB satisfies axioms** $(P1)$ **and** $(P3)$**.** Axiom $(P1)$ follows directly from the definition of a LRB and axiom $(P3)$ holds by the following proposition. It is a good exercise to check axiom $(P3)$ directly for the free LRB.

**Proposition 2.** *Let $\mathcal{F}$ be a poset with an associative product $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$. Let $\mathcal{C}$, the set of maximal elements of $\mathcal{F}$, be a left ideal in $\mathcal{F}$. Also assume that $\mathcal{F}$ satisfies axiom $(P1)$ with the additional property that $F \leq FF'$ for any $F, F' \in \mathcal{F}$. Then $\mathcal{F}$ satisfies axiom $(P3)$.*

*Proof.* Though our setting is somewhat abstract, it would help to keep in mind the picture in Figure 3. Let $C_1 \xrightarrow{F_1} C_2 \xrightarrow{F_2} \ldots \xrightarrow{F_{n-1}} C_n = D$ be a weak $C$ gallery; that is, $F_iC = C_i$ and $F_i \leq C_i, C_{i+1}$. We want to show that if $FC = D$ then $FC_1 = D$. For notational consistency, we put $F = F_n$. Let $\mathcal{G} = \{H \in \mathcal{F} \mid F_n \leq H \leq C_n = F_nC\}$. We need to show that $F_nC_1 = C_n$.

We first note three easy consequences of parts $(i)$, $(ii)$ and $(iii)$ respectively of axiom $(P1)$.

(1). If $F_n \leq H$ and $HE = C_n$ for some $E \in \mathcal{C}$ then $H \in \mathcal{G}$.
(2). For $2 \leq i \leq n$, we have $F_{i-1}F_iC = F_iC$. This is because $F_{i-1} \leq C_i = F_iC$.
(3). If $H \in \mathcal{G}$ then $HC = C_n$.

We now claim that $F_nF_i \in \mathcal{G}$ for $1 \leq i \leq n$.
Proof of the claim. We do a reverse induction on the index $i$. Clearly $F_nF_n \in \mathcal{G}$. To do one more step, note that $F_nF_{n-1} \in \mathcal{G}$. This is because $F_n \leq F_nF_{n-1}$ by our additional assumption and $F_nF_{n-1} \leq C_n$ follows from $F_{n-1}, F_n \leq C_n$ and axiom $(P1)$. Now assume that $F_nF_i \in \mathcal{G}$ for some $i$. We show that $F_nF_{i-1} \in \mathcal{G}$.

Using (3) we get $(F_nF_i)C = C_n$. Rewriting this as $F_n(F_iC) = C_n$ and using (2), we get $F_n(F_{i-1}F_iC) = C_n$ which we write as $(F_nF_{i-1})F_iC = C_n$. Note that $F_n \le F_nF_{i-1}$. Hence we apply (1) with $H = F_nF_{i-1}$ and $E = F_iC$ to conclude that $F_nF_{i-1} \in \mathcal{G}$. This completes the induction step and the claim is proved.

From the claim we get $F_nF_1 \in \mathcal{G}$. Using (3) we have $(F_nF_1)C = C_n$ which is same as $F_n(F_1C) = F_nC_1 = C_n$. This is exactly what we wanted to show. □

The discussion in this section shows that the LRBs in Section 2 satisfy the projection axioms and hence are examples of our theory. In the next section, we again consider these examples (including the non-associative ones) but from a more intuitive and geometric perspective.

## 6. The metric setup

In this section we show that if a chamber complex $\Delta$ satisfies the gate property then it satisfies our axioms. This is essentially an axiomatic restatement of a result of Scharlau; see the proposition in [27, Section 3]. (He states it in terms of shellings. Also he does not consider the compatibility axiom.) Furthermore in this special situation, the maps of Section 4 have a geometric meaning. To make this clear we first need some definitions.

6.1. **Some definitions.** Let $\Delta$ be a pure simplicial complex. The maximal simplices are called *chambers*. We say two chambers are *adjacent* if they have a common codimension 1 face. A *gallery* is a sequence of chambers such that consecutive chambers are adjacent. We say that $\Delta$ is *gallery connected* if for any two chambers $C$ and $D$, there is a gallery from $C$ to $D$. For any $C, D \in \mathcal{C}$, we then define the gallery distance $d(C, D)$ to be the minimal length of a gallery connecting $C$ and $D$. And any gallery which achieves this minimum is called a *geodesic gallery* from $C$ to $D$. Another relevant metric notion is the following.

*Gate Property.* For any face $F \in \mathcal{F}$ and chamber $C \in \mathcal{C}$, there exists a chamber $D \in \mathcal{C}_{\ge F}$ such that $d(C, D) \le d(C, E)$ for any $E \in \mathcal{C}_{\ge F}$. Furthermore $d(C, E) = d(C, D) + d(D, E)$.

The gate property implies that $\Delta$ is *strongly connected*; that is, $\mathcal{C}_{\ge F}$ is gallery connected for all $F \in \mathcal{F}$. In fact it implies that $\mathcal{C}_{\ge F}$ is a *convex* subset of $\mathcal{C}$; that is, if $D$ and $E$ are any two chambers in $\mathcal{C}_{\ge F}$ then any geodesic gallery from $D$ to $E$ lies entirely in $\mathcal{C}_{\ge F}$.

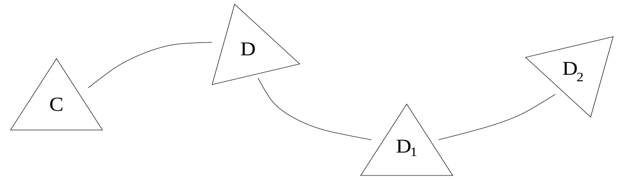



Figure 8. A geodesic gallery that illustrates shelling compatibility.

6.2. **Geometric descriptions of the relevant maps.** We now use the above metric notions to define the maps that played a role in the axiomatic setup of Section 4. Let $\Delta$ be a *chamber complex*. It is by definition a gallery connected pure simplicial complex. For every $C \in \mathcal{C}$, we define a partial order $\leq_C$. We say that $D \leq_C E$ if there is a geodesic gallery from $C$ to $E$ that passes through $D$. With this definition of the partial orders $\leq_C$, we analyze the shelling axioms. We first note that axiom $(S3)$ is a consequence of our definition. If $D \leq_C D_1 \leq_C D_2$ then we have a geodesic gallery $C - \ldots - D - \ldots - D_1 - \ldots - D_2$ as shown in Figure 8. This restricts to a geodesic gallery $D - \ldots - D_1 - \ldots - D_2$, which implies $D_1 \leq_D D_2$. Now we look at axiom $(S1)$. It says that for any $C \in \mathcal{C}$ and $E \in \mathcal{C}_{\geq F}$ there is a unique chamber $D \geq F$ such that $D \leq_C E$; that is, such that there is a geodesic gallery $C - \ldots - D - \ldots - E$. This is equivalent to saying that $\Delta$ has the gate property. Next we claim that a chamber complex $\Delta$ with the partial orders $\leq_C$ as defined above satisfies the shelling axioms $(S)$ if and only if it has the gate property. From what we have so far, we only need to show that the gate property implies axiom $(S2)$. This can be checked directly. However we will prove this using the projection axioms.

*Remark.* If $\Delta = \Sigma$ is the Coxeter complex associated to a Coxeter group $W$ then $\leq_C$ coincides with the weak Bruhat order on $W$, after we choose $C$ as the fundamental chamber and identify chambers of $\Sigma$ with elements of $W$.

Now we assume that $\Delta$ has the gate property. We show that this implies the projection axioms. Using the gate property, we define $FC$ to be the chamber containing $F$ that is closest to $C$. Axiom $(P1)$ follows. The gate property further says that a weak $C$ gallery $C_1 - C_2 - \ldots - C_n$ can be extended to a geodesic gallery $C - \ldots - C_1 - \ldots - C_2 - \ldots - C_n$. Axiom $(P3)$ now follows. To show axiom $(P2)$, we proceed by contradiction. Let $FC = E \neq D$. Choose a geodesic gallery $C - \ldots - E - \ldots - D$. Let $G$ be the facet of $D$ that is crossed by this gallery in the final step. Then $GC \neq D$, which is a contradiction. To summarize, $\Delta$ satisfies the projection axioms $(P)$ (with the closest chamber definition) if and only if it satisfies the gate property.

To complete the story we now describe the restriction maps. Recall from Section 4.2 that $R_C(D)$ is the face of the chamber $D$ spanned by vertices $v$ such that $(D \setminus v)C \neq D$. It can be equivalently described as the face of the chamber $D$ spanned by vertices $v$ which have the following property.

There is a minimal gallery from $C$ to $D$ that passes through the facet $D \setminus v$ in the final step.

We refer to the second description as $\mathrm{Des}(C, D)$. The equivalence of the two descriptions says that $R_C(D) = \mathrm{Des}(C, D)$. The terminology $\mathrm{Des}(C, D)$ is again motivated by the theory of Coxeter groups. $\mathrm{Des}(C, D)$ is the face of $D$ spanned by "the descent set of $D$ with respect to $C$". For an explanation, see the appendix by Tits to Solomon's paper [30] or the more elaborate exposition in [14, Section 9].

*Remark.* Another way to describe $\mathrm{Des}(C, D)$ is as the smallest face $F$ of $D$ such that $FC = D$. Note that this definition makes sense for any LRB that satisfies axiom $(P2')$. And we can define the partial order $\leq_C$ using the notion of a weak $C$ gallery.

We have proved the following theorem.

**Theorem 2.** *Let $\Delta$ be a chamber complex. Also let the partial orders $\leq_C$, the restriction maps $R_C$ and the projection maps be as defined in this section. Then $\Delta$ satisfies our axioms if and only if it satisfies the gate property.*

6.3. **Examples of Section 2 revisited.** Hyperplane arrangements, buildings and modular lattices are all examples of chamber complexes that satisfy the gate property. Hence by Theorem 2 we have established that they indeed satisfy our axioms.

For oriented matroids (in particular, central hyperplane arrangements), the gate property is due to Björner and Ziegler and for buildings it is due to Tits [33]. The gate property for modular lattices (in particular, distributive lattices) was proved by Abels, see [1, Proposition 2.9]. Alternatively, for distributive lattices, we can deduce the gate property by combining the following facts. If $\Delta$ satisfies the gate property then so does any subcomplex $\Delta_c$ spanned by a convex subset of chambers. The flag (order) complex $\Delta(M)$ of a distributive lattice $M$ corresponds to a convex set of chambers in $\Delta(\mathcal{B}_n)$, the complex of the braid arrangement.

*More on Example 1.* The metric notions that we described in this section can be made very explicit for this example. We prefer to restrict to the simplicial case though all statements except the last make sense and hold in general. A minimal gallery from $C$ to $D$ crosses exactly those hyperplanes that separate $C$ and $D$. Consequently the gallery distance $d(C, D)$ is given by the number of hyperplanes separating $C$ and $D$. We have $D \leq_C E$ if and only if the hyperplanes that separate $C$ and $D$ also separate $C$ and $E$. The chamber $FC$ is the unique chamber containing $F$ for which no hyperplane passing through $F$ separates $C$ from $FC$. Also $R_C(D)$ is the face of $D$ whose support is the intersection of those walls of $D$ that do not separate $D$ from $C$. Rigorous proofs of these unjustified facts can be found in any of the references cited earlier or you may accept them as "intuitively obvious".

## 7. Type selected subcomplexes

In this section, we introduce the notion of a local flag $h$ vector. This notion will also be used to motivate the discussion in the next section. For general background on the flag $h$ vector, see [34, Section 8.3]. We will present some well-known results as well as some new ones. However our approach will be different and will use projection maps. The treatment will be fairly self-contained.

A *labeling* of a pure simplicial complex $\Delta$ by a set $I$ is a function which assigns to each vertex an element of $I$, in such a way that the vertices of every chamber are mapped bijectively onto $I$. A labeled simplicial complex is also sometimes called "numbered" or "balanced".

Let $\Delta$ be a labeled simplicial complex and let $\Delta_J$ be its *type selected* subcomplex consisting of faces whose label set or *type* is contained in $J$. In Section 7.1, we study the connection between the homotopy type of $\Delta_J$ and the flag $h$ vector. In Section 7.2, we define the local flag $h$ vector. This is the dual picture for the flag $h$ vector. The fact that this viewpoint is useful will be seen from the example that we present later in Section 7.4. We will also consider the unlabeled case briefly in Section 7.3.

We start with the following result of Björner that generalizes Proposition 1.

**Proposition 3.** [9, Theorem 1.6] *For a labeled shellable complex $\Delta$, the type selected subcomplex $\Delta_J$ is shellable and homotopy equivalent to a wedge of $(|J|-1)$-spheres.*

*The number of these spheres is given by $\beta_J = |\{D \mid R(D)$ has type $J\}|$, where $R$ is the restriction map associated with the shelling.*

*Remark.* The proposition is proved by showing that the shelling order for $\Delta$ "induces" a shelling order for $\Delta_J$. This raises the following question. Suppose we assume that $\Delta$ satisfies our axioms with $R_C$ as the restriction maps. Also let $R_{J,C}$ be the maps for $\Delta_J$ induced by $R_C$. Then what kind of restriction axioms do the induced maps $R_{J,C}$ satisfy for $\Delta_J$? The axioms need to be suitably generalized because there is more than one restriction map for every chamber in $\Delta_J$.

7.1. **The restriction map and the flag $h$-vector.** We fix a chamber $C \in \mathcal{C}$ and assume that $\Delta$ satisfies the first two axioms (in each category) for $C$. Recall that $R_C(D)$ is the unique smallest face $F$ of $D$ such that $FC = D$. Proposition 3 applies to our situation because of axiom $(S2)$ which implies that $\Delta$ is shellable. The restriction map $R$ in the proposition is the map $R_C$, where $C$ is the chamber that we have fixed. Next define

$$\beta_J = |\{D \mid R_C(D) \text{ has type } J\}|.$$

We will show that the vector $(\beta(J))_{J\subseteq I}$ coincides with the "flag $h$-vector" defined below. This result goes back to Björner or even earlier.

First we define the *flag $f$-vector* of $\Delta$ by setting $f_J(\Delta)$ equal to the number of simplices of type $J$. The *flag $h$-vector* is then obtained by writing

$$f_J(\Delta) = \sum_{K\subseteq J} h_K(\Delta), \tag{7}$$

or, equivalently,

$$h_J(\Delta) = \sum_{K\subseteq J} (-1)^{|J-K|} f_K(\Delta). \tag{8}$$

**Proposition 4.** *Let $\Delta$ be as above and let $I$ be the set of types of vertices. Then for any $J \subseteq I$,*

$$\beta_J = h_J(\Delta).$$

*Proof.* Let $\mathcal{F}_J$ be the set of simplices of type $J$. There is a 1–1 map $\mathcal{F}_J \to \mathcal{C}$, given by $F \mapsto FC$, where $C$ is fixed. It is 1–1 because we can recover $F$ from $FC$ as the face of type $J$. Its image is the set of chambers $D$ for which the type of $R_C(D)$ is contained in $J$. Hence

$$f_J(\Delta) = \sum_{K\subseteq J} \beta_K.$$

The proposition now follows from (7). □

*Remark.* The above material is taken from [14]. The difference is that Ken Brown stated this proposition for Coxeter complexes. Also he worked with descent sets rather than restriction maps. We know from the discussion in Section 6 that in the metric setup, $R_C(D) = \mathrm{Des}(C, D)$.

7.2. **The dual picture.** Now we assume that $\Delta$ satisfies all three axioms. In other words, we are now allowed to vary $C \in \mathcal{C}$ and have the compatibility axiom. Note that $\beta_J$ does not depend on the choice of the chamber $C$. For example, from Proposition 3, we know that $\beta_J$ counts the number of spheres in $\Delta$ and hence is independent of the choice of $C$. Also from Proposition 4, we know that it coincides with the flag $h$ vector of $\Delta$. We also see that $h_J(\Delta) = |\{D \mid R_C(D) \text{ has type } J\}|$. This gives geometric interpretations of the flag $h$ vector, one for every $C \in \mathcal{C}$.

Now we make full use of the compatibility axiom by turning the picture around. For the map $\mathcal{F} \times \mathcal{C} \to \mathcal{C}$ that maps $(F, C)$ to $FC = D$, we fix $D$ and vary $C$. More precisely, we define the local flag $h$ vector $h_J(D) = |\{C \mid R_C(D) \text{ has type } J\}|$. Similarly define the local flag $f$ vector $f_J(D)$ to be the number of chambers $C$ such that $FC = D$, where $F$ is the face of $D$ of type $J$. This can be rewritten as $f_J(D) = |\{C \mid \text{The type of } R_C(D) \text{ is contained in } J\}|$. Now observe that

$$f_J(D) = \sum_{K \subseteq J} h_K(D). \tag{9}$$

The numbers $f_J(D)$ and $h_J(D)$ do in general depend on $D$. If we average them over all $D \in \mathcal{C}$ then we recover the usual flag vectors. To see this, let $\sigma_J$ be the sum of all faces of type $J$. In particular, $\sigma_I$ is the sum of all chambers. Note that $\sigma_J \sigma_I = \sum_{D \in \mathcal{C}} f_J(D) D$. By counting the total number of terms involved, we get $f_J(\Delta) = \frac{1}{|\mathcal{C}|} \sum_{D \in \mathcal{C}} f_J(D)$. This along with (7) and (9) says that $h_J(\Delta) = \frac{1}{|\mathcal{C}|} \sum_{D \in \mathcal{C}} h_J(D)$. Note that it is not obvious from the definition that the right hand side is an integer.

The numbers $h_J(D)$ give us a different way to understand the flag $h$ vector. A natural question that arises is: When is $h_J(\Delta) = h_J(D)$ for all $D \in \mathcal{C}$? We will address this in more detail in Section 8.

7.3. **The unlabeled case.** If $\Delta$ is not labeled then we work with the ordinary $f$ and $h$ vectors rather than the flag vectors. Let $\Delta$ be a simplicial complex of rank $n$ that satisfies our axioms. As before, we will need the compatibility axiom only when we pass to the dual picture. We define $\beta_j = |\{D \mid R_C(D) \text{ has rank } j\}|$. The unlabeled version of Proposition 3 is as follows.

**Lemma 4.** *For a shellable complex $\Delta$ of rank $n$, the $(k-1)$-skeleton $\Delta_{k-1}$ is shellable and homotopy equivalent to a wedge of $(k-1)$-spheres. The number of these spheres is given by $\sum_{i=0}^{k} \binom{n-i-1}{k-i} \beta_i$.*

The fact that shellability is inherited by $k$-skeleta appears as Corollary 10.12 in [11]. Also see the references cited therein. The formula for the number of spheres was told to me by Björner. As before, we can show that the vector $(\beta_j)_{0 \le j \le n}$ coincides with the "$h$-vector" defined below.

First we define the *$f$-vector* of $\Delta$ by setting $f_j(\Delta)$ equal to the number of simplices of rank $j$. The $h$-vector is then obtained by writing

$$f_j(\Delta) = \sum_{k=0}^{j} \binom{n-k}{n-j} h_k(\Delta). \tag{10}$$

**Proposition 5.** *Let $\Delta$ be as above. Then for any $0 \le j \le n$,*

$$\beta_j = h_j(\Delta).$$

*Proof.* Let $\mathcal{F}_j$ be the set of simplices of rank $j$. As before, we consider the map $\mathcal{F}_j \to \mathcal{C}$, given by $F \mapsto FC$, where $C$ is fixed. Recall that for a fixed $C$ and $D$, $\{F \in \mathcal{F} \mid R_C(D) \le F \le D\} = \{F \in \mathcal{F} \mid FC = D\}$. This says that the image of the above map is the set of chambers $D$ for which the rank of $R_C(D)$ is less than or equal to $j$. However this map is no longer 1–1. The number of times a fixed chamber $D$ occurs in the image is $\binom{n-k}{n-j}$. This is the number of faces of $D$ of rank $j$ that contain $R_C(D)$ (whose rank we set equal to $k$). This gives

$$f_j(\Delta) = \sum_{k=0}^{j} \binom{n-k}{n-j} \beta_k$$

and the proposition now follows. □

To get the dual picture, let $h_j(D) = |\{C \mid R_C(D) \text{ has rank } j\}|$. Similarly let $f_j(D) = |\{(F, C) \mid FC = D, \ \mathrm{rk}(F) = j\}|$. As in the proof of Proposition 5, $f_j(D) = \sum_{k=0}^{j} \binom{n-k}{n-j} h_k(D)$. Now if we let $\sigma_j$ be the sum of all faces of rank $j$ then $\sigma_j \sigma_n = \sum_{D \in \mathcal{C}} f_j(D) D$. This gives $f_j(\Delta) = \frac{1}{|\mathcal{C}|} \sum_{D \in \mathcal{C}} f_j(D)$ and then $h_j(\Delta) = \frac{1}{|\mathcal{C}|} \sum_{D \in \mathcal{C}} h_j(D)$.

*Remark.* For a labeled complex, we have $f_j = \sum_{|J|=j} f_J, h_j = \sum_{|J|=j} h_J$ and $\sigma_j = \sum_{|J|=j} \sigma_J$. So in this case, the results for the $f$ and $h$ vector follow from the corresponding result for the flag vectors. For instance, using the first two equations, one can recheck that equation (7) reduces to equation (10).

7.4. **Simplicial hyperplane arrangements.** Let $\Delta$ be a simplicial hyperplane arrangement. We now give a description of the local flag $h$ vector $h_J(D)$ in this case. This dual description given in Section 7.2 is easier to visualize than the usual one in Section 7.1. Recall that in this situation, $R_C(D)$ is the face of $D$ whose support is the intersection of those walls of $D$ that do not separate $D$ from $C$; see the description at the end of Section 6. Using this description, in the labeled case, we have the following lemma.

**Lemma 5.** *The local flag $h$ vector $h_J(D)$ is the cardinality of the set of chambers $C$ such that $D$ and $C$ lie on the same side of a wall of $D$ if and only if the wall contains the face of type $J$ of $D$.*

*The local flag $f$ vector $f_J(D)$ is the cardinality of the set of chambers $C$ such that $D$ and $C$ lie on the same side of all walls of $D$ that contain the face of type $J$ of $D$.*


Figure 9. The schematic picture in rank 3.

In Figure 9 only the three walls of the chamber $D$ have been drawn. The only chamber that is not seen in the picture is the one directly opposite to $D$. The local flag $f$ vector for $D$ is represented by various lunes formed at $D$ by its faces. In the figure on the left, the shaded part shows the lune for the face of $D$ of type $\{s, t\}$ and corresponds to $f_{s,t}(D)$. The local flag $h$ vector for $D$ is represented by the 8 regions into which the three walls divide the sphere. We have marked the region that corresponds to $h_s(D)$ and shaded the one for $h_{s,t}(D)$ in the figure on the right.

In the unlabeled case, $h_j(D)$ is the cardinality of the set of chambers $C$ such that $D$ and $C$ lie on opposite sides of exactly $j$ of the $n$ walls of $D$. In Figure 9, for example, $h_1(D)$ counts the chambers that lie in the three regions adjacent to $D$ along its facets.

*Remark.* If $\Delta$ is a finite Coxeter complex or a finite (spherical) building associated with a BN-pair then the local flag vectors coincide with the usual or global flag vector. Also see the remark after Proposition 6.

Next we give a tiny application of this description. Let $\Delta_{min}$ be the Coxeter complex of the reflection arrangement $x_i = 0$, where $1 \leq i \leq n$. In fact Figure 9 as drawn corresponds to the case $n = 3$. Observe that $h_j(D) = \binom{n}{j}$. From our interpretation, it is clear that for any other simplicial arrangement $\Delta$ of rank $n$, we get $h_j(D) \geq \binom{n}{j}$ for all $D \in \mathcal{C}$ and hence $h_j(\Delta) \geq \binom{n}{j}$.

Now consider the barycentric subdivision $\Delta'$ of $\Delta$. Then $\Delta'_{min}$ corresponds to the Coxeter arrangement of type $B_n$, namely, $x_i = \pm x_j$ and $x_i = 0$, where $1 \leq i < j \leq n$. However in general, $\Delta'$ is not a hyperplane arrangement. It is just a labeled simplicial complex. Hence we cannot directly conclude that $h_J(\Delta') \geq h_J(\Delta'_{min})$ (which is known to be true). This raises an important question. If $\Delta$ satisfies our axioms then what can we say about its barycentric subdivision $\Delta'$ ? For a positive result in this direction, see [8, Theorem 5.1] which says that if $\Delta$ is shellable then so is $\Delta'$.

## 8. Commutativity issues

In this section, we address the question raised in Section 7: When is $h_J(D) = h_J(\Delta)$ for all $D \in \mathcal{C}$? This can be rephrased as a commutativity problem. Recall from the previous section that $\sigma_J$ (resp. $\sigma_j$) is the sum of all faces of type $J$ (resp. rank $j$). Also $h_J(D)$ and $h_J(\Delta)$ are the local and global versions of the flag $h$ vector of $\Delta$. And $h_j(D)$ and $h_j(\Delta)$ are the corresponding versions of the $h$ vector of $\Delta$.

**Proposition 6.** *Let $\Delta$ be a simplicial LRB (possibly non-associative) that satisfies our axioms. Then*
*(labeled case) $h_J(\Delta) = h_J(D)$ for all $D \in \mathcal{C}$ and for all subsets $J \subseteq I \Leftrightarrow \sigma_I\sigma_J = \sigma_J\sigma_I$ for all subsets $J \subseteq I$.*
*(unlabeled case) $h_j(\Delta) = h_j(D)$ for all $D \in \mathcal{C}$ and for all $0 \leq j \leq n \Leftrightarrow \sigma_n\sigma_j = \sigma_j\sigma_n$ for all $0 \leq j \leq n$.*

*Proof.* We do only the labeled case. The unlabeled case is similar. Suppose that $\sigma_I\sigma_J = \sigma_J\sigma_I$ for all subsets $J \subseteq I$. Counting the number of times a fixed chamber $D$ occurs on both sides of the equality, we get $f_J(\Delta) = f_J(D)$, or equivalently, $h_J(\Delta) = h_J(D)$. □

*Remark.* In the proposition above, a concrete case to keep in mind is that of simplicial hyperplane arrangements. Also we required $\Delta$ to satisfy our axioms because we only defined $h_J(D)$ in that situation. And we required $\Delta$ to be a LRB so as to make sense of the product $\sigma_I\sigma_J$. Of course, we only needed to use that $CF = C$ for $C \in \mathcal{C}$ and $F \in \mathcal{F}$.

If $\Delta$ is a finite Coxeter complex or a finite (spherical) building associated with a BN-pair then there is a type and product preserving group action on $\Delta$ and it is transitive on $\mathcal{C}$. Hence the commutativity condition in Proposition 6 is automatic. And so for this case we obtain that $h_J(D) = h_J(\Delta)$ for all $D \in \mathcal{C}$. This raises the following question: Let $\Delta$ be a thin labeled chamber complex of rank greater than 3 such that $\sigma_I \sigma_J = \sigma_J \sigma_I$ for all subsets $J \subseteq I$. Then is it true that $\Delta$ is a Coxeter complex ?

8.1. **Connection with random walks.** Bidigare, Hanlon and Rockmore [5] found a natural family of random walks associated with hyperplane arrangements. These walks were studied further by Brown and Diaconis [15] and generalized to LRBs by Brown [14]. One reason this development is exciting is that the walks admit a rather complete theory. We now describe the walk.

Using the projection operators, define the following walk on the chambers $\mathcal{C}$ of $\Delta$: If the walk is in chamber $C$, choose a face $F$ of rank $j$ at random and move to the projection $D = FC$. (We assume here that the faces of rank $j$ are chosen uniformly.) Note that this random walk on chambers has a uniform stationary distribution if and only if $\sigma_n \sigma_j = \sigma_j \sigma_n$. Here $n$ is the rank of $\Delta$.

More generally, we can run a random walk on $\mathcal{C}$ using a probability distribution $\{w_F\}_{F \in \mathcal{F}}$ on $\mathcal{F}$, the faces of $\Delta$. Also if the complex is labeled then we can define the walk by choosing instead a face $F$ of type $J$ at random. Then the walk on chambers has a uniform stationary distribution if and only if $\sigma_I \sigma_J = \sigma_J \sigma_I$. However for the rest of the section, we will restrict to the unlabeled case.

8.2. **Some commutativity conditions.** Motivated by the above discussion, we look at a more general commutativity problem. We work with a graded LRB. All known examples of LRBs are graded. However we do not know whether this follows from the definition. For now, we do not make any further assumptions.

Let $S$ be a *LRB*, $L$ its associated lattice and $\operatorname{supp}: S \twoheadrightarrow L$ the support map. For any $X \in L$, let $S_{\leq X} = \{y \in S : \operatorname{supp} y \leq X\}$. Then one can check that $S_{\leq X}$ is also a LRB whose associated lattice is the interval $[\hat{0}, X]$ in $L$. Here $\hat{0}$ is the support of the identity of $S$ and hence the minimum element of $L$. Also the set of chambers $\mathcal{C}$ consists precisely of those elements of $S$ whose support is $\hat{1}$, the maximum element of $L$.

We say that a LRB satisfies the commutativity condition for $i$ and $j$ if

$$(C_{i,j}) \qquad \sigma_i \sigma_j = \sigma_j \sigma_i.$$

Similarly we say that it satisfies the uniformity condition $(U)$ if for any $i, j$ the number of times a chamber occurs in the product $\sigma_i \sigma_j$ does not depend on the chamber we choose. Loosely speaking, these conditions may be thought of as certain symmetry conditions on the LRB. We have already given some motivation for the condition $(C_{i,j})$. The motivation for condition $(U)$ becomes clear from the following proposition.

**Proposition 7.** *Let $S$ be a LRB such that $S_{\leq X}$ satisfies the uniformity condition $(U)$ for all $X \in L$. Then $S$ (and hence all $S_{\leq X}$) satisfies the commutativity condition $(C_{i,j})$ for all $i, j$.*

*Proof.* We first observe that $\operatorname{supp}(FG) = \operatorname{supp}(GF) = \operatorname{supp} F \vee \operatorname{supp} G$. In particular, for $H = FG$, we have $\operatorname{supp}(F), \operatorname{supp}(G) \leq \operatorname{supp}(H)$. Hence the coefficient of $H$ in $\sigma_i \sigma_j$ remains unchanged if we replace $S$ by the subsemigroup $S_{\leq \operatorname{supp}(H)}$. This shows that if $S$ satisfies $(C_{i,j})$ then so does $S_{\leq X}$ for all $X \in L$.

Now we show that $S$ (and hence all $S_{\leq X}$) satisfies $(C_{i,j})$ for all $i, j$. We first check that for any $D \in \mathcal{C}$, the coefficient of $D$ in $\sigma_i \sigma_j$ is same as the coefficient of $D$ in $\sigma_j \sigma_i$. Since we assume that $S$ satisfies $(U)$, we simply need to check that

$|\{(F, G) \mid \mathrm{rk}(F) = i, \mathrm{rk}(G) = j, FG \text{ is a chamber}\}| = |\{(G, F) \mid \mathrm{rk}(F) = i, \mathrm{rk}(G) = j, GF \text{ is a chamber}\}|$. This is true since $FG$ is a chamber $\Leftrightarrow$ $GF$ is a chamber. This is because of our earlier observation that $\mathrm{supp}(FG) = \mathrm{supp}(GF)$.

To do the general case, we need to check that for any $H \in \mathcal{F}$, the coefficient of $H$ in $\sigma_i\sigma_j$ is same as the coefficient of $H$ in $\sigma_j\sigma_i$. To do this computation, we replace $S$ by $S_{\leq \mathrm{supp}(H)}$. Note that $H$ is a chamber in $S_{\leq \mathrm{supp}(H)}$ and by our assumption $S_{\leq \mathrm{supp}(H)}$ satisfies $(U)$. Hence we now apply the previous argument and the proposition is proved. □

**Corollary.** *If $S$ is the Coxeter complex of type $A_{n-1}$ or $B_n$, then $S$ satisfies $(C_{i,j})$ for all $i, j$.*

*Proof.* The uniformity condition $(U)$ is automatic for a Coxeter complex $\Sigma$ because the Coxeter group acts transitively on the chambers of $\Sigma$. However to apply the above proposition, we require $(U)$ to hold for all $S_{\leq X}$. For types $A_{n-1}$ or $B_n$, given any $X \in L$, $S_{\leq X}$ is again isomorphic to a Coxeter complex of the same type but now with a smaller value of $n$. Hence the Coxeter complex of type $A_{n-1}$ or $B_n$ satisfies $(U)$ for all $S_{\leq X}$ and we can apply the proposition. □

*Remark.* For types $A_{n-1}$ and $B_n$, the semigroups and the underlying lattices can be made combinatorially explicit [14]. The fact about $S_{\leq X}$ being a Coxeter complex for types $A_{n-1}$ or $B_n$ that we used in the proof above is then quite elementary. For the case when $X$ is a hyperplane (wall), this problem has been treated in general for any Coxeter complex by Abramenko [2].

A completely different proof of this corollary can be given using the language of card shuffles, see [3, 20]. We also note that for $S$, the Coxeter complex of type $D_n$, the subcomplexes $S_{\leq X}$ are not even Coxeter complexes. This is the primary reason why the complex of type $D_n$ fails the commutativity condition in general. We have checked by explicit computation that there exists an $i, j$ such that the Coxeter complex of type $D_4$ does not satisfy $(C_{i,j})$.

8.3. **A low rank computation.** In contrast to the previous subsection, we get a less restrictive result if we work with low ranks.

**Theorem 3.** *Let $S$ be an oriented matroid of rank 3. Then $S$ satisfies $C_{1,2}$ $\Leftrightarrow$ $S$ satisfies $C_{1,3}$ $\Leftrightarrow$ $S$ satisfies $C_{2,3}$ $\Leftrightarrow$ $S$ is simplicial.*

*Proof.* A part of this theorem, namely, $S$ satisfies $C_{1,3}$ $\Leftrightarrow$ $S$ is simplicial occurs as Proposition 1 in [6]. It says that the random walk on $\mathcal{C}$ driven by a uniform probability distribution on the vertices of $S$ has a uniform stationary distribution iff $S$ is simplicial (see the discussion in Section 8.1). Here we will prove that $S$ satisfies $C_{1,2}$ $\Leftrightarrow$ $S$ is simplicial. The other cases are simpler and we leave them out. The spirit of the computation is as in [6]. Let $v$,$e$ and $f$ be the number of vertices, edges and faces respectively in $S$.

We need to look at $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Note that a vertex can never occur in either product. And the number of times an edge appears on either side is the same. (This is because we can restrict to the support of that edge, which is just a pseudoline.) So the non-trivial case is that of chambers. We fix a chamber $D$. Denote the chamber exactly opposite to $D$ by $\overline{D}$. We count the number of times that either $D$ or $\overline{D}$ occurs in $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$. (It will be evident from the computation as to why we group $D$ and $\overline{D}$ together. Note that due to the antipodal symmetry of the arrangement, we do not lose anything essential by doing this.)

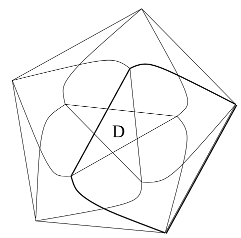



FIGURE 10. The walls of a 5-gon.

If $D$ is a $k$-gon then consider the $k$ walls of $D$ (and hence $\overline{D}$). We call an edge of $S$ an *interior* edge if it does not lie on any of these $k$ walls. Figure 10 illustrates the case when $k = 5$. One of the walls has been highlighted. The only chamber that is not seen in the picture is the pentagon $\overline{D}$. It is on the backside bounded by the five segments that bound the figure.

We first look at $\sigma_1\sigma_2$ and count the number of times that either $D$ or $\overline{D}$ occurs in this product; that is, $|\{(F, G) \mid FG = D \text{ or } FG = \overline{D}, \mathrm{rk}(F) = 1, \mathrm{rk}(G) = 2\}|$. We use the geometry of the arrangement as illustrated in the figure. Note that $F$ must be a vertex of $D$ or $\overline{D}$. We consider two cases. In the first case, let $G$ be an interior edge or an edge of either $D$ or $\overline{D}$. Then there are $k-2$ choices for $F$. The individual contributions of $D$ or $\overline{D}$ depend on the location of the edge $G$, however the net contribution is always $k-2$. In the second case, $G$ lies on one of the $k$ walls but is not a face of either $D$ or $\overline{D}$. Now there are $k-3$ choices for $F$. Hence the required count is $(k-2)e -$ (number of edges on all the walls) $+ 2k$.

Now we look at $\sigma_2\sigma_1$. We want to count $|\{(G, F) \mid GF = D \text{ or } GF = \overline{D}, \mathrm{rk}(F) = 1, \mathrm{rk}(G) = 2\}|$. The only $G$'s that we need consider are the edges of $D$ and $\overline{D}$. Given a wall $H$ of $D$ (and $\overline{D}$), the edge of $D$ that lies on $H$ and its opposite edge (which is an edge of $\overline{D}$) together can pair with all vertices except the ones that lie on $H$. Summing up over all the $k$ walls, we get the count to be $kv -$ (the sum of the no. of vertices on each of the $k$ walls).

Now we compare the counts for $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$. Since the no. of vertices is same as the no. of edges on each wall (since it is a pseudoline), we get $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow (k-2)e + 2k = kv \Leftrightarrow kf = 2e$. The last equality holds if and only if the arrangement is made entirely of $k$-gons. However, by Levi's theorem [18, p. 25]), any arrangement must have at least one triangle. Hence we get $k = 3$. □

## 9. THE OPPOSITE OR DUALITY AXIOM

Let $\Delta$ be a simplicial complex of rank $n$. Under suitable hypotheses on $\Delta$ (for example, if $\Delta$ triangulates a sphere), the $h$ vector of $\Delta$ satisfies the linear relations $h_i = h_{n-i}$ for $0 \leq i \leq n$. Similarly if $\Delta$ is a labeled simplicial complex with label set $I$ then under suitable hypotheses the flag $h$ vector of $\Delta$ satisfies the linear relations $h_J = h_{I\setminus J}$, for $J \subseteq I$. The former are called the *Dehn-Sommerville* equations and the latter the *generalized Dehn-Sommerville* equations. From now on, we will just write them as DS for short. For more information see [32, Section 3.14] and the references therein.

The DS equations are a form of Alexander duality. Our goal for this section is to formulate an axiom relevant to this notion of duality. This will also give a local version of the DS equations. As motivation we begin with a result that introduces the concept of shelling reversal.

**Theorem 4.** *Let $\Delta$ be a shellable complex such that every facet lies in at least 2 chambers. Then $\Delta$ is thin $\Rightarrow |\{D \mid R(D) = D\}| = 1 \Leftrightarrow \Delta$ is homotopy equivalent to a sphere.*

*If $\Delta$ is thin then it satisfies the DS equations.*

*Proof.* The first part is proved in [9, Theorem 1.5]. We prove the second part. Also we assume that $\Delta$ is labeled. If it is not then the same proof gives us the DS equations instead of the generalized DS equations. The proof is based on the idea of shelling reversal.

Let $\leq_S$ be a shelling order of $\Delta$, and let $\leq_{\text{Ƨ}}$ be the reverse order defined by $E \leq_{\text{Ƨ}} D \Leftrightarrow D \leq_S E$. Assume for the moment that $\leq_{\text{Ƨ}}$ also gives a shelling of $\Delta$. Now if we let $R$ (resp. Я) be the restriction map of $\leq_S$ (resp. $\leq_{\text{Ƨ}}$) then it follows that

$$\text{type of } R(D) = I \setminus \text{ type of Я}(D).$$

This along with the fact that $h_J = |\{D \mid R(D) \text{ has type } J\}|$ gives us the generalized DS equations and proves the second part.

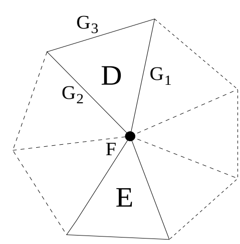



FIGURE 11. An example with $\mathcal{F}_D = \{G_1, G_2\}$ and $\mathcal{F}'_D = \{G_3\}$.

Hence we only need to show that $\leq_{\text{Ƨ}}$ gives a shelling of $\Delta$. We assumed that $\leq_S$ is a shelling of $\Delta$. This says that $D \cap (\cup_{E <_S D} E) = \cup_{G \in \mathcal{F}_D} G$, where $\mathcal{F}_D$ is a subset of the set of facets of $D$. Now let $\mathcal{F}'_D$ be the set of those facets of $D$ that are not in $\mathcal{F}_D$. We will be done once we show the following.

Claim: $D \cap (\cup_{D <_{\text{Ƨ}} E} E) = \cup_{G \in \mathcal{F}'_D} G$.

Proof of claim. ($\supseteq$) This is clear.

($\subseteq$) Let $F$ be a face of $D$ such that $F \nsubseteq RHS$. We show that $F \nsubseteq LHS$. Let $E \geq F$ such that $E \neq D$. We need to show that $E <_S D$. To put in words, we need to show that among all the chambers in $\Delta_{\geq F}$, $D$ is the last chamber to be shelled by the order $\leq_S$. By Lemma 1, we know that $\Delta_{\geq F}$, or equivalently $\mathrm{lk}(F, \Delta)$, is shellable with the induced order from $\leq_S$. The chambers of $\mathrm{lk}(F, \Delta)$ are in 1–1 correspondence with the chambers of $\Delta$ that contain $F$. Since we assume that $\Delta$ is thin, it follows that $\mathrm{lk}(F, \Delta)$ is also thin. Let $R_F$ be the restriction map of the induced shelling on $\mathrm{lk}(F, \Delta)$. Let $D_F$ be the chamber of $\mathrm{lk}(F, \Delta)$ that corresponds to $D$. We need to show that $D_F$ is the last chamber to be shelled in $\mathrm{lk}(F, \Delta)$. Since $\mathrm{lk}(F, \Delta)$ is thin, applying the first part of the theorem, we need to show that $R_F(D_F) = D_F$; also see comment after Proposition 1. However this follows by our assumption that $F \nsubseteq RHS$, which says that every facet of $D$ that contains $F$ belongs to $\mathcal{F}_D$. This proves the claim. ∎

*Remark.* For a sketch of the same argument in the context of polytopes, see [34, pg 252]. Also for the first part of the theorem, we do not know whether thinness is equivalent to the other two conditions.

**The opposite axiom.** Let $\Delta$ be a thin shellable complex. Let $C$ and $D$ be the first and last elements of a shelling $\leq_S$ of $\Delta$. Then $C$ and $D$ are the unique elements satisfying $R(C) = \emptyset$ and $R(D) = D$, where $R$ is the restriction map of $\leq_S$. As we saw in the proof of Theorem 4, the shelling $\leq_S$ is reversible and for the reverse shelling $\leq_{\reflectbox{$S$}}$, the roles of $C$ and $D$ get interchanged. Hence we would like to think of $C$ and $D$ as chambers opposite to each other.

Motivated by this discussion, we add an *opposite* axiom to the axiomatic setup of Section 4. Of course, the main case of interest is when $\Delta$ is thin. We give three equivalent definitions which correspond to the projection, restriction and shelling cases respectively. Their equivalence will be proved in Theorem 5.

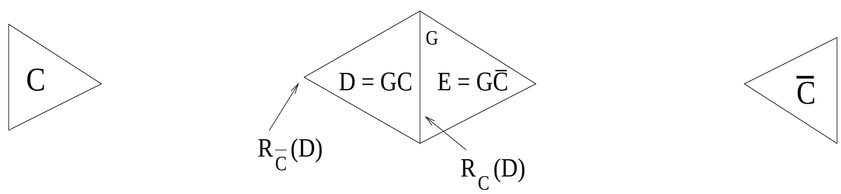



FIGURE 12. How the opposition map works.

Let $^- : \mathcal{C} \to \mathcal{C}$ be a map such that

(P4) For any facet $G$ of $\Delta$ and any $C \in \mathcal{C}$, we have $GC \neq G\overline{C}$.
(R4) For all $C, D \in \mathcal{C}$, we have type of $R_C(D) = I \setminus$ type of $R_{\overline{C}}(D)$.
(S4) For all $C \in \mathcal{C}$, the partial orders $\leq_C$ and $\leq_{\overline{C}}$ are dual to each other.

If $\Delta$ is not labeled then axiom $(R4)$ is to be interpreted as saying that $R_C(D)$ and $R_{\overline{C}}(D)$ are complementary faces of $D$. If $\Delta$ is thin then axiom $(P4)$ says that for a facet $G$, $GC$ and $G\overline{C}$ are the two distinct chambers containing $G$. Each of these axions implies that $^- : \mathcal{C} \to \mathcal{C}$ is an involution of $\Delta$. Also, using these axioms, $\overline{C}$, "the chamber opposite to $C$", can be characterized as follows in the three cases.

P. $\overline{C}$ is the unique chamber such that $FC \neq \overline{C}$ for any $F < \overline{C}$.
R. $\overline{C}$ is the unique chamber satisfying $R_C(\overline{C}) = \overline{C}$.
S. $\overline{C}$ is the unique maximal element in the partial order $\leq_C$.

We now give a local version of the DS equations for a complex that satisfies the opposite axiom.

**Lemma 6.** *If $\Delta$ satisfies the opposite axiom then $h_J(D) = h_{I \setminus J}(D)$.*

*Proof.* Recall that $h_J(D) = |\{C \mid R_C(D) \text{ has type } J\}|$. Now by axiom $(R4)$, the chamber $C$ contributes to $h_J(D)$ if and only if its opposite chamber $\overline{C}$ contributes to $h_{I \setminus J}(D)$. □

Averaging the above local equations over all $D \in \mathcal{C}$ leads to the usual generalized DS equations. (Of course, we already knew from Theorem 4 that $\Delta$ satisfied the generalized DS equations.) In the unlabeled case, the same reasoning gives a local version of the DS equations instead of the generalized DS equations.

**Theorem 5.** *Let $\Delta$ be a thin chamber complex that satisfies our (previous) axioms. Then $\Delta$ satisfies $(P4) \Leftrightarrow \Delta$ satisfies $(R4) \Leftrightarrow \Delta$ satisfies $(S4)$.*

*Proof.* The complex $\Delta$ is shellable since it satisfies our earlier axioms. And a shellable complex is automatically gallery connected. So really the only additional assumption on $\Delta$ is that of thinness. To go from one set of axioms to another, we again use the formal connections outlined in Section 4.2.

$(P4) \Rightarrow (R4)$**.** Let $D \in \mathcal{C}$. Apply axiom $(P4)$ to every facet of $D$. By the definition of $R_C$ and $R_{\overline{C}}$, we get that the type of $R_C(D) = I \setminus$ type of $R_{\overline{C}}(D)$.

$(R4) \Rightarrow (S4)$**.** For a thin shellable complex, Lemma 2 gives a simpler description of $\leq_C$. It is the transitive closure of the relation: $E \leq_C D$ if $E$ is adjacent to $D$ and $R_C(E) \leq D$.

Let $D$ and $E$ be adjacent chambers and $D \neq E$. Let $G$ be the common facet. By thinness of $\Delta$ and axiom $(R2)$, either $R_{\overline{C}}(D) \leq G$ or $R_{\overline{C}}(E) \leq G$. This implies that either $R_{\overline{C}}(D) \leq E$ or $R_{\overline{C}}(E) \leq D$.

In order to prove axiom $(S4)$, we need to show that $D \leq_C E \Leftrightarrow E \leq_{\overline{C}} D$. We show that $D \leq_C E \Rightarrow E \leq_{\overline{C}} D$. The other implication follows by symmetry. Suppose that $D$ and $E$ are adjacent and $R_C(D) \leq E$. Then by axiom $(R4)$, we have $R_{\overline{C}}(D) \not\leq E$. Hence $R_{\overline{C}}(E) \leq D$ which implies $E \leq_{\overline{C}} D$.

$(S4) \Rightarrow (P4)$**.** Let $G$ be any facet of $\Delta$. Since $\Delta$ is thin, there are exactly two chambers in $\Delta_{\geq G}$. Axiom $(S1)$ says that for any $C \in \mathcal{C}$, the partial order $\leq_C$ restricted to $\mathcal{C}_{\geq G}$ has a minimum. So the two chambers in $\Delta_{\geq G}$ get ordered by $\leq_C$. Now axiom $(S4)$ says that the order obtained using $\leq_{\overline{C}}$ is reverse of the one got using $\leq_C$ and hence axiom $(P4)$ follows by the definition of projection maps. □

A simplicial hyperplane arrangement clearly satisfies axiom $(P4)$. Axiom $(R4)$ can also be checked directly using the discussion in Section 7.4. We record this formally as a simple corollary.

**Corollary.** *A simplicial hyperplane arrangement satisfies all three opposite axioms $(P4)$, $(R4)$ and $(S4)$.*

## 10. Duality in Buildings

In the previous section, we axiomatized duality in a way that was relevant to thin chamber complexes. Now we prove a somewhat isolated result on duality in buildings and illustrate it with the building of type $A_{n-1}$. A suitable generalization of the opposite axiom to buildings may be possible. An appropriate setting may be gated chamber complexes of spherical type [21]. Also the duality result on buildings (Theorem 6) may generalize to Moufang complexes [22, Section 5].

We first set up some notation. Let $\Delta$ be a finite (spherical) building and $\mathcal{A}$ be the complete set of apartments of $\Delta$. For chambers $C, D$, we denote by $\mathcal{A}_C$ (resp. $\mathcal{A}_{C,D}$) the set of apartments in $\mathcal{A}$ that contain $C$ (resp. $C$ and $D$). Also, for a chamber $C$, let $C^{op}$ denote the set of all chambers that are opposite to $C$. Observe that there is a bijection between the sets $C^{op}$ and $\mathcal{A}_C$ as follows. We associate to $\overline{C} \in C^{op}$, the apartment $\Sigma \in \mathcal{A}_C$, which is the convex hull of $C$ and $\overline{C}$. Now let $\mathcal{A}'_{C,D} = \{E \in C^{op} \mid D \text{ lies in the apartment determined by } C \text{ and } E\}$. Observe that the bijection between $\mathcal{A}_C$ and $C^{op}$ restricts to a bijection between the sets $\mathcal{A}_{C,D}$ and $\mathcal{A}'_{C,D}$. Also we let $\rho_{\Sigma,C} : \Delta \to \Sigma$ denote the usual retraction.

**Lemma 7.** *Let $C$ and $\overline{C}$ be a pair of opposite chambers in $\Delta$ and let $D$ be any chamber in the apartment determined by $C$ and $\overline{C}$. Then $|\mathcal{A}_{C,D}||\mathcal{A}_{D,\overline{C}}| = |\mathcal{A}_D|$.*

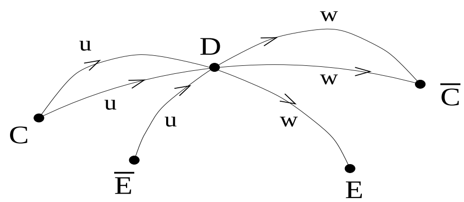



FIGURE 13. A schematic picture.

*Proof.* Let $\delta : \mathcal{C} \times \mathcal{C} \to W$ be the $W$-valued distance function on $\mathcal{C}$, where $W$ is the Coxeter group of $\Delta$. Since $\Delta$ is spherical the Coxeter group $W$ is finite. Denote the longest word in $W$ by $w_0$. Let $C, D$ and $\overline{C}$ be as in the statement of the lemma. Also let $\delta(C, D) = u$ and $\delta(D, \overline{C}) = w$. Then $uw = w_0$ since $C$ and $\overline{C}$ are opposite and $C, D$ and $\overline{C}$ lie in the same apartment. Note that $\mathcal{A}'_{C,D} = \{E \mid \delta(D, E) = w\}$. This is because $l(uw) = l(u) + l(w)$. We now define a bijection $\mathcal{A}'_{C,D} \times \mathcal{A}'_{D,\overline{C}} \to \mathcal{A}_D$ as follows. If $E \in \mathcal{A}'_{C,D}$ and $\overline{E} \in \mathcal{A}'_{D,\overline{C}}$ then $\delta(\overline{E}, E) = uw = w_0$ again because $l(uw) = l(u) + l(w)$. Hence $E$ and $\overline{E}$ are opposite. So we map $(E, \overline{E})$ to the unique apartment containing $E$ and $\overline{E}$. The map is clearly a bijection.

□

Let $\Delta$ be a finite *irreducible* Moufang building. By irreducible, we mean that the Coxeter group $W$ associated to $\Delta$ is irreducible. We will not state the Moufang condition here; see [26, Chapter 6]. We only mention that all buildings of rank greater or equal to 3 are Moufang. Hence it is a restriction only when $\mathrm{rk}(\Delta) = 2$. By a theorem of Tits [33, Theorem 11.4], if $\Delta$ is irreducible and Moufang then it is the building of an algebraic group $G$ over a finite field $\mathbb{F}_q$. The group $G(\mathbb{F}_q)$ acts by simplicial type-preserving automorphisms on $\Delta$. Also the action is transitive on pairs $(\Sigma, C)$ consisting of an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$. In particular, the stabilizer of a fixed chamber $C$, which we denote $\mathrm{Stab}(C)$ acts transitively on $\mathcal{A}_C$. It follows from the properties of buildings that for $g \in \mathrm{Stab}(C)$, $\rho_{\Sigma,C}(gD) = \rho_{\Sigma,C}(D)$ for any chamber $D \in \mathcal{C}$.

It is known that for any chamber $C$ of $\Delta$, the number of chambers adjacent to $C$ along a fixed facet is a power of $q$. Here $q$ is the cardinality of the finite field $\mathbb{F}_q$. It follows that for chambers $C, D$, the number $|\mathcal{A}_D|$ (and also $|\mathcal{A}_{C,D}|$) is a power of $q$. Using this fact, $h_J(\Delta)$, the flag $h$ vector of $\Delta$, defined in Section 7 can be written as a polynomial in $q$. In the course of proving Theorem 6, we will recall the definition of $h_J(\Delta)$ and give a precise definition of this polynomial. To emphasize the dependence on $q$, we write $h_J(q)$ instead of $h_J(\Delta)$. It will also follow that if $\Sigma$ is any apartment of $\Delta$ then $h_J(\Sigma) = h_J(q)\ |_{q=1}$. The geometric significance of $h_J(\Delta)$ was explained in Propositions 3 and 4. It counts the number of spheres in the homotopy type of $\Delta_J$, the type selected subcomplex of $\Delta$. Now we prove the main result of this section. It gives the precise relation between the homotopy types of $\Delta_J$ and $\Delta_{I \setminus J}$.

**Theorem 6.** *Let $\Delta$ be a finite, irreducible and Moufang building. Then with the notation as above $h_J(q) = h_I(q) h_{I \setminus J}(q^{-1})$.*

*Proof.* We start with a philosophical comment. "As $q \to 1$, the building $\Delta$ degenerates to a single apartment $\Sigma$. The equation of the theorem then reduces to the generalized DS equation $h_J(\Sigma) = h_{I\setminus J}(\Sigma)$." Hence to prove the theorem, we will go in the opposite direction. We will start with the relation $h_J(\Sigma) = h_{I\setminus J}(\Sigma)$. Then for a fixed pair of opposite chambers $C, \overline{C} \in \Sigma$, we will use the retractions $\rho_{\Sigma,C}$ and $\rho_{\Sigma,\overline{C}}$ to generalize this relation to $\Delta$. We now start with the formal proof.

We fix an apartment $\Sigma$ and a chamber $C$ that lies in it. Recall that $R_C(D)$ is the smallest face $F$ of $D$ such that $FC = D$ (Section 4.2). Now set $H_J(\Delta) = \{D \in \Delta \mid R_C(D) \text{ has type } J\}$. Hence by definition, the cardinality of this set is $h_J(\Delta)$. Similarly let $H_J(\Sigma, C) = \{D \in \Sigma \mid R_C(D) \text{ has type } J\}$. We make a few observations. The restriction map $R_C$ for $\Delta$ when restricted to the apartment $\Sigma$ gives the restriction map for $\Sigma$. This follows, for example, from the fact that the projection maps for $\Delta$ are defined using the projection maps for the apartments. Hence the cardinality of $H_J(\Sigma, C)$ is $h_J(\Sigma)$.

Next note that $\Sigma$ is a Coxeter complex and hence corresponds to a simplicial hyperplane arrangement. So it satisfies the opposite axiom $(R4)$; see the Corollary to Theorem 5 and the preceding comments. Hence applying axiom $(R4)$ to it, we get $H_J(\Sigma, C) = H_{I\setminus J}(\Sigma, \overline{C})$. This is a more precise set theoretic version of the generalized DS equation. The cardinalities of the two sets are $h_J(\Sigma)$ and $h_{I\setminus J}(\Sigma)$ respectively and do not depend on $C$ and $\overline{C}$. We next claim that

$$H_J(\Delta) = \sqcup_{D \in H_J(\Sigma,C)} \rho^{-1}_{\Sigma,C}(D).$$

To see this, note that $\rho_{\Sigma,C}(F)C = \rho_{\Sigma,C}(FC)$ for any face $F$ of $\Delta$. Hence $F = R_C(D) \Leftrightarrow \rho_{\Sigma,C}(F) = R_C(\rho_{\Sigma,C}(D))$ which proves the claim. Let $\overline{C}$ be the chamber opposite to $C$ in $\Sigma$. Then $\rho^{-1}_{\Sigma,C}(\overline{C}) = C^{op}$, the set of chambers opposite to $C$. Also note that $h_I(\Sigma) = |H_I(\Sigma, C)| = 1$ with $\overline{C}$ as the only element of $H_I(\Sigma, C)$. Applying the claim for $J = I$, we get $h_I(\Delta) =$ the number of spheres in the homotopy type of $\Delta = |C^{op}| = |\mathcal{A}_C|$. We now have the following identities.

$$h_J(q) = \sum_{D \in H_J(\Sigma,C)} |\rho^{-1}_{\Sigma,C}(D)| = \sum_{D \in H_J(\Sigma,C)} \frac{|\mathcal{A}_D|}{|\mathcal{A}_{C,D}|}.$$

$$h_{I\setminus J}(q) = \sum_{D \in H_{I\setminus J}(\Sigma,\overline{C})} |\rho^{-1}_{\Sigma,\overline{C}}(D)| = \sum_{D \in H_J(\Sigma,C)} \frac{|\mathcal{A}_D|}{|\mathcal{A}_{D,\overline{C}}|}.$$

The first equality is clear. For the second equality we apply Tits theorem to get a group $G(\mathbb{F}_q)$ associated to $\Delta$. The group serves two purposes. It gives us the number $q$ and it acts nicely on $\Delta$. As already noted $\mathrm{Stab}(C)$ (resp. $\mathrm{Stab}(\overline{C})$) acts transitively on $\mathcal{A}_C$ (resp. $\mathcal{A}_{\overline{C}}$) and is compatible with the retraction $\rho_{\Sigma,C}$ (resp. $\rho_{\Sigma,\overline{C}}$). This along with the fact $|\mathcal{A}_C| = |\mathcal{A}_D| = |\mathcal{A}_{\overline{C}}|$ gives us the second equality. The formulas show that $h_J(q)$ can be written as a polynomial in $q$ as claimed earlier. Furthermore due to the group action, it does not depend on the choice of $\Sigma$ and $C$. It is also clear that if $\Sigma$ is any apartment of $\Delta$ then $h_J(\Sigma) = h_J(q)\mid_{q=1}$.

Recall that $h_I(q) =$ the number of spheres in the homotopy type of $\Delta = |\mathcal{A}_D|$. Now if we replace $q$ by $q^{-1}$ in the second equation and multiply by $h_I(q)$, we get

$$h_I(q)h_{I\setminus J}(q^{-1}) = |\mathcal{A}_D| \sum_{D \in H_J(\Sigma,C)} \frac{|\mathcal{A}_{D,\overline{C}}|}{|\mathcal{A}_D|} = \sum_{D \in H_J(\Sigma,C)} |\mathcal{A}_{D,\overline{C}}|.$$

The theorem now follows from Lemma 7. $\square$

*Example* 5. Building of type $A_{n-1}$: Let $V$ be the $n$-dimensional vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the field with $q$ elements. Let $\mathcal{L}_n$ be the lattice of subspaces of $V$ under inclusion. Also let $\mathcal{B}_n$ be the Boolean lattice. Note that the flag (order) complex $\Delta(\mathcal{B}_n)$ is the Coxeter complex of type $A_{n-1}$ and corresponds to the braid arrangement. Also note that a choice of a basis for $V$ gives an embedding of $\mathcal{B}_n$ into $\mathcal{L}_n$. The building of type $A_{n-1}$ over $\mathbb{F}_q$ is defined simply as the flag (order) complex $\Delta(\mathcal{L}_n)$. The subcomplexes $\Delta(\mathcal{B}_n)$, for various embeddings $\mathcal{B}_n \hookrightarrow \mathcal{L}_n$, play the role of apartments. The algebraic group $G(\mathbb{F}_q)$ in this case can be taken to be $GL_n(\mathbb{F}_q)$ or $SL_n(\mathbb{F}_q)$.

For any chamber $D$, one may directly check that $|\mathcal{A}_D| = h_I(q) = q^{\binom{n}{2}}$. Now fix an apartment $\Sigma$ corresponding to a basis $e_1, e_2, \ldots, e_n$ of $V$. Let $C = [e_1] < [e_1, e_2] < \ldots < [e_1, \ldots, e_n]$. Then the chamber opposite to $C$ in $\Sigma$, namely $\overline{C} = [e_n] < [e_n, e_{n-1}] < \ldots < [e_n, \ldots, e_1]$. If we let $D = [e_{i_1}] < [e_{i_1}, e_{i_2}] < [e_{i_1}, \ldots, e_{i_n}]$ be any chamber in $\Sigma$ then a simple counting argument shows that

$$|\mathcal{A}_{C,D}| = q^{|\{(j,k)|j<k \text{ and } i_j<i_k\}|}$$

$$|\mathcal{A}_{D,\overline{C}}| = q^{|\{(j,k)|j<k \text{ and } i_j>i_k\}|}.$$

Hence we can see directly that $|\mathcal{A}_{C,D}||\mathcal{A}_{D,\overline{C}}| = |\mathcal{A}_D|$. This proves Lemma 7 for this example. To unwind Theorem 6, we need to understand how retractions work. We leave that out and instead just give some explicit computations that illustrate the theorem. For type $A_3$, $h_\emptyset = 1$, $h_{\{1\}} = h_{\{3\}} = q(1+q+q^2)$, $h_{\{2\}} = q(1+2q+q^2+q^3)$, $h_{\{1,2\}} = h_{\{2,3\}} = q^3(1+q+q^2)$, $h_{\{1,3\}} = q^2(1+q+2q^2+q^3)$, $h_{\{1,2,3\}} = q^6$. We mention that these polynomials can also be described using descent sets without any reference to retractions, see [32, Theorem 3.12.3].

## 11. Future prospects

In this paper we presented a theory of projection maps and compatible shellings. We now suggest some problems for future study.

**The space of all shellings.** As we saw, compatible shellings on a complex $\Delta$ consist of compatible partial orders $\leq_C$, one for every chamber $C$ of $\Delta$. Using this information, we may construct a geometric object $S(\Delta)$ containing a lot of shelling information about $\Delta$. It would be worthwhile to make this idea precise since the object $S(\Delta)$ would be important for studying the space of all shellings of $\Delta$. In the example of Section 4.3, a candidate for $S(\Delta)$ is a line with vertices at the integer points. The group $\mathbb{Z}$ acts on $S(\Delta)$ by translations. The complex $\Delta$ is a quotient of $S(\Delta)$ by the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$ whose generator shifts a vertex $n$ units to the right.

In this regard, the following formalism could be useful. Let $\Delta$ be a pure simplicial complex that satisfies our axioms. Let $\mathcal{S}$ be the space of all partial orders on $\mathcal{C}$ that extend $\leq_C$ for some $C \in \mathcal{C}$. Then $\mathcal{S}$ is a poset with the order relation given by extension with the partial orders $\leq_C$ as the minimal elements. We also think of $\mathcal{C}$ as a poset with the trivial partial order. Then $\mathcal{S}$ and $\mathcal{C}$ are posets related by a Galois connection. Denote a typical element of $\mathcal{S}$ by $\leq_S$. We are slightly abusing notation because in our earlier usage, $\leq_S$ was always a linear order. The map $\mathcal{S} \to \mathcal{C}$ sends $\leq_S$ to the chamber that appears first in $\leq_S$. And the map $\mathcal{C} \to \mathcal{S}$ sends $C$ to $\leq_C$.

**Other questions.** In contrast to the situation in Section 6, a shelling of a simplicial complex $\Delta$ may have nothing to do with its gallery metric; see the example of Section 4.3. We ask for other non-metrical examples of $\Delta$ that satisfy our axioms.

While we explained some tiny applications and connections of our theory to the flag $h$ vector, random walks and buildings, other applications of this circle of ideas still remain to be seen. As a possibility, we suggest that a general framework to study Solomon's descent algebra [30] would be to start with a labeled simplicial complex $\Delta$ that satisfies our axioms. If $\Delta$ is a Coxeter complex then we would recover the usual descent algebra; see [4, 14, 20] for more information on this geometric way of thinking.

As shown by the examples in Section 2 our theory is closely related to LRBs. We ask if this connection can be clarified. For example, we may ask the converse question to the discussion in Section 5. Given a simplicial complex that satisfies our axioms, how close is it to being a simplicial LRB (see the remark in Section 4.3)?

**More examples.** More examples of the theory that we presented may be possible. A possibility already mentioned is that of shellable complexes with group actions. Another possibility is that of simplicial polytopes equipped with a suitable class of shellings like line shellings. However a better alternative seems to be to generalize the theory itself. We conclude by suggesting some approaches through examples.

($i$) Recall that in the metric setup, $FC$ was the chamber containing $F$ closest to $C$. What happens if there is no closest chamber? We present a building-like example where this occurs.

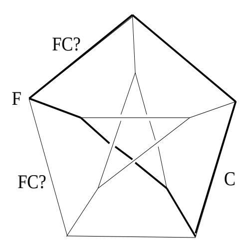
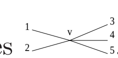
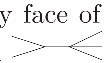
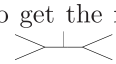
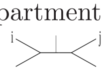



FIGURE 14. The Petersen graph - an almost-building?

The Petersen graph $\Delta$: Consider a node $v$ with 5 labeled leaves 

. This corresponds to the empty face of $\Delta$. To get the rest of $\Delta$, we do a "blowup" at $v$. The trees of the form  (resp.  ) are the vertices (resp. edges) of $\Delta$. The 5 leaves of each tree are labeled $1, 2, 3, 4$ and $5$ in some order. Define an apartment to be the subcomplex made of those edges (chambers) that have the form 

for a fixed $i, j$. There are 10 apartments, 15 chambers and every chamber lies in exactly 4 apartments. Furthermore each apartment is an hexagon and given two chambers, there is always an apartment $\Sigma$ containing them. So we may first define projection maps in $\Sigma$ as usual and then try to extend them to $\Delta$. However, in some cases, there are two apartments containing a face $F$ and a chamber $C$ and the products are not always consistent. This happens exactly when $F$ and $C$ are opposite to each other in some pentagon as shown in Figure 14. Then there are exactly two apartments containing $F$ and $C$ one of which is indicated in the figure. And the two candidates for $FC$ are indeed the two chambers containing $F$ that are closest to $C$.

(*ii*) In [8], Björner constructed many examples of shellable complexes $\Delta$. For example, he shows that all finite semimodular and supersolvable lattices are shellable. As a very special example, we would also like to mention the Whitehead poset which has found an application to group theory [12]. In all these examples, it is clear that $\Delta$ has many shellings. It would be worthwhile to understand the precise compatibility relations among these shellings.

(*iii*) New examples from old: Let $\Delta$ be a labeled simplicial complex that satisfies our axioms. Then what can be said about its type selected subcomplex $\Delta_J$ or its $k$-skeleton $\Delta_k$ (if $\Delta$ is not labeled)? We may ask the same question for the barycentric subdivision $\Delta'$. These complexes again have many shellings but they do not directly give us new examples. However they might point to an appropriate generalization of the theory.

(*iv*) Finally, we mention non-simplicial examples like polytopes and hyperplane arrangements. The second situation may be easier to handle since we already have projection maps in that case. The other non-simplicial examples to consider are LRBs as explained in Section 5. We mention that there is a notion of shelling for a poset. It is called a recursive coatom ordering [10]. This would be the relevant shelling concept in this generality.

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## References

[1] Herbert Abels, *The geometry of the chamber system of a semimodular lattice*, Order **8** (1991), no. 2, 143–158. 4, 5, 17

[2] Peter Abramenko, *Walls in Coxeter complexes*, Geom. Dedicata **49** (1994), no. 1, 71–84. 23

[3] Dave Bayer and Persi Diaconis, *Trailing the dovetail shuffle to its lair*, Ann. Appl. Probab. **2** (1992), no. 2, 294–313. 23

[4] T. Patrick Bidigare, *Hyperplane arrangement face algebras and their associated Markov chains*, Ph.D. thesis, University of Michigan, 1997. 31

[5] T. Patrick Bidigare, Phil Hanlon, and Daniel N. Rockmore, *A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements*, Duke Math. J. **99** (1999), no. 1, 135–174. 1, 2, 3, 22

[6] Louis J. Billera, Kenneth S. Brown, and Persi Diaconis, *Random walks and plane arrangements in three dimensions*, Amer. Math. Monthly **106** (1999), no. 6, 502–524. 2, 3, 23

[7] Garrett Birkhoff, *Lattice theory*, third ed., American Mathematical Society, Providence, R.I., 1979. 5

[8] Anders Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 159–183. 21, 32

[9] ———, *Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings*, Adv. in Math. **52** (1984), no. 3, 173–212. 1, 2, 8, 17, 25

[10] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler, *Oriented matroids*, Encyclopedia of Mathematics and its Applications, vol. 46, Cambridge University Press, Cambridge, 1993. 1, 3, 32

[11] Anders Björner and Michelle L. Wachs, *Shellable nonpure complexes and posets. II*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 3945–3975. 19

[12] Noel Brady, Jon McCammond, John Meier, and Andy Miller, *The pure symmetric automorphisms of a free group form a duality group*, Journal of Algebra, to appear. 32

[13] Kenneth S. Brown, *Buildings*, Springer-Verlag, New York, 1989. 3, 4
[14] ______, *Semigroups, rings, and Markov chains*, J. Theoret. Probab. **13** (2000), no. 3, 871–938. 3, 4, 16, 18, 22, 23, 31
[15] Kenneth S. Brown and Persi Diaconis, *Random walks and hyperplane arrangements*, Ann. Probab. **26** (1998), no. 4, 1813–1854. 3, 22
[16] Andreas W. M. Dress and Rudolf Scharlau, *Gated sets in metric spaces*, Aequationes Math. **34** (1987), no. 1, 112–120. 1
[17] Pierre Antoine Grillet, *Semigroups. an introduction to the structure theory*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 193, Marcel Dekker Inc., New York, 1995. 3
[18] Branko Grünbaum, *Arrangements and spreads*, CBMS Regional Conference Series in Mathematics, vol. 10, American Mathematical Society, Providence, R.I., 1972. 24
[19] Fritz Klein-Barmen, *Über eine weitere Verallgemeinerung des Verbandsbegriffes*, Math. Z. **46** (1940), 472–480. 3
[20] Swapneel Mahajan, *Shuffles on Coxeter groups*, arXiv:math.PR/0108094. 23, 31
[21] Bernhard Mühlherr, *Some contributions to the theory of buildings based on the gate property*, Ph.D. thesis, Eberhard-Karls-Universität zu Tübingen, 1994. 1, 27
[22] Bernhard Mühlherr and Peter Schmid, *The extended Steinberg character*, J. Algebra **175** (1995), no. 1, 373–384. 27
[23] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer-Verlag, Berlin, 1992. 3
[24] Mario Petrich, *A construction and a classification of bands*, Math. Nachr. **48** (1971), 263–274. 3
[25] ______, *Lectures in semigroups*, John Wiley & Sons, London-New York-Sydney, 1977. 3
[26] Mark Ronan, *Lectures on buildings*, Perspectives in Mathematics, vol. 7, Academic Press Inc., Boston, MA, 1989. 28
[27] Rudolf Scharlau, *Metrical shellings of simplicial complexes*, European J. Combin. **6** (1985), no. 3, 265–269. 1, 15
[28] ______, *Buildings*, Handbook of incidence geometry, North-Holland, Amsterdam, 1995, pp. 477–645. 4
[29] Maurice-Paul Schützenberger, *Sur certains treillis gauches*, C. R. Acad. Sci. Paris **224** (1947), 776–778. 3
[30] Louis Solomon, *A Mackey formula in the group ring of a Coxeter group*, J. Algebra **41** (1976), no. 2, 255–264. 16, 31
[31] R. P. Stanley, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197–217. 5
[32] Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, with a foreword by Gian-Carlo Rota, corrected reprint of the 1986 original. 24, 30
[33] Jacques Tits, *Buildings of spherical type and finite BN-pairs*, Springer-Verlag, Berlin, 1974, Lecture Notes in Mathematics, Vol. 386. 1, 4, 17, 28
[34] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. 3, 17, 26

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