SHUFFLES ON COXETER GROUPS

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ABSTRACT. The random-to-top and the riffle shuffle are two well-studied methods for shuffling a deck of cards. These correspond to the symmetric group S_n, i.e., the Coxeter group of type A_{n-1}. In this paper, we give analogous shuffles for the Coxeter groups of type B_n and D_n. These can be interpreted as shuffles on a "signed" deck of cards. With these examples as motivation, we abstract the notion of a shuffle algebra which captures the connection between the algebraic structure of the shuffles and the geometry of the Coxeter groups. We also give new joker shuffles of type A_{n-1} and briefly discuss the generalisation to buildings which leads to q-analogues.

CONTENTS

1. INTRODUCTION

In a recent work, Ken Brown [10] used algebraic methods to analyze random walks on a class of semigroups called "left-regular bands". These walks include the hyperplane chamber walks of Bidigare, Hanlon and Rockmore [4, 5]. In this paper, we look at the special case of reflection arrangements that arise in the study of Coxeter groups. The random walks that we look at can be thought of as shuffles on the chambers of a Coxeter complex. The motivating examples are the well studied riffle shuffle and the random-to-top shuffle on a deck of n cards [2, 11, 13, 14, 20]. These correspond to the symmetric group S_n, i.e., the Coxeter group of type A_{n-1}.

We define analogous shuffles for the Coxeter groups of type B_n and D_n; also see [3, 16]. Our examples are motivated by shuffle considerations on the one hand and the geometry of the Coxeter complex on the other. This is best understood in

the framework of a shuffle algebra, a notion we introduce. The striking similarity among the examples can be traced to maps among the three Coxeter complexes of type A_{n-1}, B_n and D_n. We also give new shuffles of type A_{n-1}, A_{2n} and A_{2n-1} and generalize some of the random walks to buildings.

The only Coxeter groups that we deal with are the ones of type A_{n-1}, B_n and D_n. To make this paper accessible to readers unfamiliar with Coxeter groups we have included two appendices. Appendix A reviews the facts we need about hyperplane arrangements. In Appendix B, we give a brief review of Coxeter groups and then explain everything in concrete terms for the three cases mentioned above. For the general theory of Coxeter groups, we refer the reader to [9, 17, 18, 24].

A version of this paper is on the preprint server at http://xxx.lanl.gov/. We note some changes. Section 7 on shuffles on cards with an involution and a joker and Section 3.4 have been added. Section 2 and the concluding section on future prospects has been rewritten. Also some random walk descriptions have been added or corrected.

1.1. **The random walk and the method of analysis.** Let Σ be the (simplicial) Coxeter complex associated to a Coxeter group W. Also let $\mathcal{C}$ be the set of chambers (or maximal simplices) of Σ. Then Σ is a semigroup containing $\mathcal{C}$ as an ideal. The product in Σ is given by the projection maps as explained in Appendix A. We now describe the walk. Let $\{w_x\}_{x \in \Sigma}$ be a probability distribution on Σ. If the walk is in chamber $c \in \mathcal{C}$, then move to the chamber xc, where $x \in \Sigma$ is chosen with probability w_x. The product xc is again a chamber because $\mathcal{C}$ is an ideal in Σ.

Next we describe the walk more algebraically. Fix a commutative ring k and consider the semigroup algebra $k\Sigma$. The k-module $k\mathcal{C}$ spanned by the chambers is an ideal in $k\Sigma$. In particular, it is a module over $k\Sigma$; we therefore obtain a homomorphism

$$k\Sigma \to \mathrm{End}_k(k\mathcal{C}),$$

the latter being the ring of k-endomorphisms of $k\mathcal{C}$. This map is in fact an inclusion; so one can regard elements of $k\Sigma$ as operators on $k\mathcal{C}$. Let $\sigma = \sum_{x \in \Sigma} w_x x$ be such that $\sum_{x \in \Sigma} w_x = 1$. It is straightforward to check that the transition matrix of the random walk determined by $\{w_x\}$ is simply the matrix of the operator "left multiplication by σ". In other words, it is the image of σ under the inclusion $k\Sigma \hookrightarrow \mathrm{End}_k(k\mathcal{C})$.

A way to analyze the random walk is to focus on $k\Sigma$ and understand the structure of the subalgebra $k[\sigma]$ that σ generates in $k\Sigma$. Because of the inclusion map above, we can also analyze the structure of $k[\sigma]$ in $\mathrm{End}_k(k\mathcal{C})$. We will make use of both viewpoints in our examples. The reader may be more familiar with this algebraic approach in the context of groups [12] rather than semigroups [10].

1.2. **Nature of our examples.** In our examples, the element σ will never be normalized. In other words, if we write $\sigma = \sum_{x \in \Sigma} w_x x$ then $\sum_{x \in \Sigma} w_x \neq 1$. So, strictly speaking, we will not have a probability distribution on Σ. The main reason for doing this is to simplify the algebra. So every time we describe the random walk associated to σ, the description will be correct up to a normalization factor. This will also be true for all other objects that will show up in the analysis. To get rid of this anomaly, we adopt the following convention. If the random walk description says "Do xxx at random." then the normalization factor is "the number of ways of doing xxx". In general, the random walk description will involve a sequence of

independent random acts in which case the normalization factor will be the product of the individual factors. In most examples, we will prefer to first define relevant elements of $k\Sigma$ and then to provide motivation by interpreting the associated random walks. Making this translation will always involve the normalization factor defined above.

Further in all examples, except those in Section 7, the element σ will lie in the subalgebra of W-invariants of $k\Sigma$. We now discuss this subalgebra. The W-invariants of $k\Sigma$ under the natural W-action form a k-algebra $(k\Sigma)^W$. As a k-module, $(k\Sigma)^W$ is free with one basis element for each W-orbit in Σ, that basis element being the sum of the simplices in the orbit. Since orbits correspond to types of simplices, we obtain a basis vector

$$\sigma_J = \sum_{F \in \Sigma_J} F$$

for each $J \subseteq I$, where Σ_J is the set of simplices of type J and I is the set of all labels or types of vertices of Σ. Also let Σ_j be the set of all simplices of rank j, irrespective of their type.

As already mentioned, in most of our examples $\sigma \in (k\Sigma)^W$ and hence $k[\sigma]$ is a certain commutative subalgebra of $(k\Sigma)^W$. Since the element σ is always motivated by some shuffle considerations, we will call $k[\sigma]$ a shuffle algebra. For a precise definition, see Section 2.1. Bidigare [4] proved that $(k\Sigma)^W$ is anti-isomorphic to Solomon's descent algebra [22], which is a certain subalgebra of the group algebra kW. So most of the shuffle algebras we consider are anti-subalgebras of Solomon's descent algebra. And since they are commutative, they are in fact subalgebras of the descent algebra.

1.3. **Organization of the paper.** In Section 2, we define shuffle algebras, state a theorem that gives examples of such algebras and then provide a method for proving it. In the next three sections, we prove the theorem by studying the examples of shuffle algebras of type A_{n-1}, B_n and D_n respectively that it contains. These sections should be read in conjunction with Appendices B.4, B.5 and B.6 respectively. The maps between Coxeter complexes which explain the close relation between different examples are explained in Section 6. This section gives us a good overall picture of how things fit together and can play an important role in the further development of the theory. We see examples of this in the next two sections, where we define some new shuffles of type A_n, A_{2n} and A_{2n-1} and briefly discuss the generalization of the random walks to buildings.

2. The abstract setup

In this section, we first abstract the notion of a shuffle algebra, then state a theorem that gives examples of such algebras and finally provide a method for proving it. It is best to skim over this section first and then to go over the details later with a concrete example in mind.

2.1. **Shuffle algebras.** We first need a preliminary definition. An additive (resp. multiplicative) *shuffle semigroup* is a subsemigroup of the semigroup of non-negative (resp. positive) integers under addition (resp. multiplication). Note that there is only one additive shuffle semigroup up to isomorphism. However in the multiplicative case, there could be many. Now we are ready to define a shuffle algebra.

TABLE 1. Examples of shuffle algebras.

Type	Additive	Multiplicative
A_{n-1}	$\sigma_{s_1}, \ \ \sigma_{s_1} + \sigma_{s_{n-1}}$	$\sigma_{s_1} + \ldots + \sigma_{s_{n-1}}$
B_n	σ_{s_1}	$\sigma_t, \ \ \sigma_{s_1} + \ldots + \sigma_{s_{n-1}}$
D_n	σ_{s_1}	$\sigma_u + \sigma_v, \ \ \sigma_u + \sigma_v + \sigma_{s_1} + \ldots + \sigma_{s_{n-2}} + \sigma_{u,v}$

We say that a subalgebra $A \subseteq k\Sigma$ is an additive (resp. multiplicative) *shuffle algebra* if it satisfies the following conditions.

(1) $\dim \Sigma + 1 \leq \dim_k A \leq \dim \Sigma + 2$.
(2) A has a basis of the form $\sigma_0 = 1, \sigma_1, \sigma_2, \ldots$ where σ_j is an element of $k\Sigma_j$, the span of simplices of rank j.
(3) $A = k[\sigma_1] \subseteq k\Sigma$.
(4) A contains an additive (resp. multiplicative) shuffle semigroup S as a spanning set.

An alternate way to express condition (4) is to say that there exists an injective semigroup map $S \hookrightarrow A$ such that the image spans A. Note that the definition of a shuffle algebra A depends on the Coxeter complex Σ under consideration. In fact, conditions $(1), (2)$ and (3) say that the algebra A is in tune with the geometry of Σ. Hence it is more correct to write a shuffle algebra as a pair (A, Σ). But we will not bother with this since the Coxeter complex Σ will always be clear from context.

The above definition is mainly motivated by the infinite families A_{n-1}, B_n and D_n. The significance of this definition for the sporadic Coxeter groups of type E_6, E_7, etc. is not clear. We have imposed the strictest possible conditions that our examples led us to. As a result, some of the algebras we consider are not shuffle algebras; for example, the double shuffle constructions in Sections 4.2 and 5.2. Hence a more flexible definition might work better, say for the classification of shuffle algebras.

Ken Brown [10] showed that if k has characteristic 0 and σ is a non-negative integral linear combination in the canonical basis of $k\Sigma$ consisting of all simplices then $k[\sigma]$ is *split-semisimple*, that is, $k[\sigma] \cong k^n$ for $n = \dim_k k[\sigma]$. To guarantee this result for an arbitrary field he introduced an additional condition on σ. We will always assume that k has characteristic 0. Also σ will always be a non-negative integral linear combination in the canonical basis of $k\Sigma$. Hence it follows by condition (3) that the shuffle algebras we consider are split-semisimple.

2.2. **The main result and method of proof.** For notation, refer to Section 1.2 and Figures 2,3 and 4 in Appendix B.3.

Theorem 1. *For the choices of σ shown in Table 1, the algebra $k[\sigma]$ is a shuffle algebra.*

Note that $\sigma_u + \sigma_v + \sigma_{s_1} + \ldots + \sigma_{s_{n-2}} + \sigma_{u,v} \notin k\Sigma_1$. Hence for this σ, the algebra $A = k[\sigma]$ violates condition (2). However we regard this as a minor point. The other examples not covered by this theorem are the joker shuffles which will be discussed in Section 7.

To prove the theorem, we need to define elements $\sigma_j \in k\Sigma_j$ and a shuffle semigroup S for each σ. These will be explicitly constructed in the next three sections.

They will always have a nice interpretation in terms of card shuffles. This explains our terminology of shuffle semigroups.

Proposition 1. *Let A be an n-dim algebra containing a spanning semigroup S. Suppose that there exist characters $\chi_1,\ldots,\chi_n$ of S and elements $e_1,\ldots,e_n$ of A such that for every $s \in S$, we have*

$$(*) \qquad s = \sum_{i=1}^{n} \chi_i(s)e_i.$$

Then $A \xrightarrow{\cong} k^n$ with e_i as the primitive idempotents. Also the χ_i's extend to characters of A.

Proof. The elements $e_1,\ldots,e_n$ of A span S which in turn spans A. Since $\dim_k A = n$ this implies that $e_1,\ldots,e_n$ form a basis for A. This yields a vector space isomorphism $\Phi : A \xrightarrow{\cong} k^n$, where $\sum_{i=1}^{n} a_i e_i \mapsto (a_1,\ldots,a_n)$. Hence for $s \in S$, $s \mapsto (\chi_1(s),\ldots,\chi_n(s))$. Since $\chi_1,\ldots,\chi_n$ are characters of S, we have $\Phi(s_1 s_2) = \Phi(s_1)\Phi(s_2)$ for $s_1, s_2 \in S$. The fact that Φ respects the algebra structures on a spanning set S of A implies that it is in fact an algebra homomorphism. $\square$

This proposition is more relevant to the multiplicative examples. It also applies to the additive examples. But since they are simpler they are tractable by other means also. To prove Theorem 1, we construct by hand, elements $\sigma_j \in k\Sigma_j$ and a spanning shuffle semigroup S in $A = k[\sigma_1,\ldots,\sigma_n]$. The characters χ of S that we will use will be of the form $\chi(a) = c^a$ if S is additive and $\chi(a) = a^c$ if S is multiplicative. Here c is a fixed non-negative integer. The specific values of c that we need to choose depend on the example at hand. Proposition 1 will then give an explicit isomorphism of A with k^n. This will imply that in fact $A = k[\sigma_1]$ and hence is a shuffle algebra. Note that along the way, we will also obtain that A is split-semisimple and formulas for the primitive idempotents in A.

The Coxeter complexes Σ of type A_{n-1}, B_n and D_n can be described using the language of "ordered partitions". Also, chambers of Σ can be described as a "deck of cards". This is explained in Appendices B.4, B.5 and B.6 respectively. The descriptions for type B and D are not in the literature. So we give full details. We follow the convention that "left to right" for an ordered partition is "top to bottom" for a deck of cards. It is essential to be familiar with this language to fully understand the examples. In Sections 3.1 and 3.3, we explain the side shuffle ($\sigma = \sigma_{s_1}$) and riffle shuffle ($\sigma = \sigma_{s_1} + \ldots + \sigma_{s_{n-1}}$) of type A_{n-1}. They are the simplest of the additive and multiplicative examples respectively and serve to illustrate the method in each case.

3. Examples of type A_{n-1}

We consider three examples; the side shuffle, the two sided shuffle and the riffle shuffle. The first two are additive in nature while the third is multiplicative. We also give involutive and joker analogues of two of these shuffles in Section 7. Elements of the label set I will be written as $s_1, s_2, \ldots, s_{n-1}$; see Figure 2 in Appendix B.3. Recall that σ_J, the sum of simplices of type J, for $J \subseteq I$ form a basis for $(k\Sigma)^W$. Hence these elements will play a key role in the analysis.

3.1. **The side shuffle.** This is more commonly known as the random-to-top shuffle or the Tsetlin library. We may also call it the one sided shuffle. It arises in the study of dynamic list-management in computer science. See Fill [14] and the references cited there. The eigenvalues for this random walk were first found by Phatarfod [20]; see also [4, 5, 10, 11, 13, 14] for other proofs.

The element of interest is $\sigma = \sigma_{s_1}$, that is, it is the sum of all vertices of type s_1. In terms of ordered partitions, σ_{s_1} is the sum of all ordered two block partitions of $[n]$ such that the first block is a singleton. As explained in Section 1.1, there is a random walk on a deck of n cards associated to σ. It consists of removing a card at random and replacing it on top. When $n = 4$, for example, $(\{2\}, \{1, 3, 4\})$ is a typical summand of σ. Its product with the deck $(\{1\}, \{2\}, \{3\}, \{4\})$ gives $(\{2\}, \{1\}, \{3\}, \{4\})$, that is, the overall effect is to remove the card labeled 2 and put it on top. Note that the normalization factor in this case is n, consistent with the convention of Section 1.2.

In addition to $\sigma = \sigma_1$, we define $\sigma_j = \sigma_{J_j}$ where $J_j = \{s_1, \ldots, s_j\} \subseteq I$ for $j = 1, \ldots, n-1$. To explain in words, σ_1 is the sum of all the vertices of type s_1, σ_2 is the sum of all the edges of type $s_1 s_2$ and so on till σ_{n-1} which is the sum of all chambers of Σ. As ordered partitions, σ_j is the sum of all ordered $(j+1)$ block partitions of $[n]$ such that the first j blocks are singletons. Also for convenience, we define $\sigma_n = \sigma_{n-1}$. Just like σ_1, we can describe the random walk on a deck of n cards associated to σ_j.

σ_j : Choose j distinct cards at random and put them on top in a random order. Note that σ_n was previously defined in an artificial way. But the random walk description for σ_n makes perfect sense and we now see the motivation in setting $\sigma_n = \sigma_{n-1}$. The two are identical as random walks and by our convention have the same normalization factor of $n!$. From now on, we will leave out the discussion on normalization. We now present two methods to analyze $k[\sigma_1]$.

The shuffle method.

Put $\sigma_0 = 1$. Define a-shuffles S_a for $a \geq 0$ by

$$(1) \qquad\qquad S_a = \sum_{j=0}^{n} S(a, j)\sigma_j,$$

where $S(a, j)$ are the Stirling numbers of the second kind. They count the number of ways in which a elements can be divided into j non-empty subsets. We make the convention that $S(a, j) = 0$ for $a < j$. Also $S(0, 0) = 1$ and $S(a, 0) = 0$ for $a > 0$. Note that $S_0 = 1$ and $S_1 = \sigma_1$. The motivation behind this definition will soon become clear. Unlike for σ_j, the description of S_a in terms of ordered partitions is not clear. However, the associated random walk on a deck of n cards can be readily described.

S_a : Pick a card at random and mark it. Repeat this process a times. There is no restriction on the number of times a given card may get marked. Move all the marked cards to the top in a random order.

To see the equivalence of the two definitions, note that S_a can be expressed as a sum indexed by the number of distinct cards that get marked. Suppose that j distinct cards get marked. By our marking scheme for S_a, there are exactly $S(a, j)$ ways in which the same set of j cards gets marked. This leads to equation (1). With the probabilistic interpretation for S_a, one obtains that $S_a S_b = S_{a+b}$. This relation was the main motivation behind the definition of S_a. Since $S_1 = \sigma_1$, it implies that

$S_a = \sigma_1^a$. Thus we obtain an additive shuffle semigroup $S = \{S_a : a \geq 0\}$ which spans $A = k[\sigma_1] = k[S_1]$. We can deduce from equation (1) that $\sigma_0 = 1, \sigma_1, \ldots, \sigma_{n-1}$ lie in A. Also they span S and hence A. Since they are linearly independent in $(k\Sigma)^W$, it now follows that they form a basis for A. Hence A is a shuffle algebra.

The analysis so far shows that the choice of the element $\sigma_j \in \Sigma_j$ was forced on us by σ_1. We will see this more directly in the second method. Now we show that $A \cong k^n$ and compute the primitive idempotents in A. We do this by molding equation (1) in the shape of equation (*) of Proposition 1. The key step is to use the following explicit formula for the Stirling numbers $S(a, j)$; see [23, pg 34].

$$S(a, j) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \frac{i^a}{j!},$$

with the convention that $0^0 = 1$. Substituting the above expression in the formula for S_a and rearranging terms, we get for $a \geq 0$

$$S_a = \sum_{i=0}^{n} i^a e_i, \quad \text{where} \quad e_i = \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i} \frac{\sigma_j}{j!}.$$

Observe that $e_{n-1} = 0$ because $\sigma_{n-1} = \sigma_n$. The remaining e_i's are clearly non-zero. Now apply Proposition 1 to S and A along with the n characters of S given by $\chi_i(S_a) = i^a$ for $i = 0, 1, 2, \ldots, n-2, n$. This gives $A \xrightarrow{\cong} k^n$. The isomorphism maps S_a to $(0^a, 1^a, 2^a, \ldots, (n-2)^a, n^a)$. These give the possible set of eigenvalues of S_a considered as an operator on any A-module. The formulas for the primitive idempotents e_i's are identical to those obtained in [10]. The minimal polynomial for S_a is given by $x(x-1)(x-2^a)\cdots(x-(n-2)^a)(x-n^a)$. If we expand this polynomial, substitute $x = S_a$ and use equation (1) then we obtain some identities involving the Stirling numbers.

The direct method.

We first claim that $\sigma_j \sigma_1 = j\sigma_j + \sigma_{j+1}$ if $j < n-1$ and $\sigma_{n-1}\sigma_1 = n\sigma_{n-1}$. We see this from the ordered partition description of the σ's. A typical summand of σ_j is $(\{1\}, \{2\}, \cdots, \{j\}, \{j+1, \cdots, n\})$. This term appears j times in the product $\sigma_j \sigma_1$; the j summands of σ_1 that contribute being the ones in which the element in the singleton block is one of $1, 2, \ldots, j$. This yields $j\sigma_j$. The remaining terms yield σ_{j+1} and the claim follows. For $j = 1$, the claim says that $\sigma_1^2 = \sigma_1 + \sigma_2$. Thus $\sigma_2 = \sigma_1^2 - \sigma_1$ is determined by σ_1 and the same is true for σ_3 and so on. More precisely, by induction we obtain

$$(2) \qquad\qquad \sigma_{j+1} = \sigma_1(\sigma_1 - 1)\ldots(\sigma_1 - j).$$

This along with $\sigma_{n-1}\sigma_1 = n\sigma_{n-1}$ implies that both $\sigma_0 = 1, \sigma_1, \ldots, \sigma_{n-1}$ and $\sigma_1^0, \sigma_1^1, \ldots, \sigma_1^{n-1}$ form a basis for A. The basic relation satisfied by σ_1 is $\sigma_{n-1}(\sigma_1 - n) = 0$, that is, $\sigma_1(\sigma_1 - 1)\ldots(\sigma_1 - n + 2)(\sigma_1 - n) = 0$, which is a polynomial of degree n in σ_1. Since $\dim_k A = n$, it follows that $A \xrightarrow{\cong} \frac{k[x]}{x(x-1)\ldots(x-n+2)(x-n)}$, where the map sends σ_1 to x. This shows that A is split semisimple. Note that this time we did not use Proposition 1 to arrive at this conclusion.

We now give algebraic motivation for equation (1) by rederiving it here. Right now σ_j is defined only for $0 \leq j \leq n-1$. But we may extend the definition of σ_j to any j using equation (2). With this extension, one may check that $\sigma_n = \sigma_{n-1}$ and

$\sigma_j = 0$ for $j > n$. We now find a formula for σ_1^j by inverting equation (2) formally. Write

$$\sigma_1^a = \sum_{j=0}^{a} S(a,j)\sigma_j$$

for some constants $S(a,j)$. Multiply both sides on the right by σ_1 and use the relation $\sigma_j\sigma_1 = j\sigma_j + \sigma_{j+1}$ to obtain the recursion: $S(a,j) = jS(a-1,j) + S(a-1,j-1)$ with $S(a,a) = S(a,1) = 1$. The recursion and the initial conditions show that $S(a,j)$ are the Stirling numbers of the second kind. We now define the a-shuffle S_a to be σ_1^a. This gives equation (1). By our definition, it follows directly that $S_aS_b = S_{a+b}$. Note that this time we did not rely on any probabilistic interpretation of S_a to derive this additive relation.

Remark. Consider the map $k[x] \twoheadrightarrow A$ which sends x to σ_1. Along with $\{x^j\}_{j\geq 0}$, the other sequence which played a prominent role in our analysis was $\{x_{(j)}\}_{j\geq 0}$, where $x_{(j+1)} = x(x-1)\dots(x-j)$. Both these sequences are polynomial sequences of binomial type [1]. This kind of structure seems to be common to all the additive examples. The first sequence is always the same but the second sequence varies. And the relation between the two gives us various analogues of the Stirling numbers.

It is possible to write down an explicit formula for $\sigma_i\sigma_j$. We state it here for completeness.

$$\sigma_i\sigma_j = \sum_{k=0}^{min(i,j)} k!\binom{i}{k}\binom{j}{k}\sigma_{i+j-k}.$$

There is also a q-analogue of the side shuffle which we will explain in Section 8.

Remark. The direct method works for all additive examples, including the generalization to buildings (Section 8). However it fails on multiplicative examples like riffle shuffles. On the other hand, the shuffle method works for all examples except the ones on buildings.

3.2. **The two sided shuffle.** This example is similar to the side shuffle and it will be useful to keep the analogy in mind. It does not seem to have been considered before. The element of interest is $\sigma = \sigma_{s_1} + \sigma_{s_{n-1}}$. In terms of ordered partitions, σ is the sum of all ordered two block partitions of $[n]$ such that either the first block or the second block is a singleton. The associated random walk on a deck of cards consists of removing a card at random and replacing it either on top or at the bottom. In other words, we make use of both sides of the deck instead of just one; whence the name.

In addition to $\sigma = \sigma_1$, we define $\sigma_j = \sum_{k=0}^{j}\binom{j}{k}\sigma_{J_{j,k}}$, where $J_{j,k} = \{s_1, \dots, s_k\} \cup \{s_{n-(j-k)}, \dots, s_{n-1}\} \subseteq I$ for $j = 1, \dots, n-1$. Note that $J_{j,k}$ is the union of the first k and last $j-k$ elements of the label set. Hence the cardinality of $J_{j,k}$ is always j. Also as ordered partitions, $\sigma_{J_{j,k}}$ is the sum of all ordered $(j+1)$ block partitions of $[n]$ such that the first k and the last $j-k$ blocks are singletons. As is evident from the formula, expressing σ_j in this language is not pleasant. However the probability description will be simple as we will see. Also put $\sigma_0 = 1$ and $\sigma_n = 2\sigma_{n-1}$. Now define a-shuffles S_a for $a \geq 0$ by

$$(3) \qquad\qquad S_a = \sum_{j=0}^{n} S(a,j)\sigma_j,$$

where $S(a, j)$ counts the number of ways in which a set of a elements can be divided into j non-empty subsets, where in each subset the elements are further divided into two subsets. We will call the $S(a, j)$'s the *signed* Stirling numbers. They satisfy the recursion: $S(a, j) = 2jS(a - 1, j) + S(a - 1, j - 1)$ with $S(a, a) = 1$ and $S(a, 1) = 2^{a-1}$. An explicit formula is as under.

$$S(a, j) = \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}\frac{(2i)^a}{2^j j!} .$$

Note that $S_0 = 1$ and $S_1 = \sigma_1$. These definitions are motivated by the following interpretations of the random walks associated to σ_j and S_a.

σ_j : Choose j cards at random. Split these into two distinct piles such that the split $(k, j - k)$ is given weight $\binom{j}{k}$. Move the cards in the first pile to the top in a random order and those in the second pile to the bottom in a random order.

S_a : Pick a card at random and mark it T or B. Repeat this process a times. If at any time we choose a card that has already been marked then we overwrite that mark. We weight this event by $\binom{j}{k}$, where j = the number of marked cards, and k = the number of cards marked T. Now move all cards marked T to the top and those marked B to the bottom in a random order.

First check that $\sigma_n = 2\sigma_{n-1}$. Next note that σ_j can be expressed as a sum depending on the number of cards k that are moved to the top. Since the splits are weighted, we pick a factor of $\binom{j}{k}$. This leads us to the equation for σ_j in terms of the $\sigma_{J_{j,k}}$'s that we wrote earlier. The interpretation of S_a is slightly subtle. As for the side shuffle, express S_a as a sum indexed by the number of distinct cards that get marked. Suppose that j distinct cards get marked. By our marking scheme for S_a, there are exactly $2^j S(a, j)$ ways in which the same set of j cards gets marked. This is because we use distinct labels T and B. However, the extra factor of 2^j is compensated in the next step where we use the last label on a marked card to decide whether it goes to the top or the bottom.

As for the previous side shuffle example, the probabilistic description implies the relation $S_a S_b = S_{a+b}$. And $A = k[\sigma_1] = k[S_1]$ is a shuffle algebra containing the additive shuffle semigroup $S = \{S_a : a \geq 0\}$. Further $A \xrightarrow{\cong} k^n$, where S_a maps to $(0^a, 2^a, 4^a, \ldots, (2n-4)^a, (2n)^a)$. The formula for the primitive idempotents in A is given by

$$e_i = \sum_{j=i}^{n}(-1)^{j-i}\binom{j}{i}\frac{\sigma_j}{2^j j!} .$$

This time the relation $\sigma_n = 2\sigma_{n-1}$ forces $e_{n-1} = 0$. Also equation (3) can be rewritten as

$$S_a = \sum_{i=0}^{n}(2i)^a e_i.$$

We now sketch the direct method. The description of the σ's as ordered partitions is not very pleasant. However one can still use it to see that $\sigma_j \sigma_1 = 2j\sigma_j + \sigma_{j+1}$ if $j < n - 1$ and $\sigma_{n-1}\sigma_1 = 2n\sigma_{n-1}$. By induction we obtain

(4) $$\sigma_{j+1} = \sigma_1(\sigma_1 - 2)\ldots(\sigma_1 - 2j).$$

This along with $\sigma_{n-1}\sigma_1 = 2n\sigma_{n-1}$ implies that both $\sigma_0 = 1, \sigma_1, \ldots, \sigma_{n-1}$ and $\sigma_1^0 = 1, \sigma_1^1, \ldots, \sigma_1^{n-1}$ form a basis for A. The basic relation satisfied by σ_1 is $\sigma_1(\sigma_1 - 2)\ldots(\sigma_1 - 2n + 4)(\sigma_1 - 2n) = 0$. Hence $A \xrightarrow{\cong} \frac{k[x]}{x(x-2)\ldots(x-2n+4)(x-2n)}$, where σ_1

maps to x. This shows that A is split semisimple. Inverting equation (4) formally and using the relation $\sigma_j\sigma_1 = 2j\sigma_j + \sigma_{j+1}$ leads to the recursion of the signed Stirling numbers written earlier and hence to equation (3), where $S_a = \sigma_1^a$.

3.3. **Riffle shuffle.** A *riffle shuffle* is a common method people use for shuffling a deck of cards. The analysis we give parallels that of Bayer and Diaconis [2]. The difference is that we prefer to work with $(k\Sigma)^W$, which is the more geometric description of Solomon's descent algebra. Also they work with a different basis of the descent algebra.

The Gilbert-Shannon-Reeds mathematical model for riffle shuffling consists of first cutting the deck into two parts, making the cut according to a binomial probability distribution, and then interleaving the two parts in such a way that every interleaving is equally likely. A natural generalization is the a-shuffle in which we cut the deck into a parts (rather than just 2) and then interleave them.

From our point of view, it is more natural to consider the inverse riffle shuffle S_2 and more generally the inverse a-shuffle S_a. It has the following description as a random walk on a deck of n cards.

S_a : Label each card randomly with an integer from 1 to a. Move all the cards labeled 1 to the bottom of the deck, preserving their relative order. Next move all the cards labeled 2 above these again preserving their relative order and so on.

We can understand the inverse a-shuffle S_a as an element of $(k\Sigma)^W$. We begin with the element $\sigma_1 = \sigma_{s_1} + \cdots + \sigma_{s_{n-1}}$; that is, σ_1 is the sum of all the vertices in Σ. In terms of ordered partitions, it is the sum of all ordered two block partitions of $[n]$. This is closely related to the inverse riffle shuffle, in fact, $S_2 = \sigma_1 + 2\sigma_0$, where $\sigma_0 = 1$ is the one block partition. We will show this in more generality.

Let σ_i be the sum of all the simplices of rank i in Σ for $i = 1, \ldots, n-1$. Then define shuffles S_a for $a \geq 1$ by

$$(5) \qquad\qquad S_a = \sum_{j=1}^{n} \binom{a}{j} \sigma_{j-1}.$$

Note that $S_1 = \sigma_0$. In terms of ordered partitions, σ_{j-1} is the sum of all ordered j block partitions of $[n]$. To express S_a in this language, we need the notion of a weakly ordered partition of $[n]$. It is an ordered partition of $[n]$ in which the blocks are allowed to be empty. Note that there are exactly $\binom{a}{j}$ weakly ordered a block partitions that give the same ordered j block partition of $[n]$. It follows that S_a is the sum of all weakly ordered a block partitions of $[n]$. If we assign labels $1, 2, \ldots, a$ to these weakly ordered a blocks such that the leftmost block is labeled a and the rightmost is labeled 1, then we obtain the random walk description for S_a that we started with. It is clear from both viewpoints that $S_a S_b = S_{ab}$. This gives us a multiplicative shuffle semigroup $S = \{S_a : a \geq 1\}$ contained in $A = k[\sigma_1, \ldots, \sigma_{n-1}]$. It now follows that $\sigma_0 = 1, \sigma_1, \ldots, \sigma_{n-1}$ is a basis for A. In fact we will show that $A = k[\sigma_1] = k[S_2]$.

First we show that $A \cong k^n$. Expand each $\binom{a}{j}$ as a polynomial in a (with no constant term).

$$\binom{a}{j} = \frac{a(a-1)\ldots(a-j+1)}{j!} = \frac{1}{j!}\sum_{i=1}^{j} s(j,i)a^i$$

for some constants $s(j, i)$. To be precise, $s(j, i)$ is the coefficient of x^i in the polynomial $x(x - 1) \cdots (x - j + 1)$. These are the Stirling numbers of the first kind. Substituting the above expression in the formula for S_a and rearranging terms, we get for $a \geq 1$

$$S_a = \sum_{i=1}^{n} a^i e_i, \quad \text{where} \quad e_i = \sum_{j=i}^{n} s(j, i) \frac{\sigma_{j-1}}{j!}.$$

All the e_i's are clearly non-zero. Now apply Proposition 1 to S and A along with the n characters of S given by $\chi_i(S_a) = a^i$ for $i = 1, 2, \ldots, n$. This gives $A \xrightarrow{\cong} k^n$. The isomorphism maps S_a to $(a^1, a^2, \ldots, a^n)$. These give the possible set of eigenvalues of S_a considered as an operator on any A-module. They are distinct for $a \neq 1$ and hence S_a generates A for all $a \geq 2$. In particular, we obtain $A = k[\sigma_1] = k[S_2]$. Hence A is a shuffle algebra.

Also the minimal polynomial for S_a on any faithful A-module is given by $(x - a)(x - a^2) \cdots (x - a^n)$. Write it as $\sum_{i=0}^{n} P_{n,i} x^i$. Now if we substitute $x = S_a$ and use equation (5) then we obtain identities involving binomial coefficients, namely $\sum_{i=0}^{n} P_{n,i} \binom{a^i}{j} = 0$ for $j = 1, 2, \ldots, n$. It would be nice to have an explicit formula for the coefficients $P_{n,i}$ also.

The analysis that we gave for this example was the analogue of the shuffle method of the previous two examples. The direct computational method of those examples is not feasible here. Writing a formula for $\sigma_j \sigma_1$ is not easy because of the multiplicative nature of the example.

3.4. **Connection with Fulman's work.** Let W be a finite irreducible Coxeter group of rank n. Let Σ be its Coxeter complex, L its associated lattice and supp : $\Sigma \to L$ the support map (Appendix A.4). Fulman [16] considered the following weight distribution (for $x \neq 0$ a real number) on the faces of Σ.

$$w_F = \frac{\chi(L_{\leq \text{supp} F}, x)}{x^n |\text{supp}^{-1}(\text{supp} F)|}.$$

Here the numerator is the characteristic polynomial of the sublattice $L_{\leq \text{supp} F}$ of L consisting of elements whose support is less than $\text{supp} F$. The formula above is a restatement of [16, Theorem 6] in geometric language. Further Fulman shows that the random walk associated to this weight distribution has eigenvalues $\frac{1}{x^i}$ for $0 \leq i \leq n$ [16, Theorem 5].

Two of our examples, namely the ones in Sections 3.3 and 4.2(odd part), arise in this way. We only explain the first example. The second one is similar. We begin by noting that for the Coxeter groups of types A and B, the characteristic polynomial has a simple form, namely,

$$\chi(L_{\leq \text{supp} F}, x) = \Pi_{i=0}^{\text{rk} F - 1}(x - (ki + 1)),$$

where we put $k = 1$ for type A and $k = 2$ for type B. We concentrate on the A_{n-1} case. We substitute $x = a$, a positive integer. From the above discussion and the simple description of the support map (Appendix B.4), we have

$$w_F = \frac{(a - 1)(a - 2) \ldots (a - \text{rk} F)}{a^{n-1}(\text{rk} F + 1)!} = \frac{1}{a^n} \binom{a}{\text{rk} F + 1}.$$

If we ignore the normalization factor a^n then we see that

$$\sum_{F \in \Sigma} w_F F = \sum_{j=1}^{n} \binom{a}{j} \sigma_{j-1}.$$

From equation (5), we see that these are the inverse a-shuffles S_a of Section 3.3. Also note that the eigenvalues we got, namely, a^i for $1 \leq i \leq n$ coincide with Fulman's, $\frac{1}{a^i}$ for $0 \leq i \leq n-1$, up to the normalization factor a^n.

4. Examples of type B_n

We consider two examples, the side shuffle and the riffle shuffle. They should not be regarded as mere generalizations of the ones discussed so far. In fact, we will see in Section 6 that they are in a sense more fundamental and have a right to exist on their own. In this section, the term partition always refers to a partition of type B_n; see Appendix B.5. Also elements of the label set I will be written as $s_1, s_2, \ldots, s_n = t$; see Figure 3 in Appendix B.3. We recall that the random walk operates on a signed deck of cards. It is a deck where each card is either face up or face down.

4.1. **The side shuffle.** The element of interest is $\sigma = \sigma_{s_1}$. In terms of partitions, σ_{s_1} is the sum of all three block partitions such that the first (and hence the last) block is a singleton. The associated random walk on a deck of n signed cards consists of removing a card at random and replacing it on top with either the same sign or its reverse both with equal probability. Note that the normalization factor is $2n$.

In addition to $\sigma = \sigma_1$, we define $\sigma_j = \sigma_{J_j}$ where $J_j = \{s_1, \ldots, s_j\} \subseteq I$ for $j = 1, \ldots, n$. To explain in words, σ_1 is the sum of all the vertices of type s_1, σ_2 is the sum of all the edges of type $s_1 s_2$ and so on till σ_n which is the sum of all chambers of Σ. As partitions, σ_j is the sum of all $(2j+1)$ block partitions such that the first (and hence the last) j blocks are singletons. Define a-shuffles for $a \geq 0$ by

$$S_a = \sum_{j=0}^{n} S(a, j) \sigma_j,$$

where $S(a, j)$ are the signed Stirling numbers defined in Section 3.2. The associated random walks on a deck of n signed cards are as follows:

σ_j : Choose j cards at random. Move them to the top in a random order. Now with equal probability either flip or do not flip the sign of the chosen cards.

S_a : Pick a card at random and with equal probability either flip or do not flip its sign. Repeat this process a times. There is no restriction on the number of times the same card may get picked. Move all the picked cards to the top in a random order.

The interpretation of S_a involves the same kind of subtlety as the two sided shuffle of type A_{n-1} discussed in Section 3.2. We observe that $S_a S_b = S_{a+b}$. After this, the analysis works exactly like the two sided shuffle for type A_{n-1}. The only difference is that $e_{n-1} \neq 0$ since we do not have any relation like $\sigma_n = 2\sigma_{n-1}$. The algebra $A = k[\sigma_1]$ is a shuffle algebra and $A \xrightarrow{\cong} k^{n+1}$, where S_a maps to $(0^a, 2^a, 4^a, \ldots, (2n-2)^a, (2n)^a)$.

Like the additive examples of type A_{n-1}, we can also use the direct method here. Using the partition description of the σ's, it is easy to see that $\sigma_j\sigma_1 = 2j\sigma_j + \sigma_{j+1}$ if $j \leq n-1$ and $\sigma_n\sigma_1 = 2n\sigma_n$ and the rest is similar.

4.2. **The riffle shuffle.** This example is motivated by the riffle shuffle of type A_{n-1}. For riffle shuffling a signed deck of cards, we first cut the deck into two parts, making the cut according to a binomial probability distribution, then turn the second part face up and then interleave the two parts in such a way that every interleaving is equally likely. For the a-shuffle we cut the deck into a parts, then turn the parts in even positions face up and then interleave them.

The algebra generated by these shuffles has been considered by F. Bergeron and N. Bergeron [3, Section 6]. They also gave an explicit description of the 3-shuffle.

We now describe the inverse a-shuffles S_a. It is convenient to split the description into two cases depending on the parity of a. They can be described using the inverse a-shuffles of Section 3.3 and a special shuffle S_2. We will call S_2 the inverse *signed* riffle shuffle or the inverse riffle shuffle of type B_n.

S_2: For every card, we either flip or do not flip its sign with equal probability. The cards with unchanged signs move to the top in the same relative order and the rest move to the bottom in the reverse relative order.

S_{2a}: We do a usual inverse a-shuffle with labels $1, 2, \ldots, a$. Then within each of the a blocks with a fixed label we do an inverse signed riffle shuffle S_2.

S_{2a+1}: We do a usual inverse $(a+1)$-shuffle with labels $0, 1, 2, \ldots, a$. Then we do an inverse signed riffle shuffle S_2 on each block except the one labeled 0.

These shuffles are multiplicative. One can check directly that $S_a S_b = S_{ab}$ for any $a, b \geq 1$ irrespective of parity. However, we will derive this relation by considering the shuffles S_a as elements of $(k\Sigma)^W$. It is natural to first analyze the even and odd shuffles separately and then to put them together later. This will yield two shuffle algebras and a double shuffle algebra.

The even part.

The element of interest is $\sigma_1 = \sigma_t$. In terms of partitions, σ_t is the sum of all three block partitions such that the second (zero) block is empty.

If we try to analyze σ_1^2, σ_1^3 and so on then we observe that they involve only those partitions that have an empty zero block. In geometric language, they involve only those faces whose type contains the letter t. This motivates the definition of σ_j which we now give. Put $\sigma_j = \sum_{|J|=j, t\in J} \sigma_J$ for $j = 1, 2, \ldots, n$. In terms of partitions, σ_j is the sum of all $(2j + 1)$ block partitions such that the zero block is empty. In the spirit of the riffle shuffle of type A_{n-1}, we define shuffles S_{2a} for $a \geq 1$ by

$$(6) \qquad S_{2a} = \sum_{j=1}^{n} \binom{a}{j}\sigma_j.$$

Note that $S_2 = \sigma_1$. Also put $S_1 = \sigma_0 = 1$. To express S_{2a} in words, we use the notion of a weak partition that we defined in Appendix B.5. It is a partition in which the signed blocks are also allowed to be empty. Since the partitions are anti-symmetric, there are exactly $\binom{a}{j}$ weak $(2a + 1)$ block partitions that give the same $(2j + 1)$ block partition. It follows that S_{2a} is the sum of all weak $(2a + 1)$ block partitions such that the zero block is always empty; so we have up to $2a$ non-empty blocks. This explains the term "$2a$-shuffle". If we assign labels $1, 2, \ldots, a$

in decreasing order to the first a blocks with the leftmost (signed) block labeled a, then we obtain the random walk description for S_{2a} that we started with.

With the partition description, it is immediate that $S_{2a}S_{2b} = S_{4ab}$. This gives us a multiplicative shuffle semigroup $\{S_{2a} : a > 0\} \cup S_1$ contained in $k[\sigma_1, \ldots, \sigma_n]$. It now follows that $\sigma_0 = 1, \sigma_1, \ldots, \sigma_n$ is a basis for $k[\sigma_1, \ldots, \sigma_n]$.

The odd part.

This case is completely analogous to the even case, the only difference being that now we put no restriction on the zero block. We begin with the element $\sigma_1' = \sigma_{s_1} + \cdots + \sigma_{s_n}$; that is, σ_1' is the sum of all the vertices in Σ. We also define σ_j' as the sum of all the simplices of rank j in Σ for $j = 1, \ldots, n$. In terms of partitions, σ_j' is the sum of all $(2j + 1)$ block partitions. Note that $\sigma_n' = \sigma_n$. Also put $\sigma_0' = \sigma_0 = 1$. Next define shuffles S_{2a+1} for $a \geq 0$ by

$$(7) \qquad\qquad S_{2a+1} = \sum_{j=0}^{n} \binom{a}{j} \sigma_j'.$$

To express S_{2a+1} in words, we again use the notion of a weak partition. Exactly as in the even case, it follows that S_{2a+1} is the sum of all weak $(2a + 1)$ block partitions. The random walk description is obtained by assigning labels $0, 1, \ldots, a$ in decreasing order to the first $(a+1)$ blocks with the leftmost (signed) block labeled a.

With the partition description it is immediate that $S_{2a+1}S_{2b+1} = S_{(2a+1)(2b+1)}$. This gives us a multiplicative shuffle semigroup $\{S_{2a+1} : a \geq 0\}$ contained in $k[\sigma_1', \ldots, \sigma_n']$. It now follows that $\sigma_0' = 1, \sigma_1', \ldots, \sigma_n'$ is a basis for $k[\sigma_1', \ldots, \sigma_n']$.

Even + odd.

We now put the even and odd parts together. Define $S = \{S_{2a} : a \geq 1\} \cup \{S_{2a+1} : a \geq 0\}$. It is immediate from the descriptions of S_{2a} and S_{2a+1} in terms of partitions that $S_a S_b = S_{ab}$ for all $a, b \geq 1$ irrespective of parity. Thus we get a multiplicative shuffle semigroup S as a spanning set in $A = k[\sigma_1, \ldots, \sigma_n, \sigma_1', \ldots, \sigma_n']$. Hence A is a $2n$-dimensional algebra with basis $\sigma_0 = 1, \sigma_1, \sigma_1', \ldots, \sigma_{n-1}, \sigma_{n-1}', \sigma_n = \sigma_n'$.

We now show that $A \cong k^{2n}$. Following the example of the riffle shuffle of type A_{n-1}, we write equation (6) as

$$(8) \qquad S_{2a} = \sum_{i=1}^{n} (2a)^i e_i, \quad \text{where} \quad e_i = \sum_{j=i}^{n} s(j, i) \frac{\sigma_j}{2^j j!}.$$

Here $s(j, i)$ is the coefficient of x^i in the polynomial $x(x-2)(x-4)\cdots(x-2(j-1))$. It is 2^{j-i} times the $s(j, i)$ that occurred in the riffle shuffle of type A_{n-1}. For notational convenience, we set $e_0 = 0$. Similarly, we rewrite equation (7) as

$$(9) \qquad S_{2a+1} = \sum_{i=0}^{n} (2a + 1)^i e_i', \quad \text{where} \quad e_i' = \sum_{j=i}^{n} s(j, i)' \frac{\sigma_j'}{2^j j!}.$$

Here $s(j, i)'$ is the coefficient of x^i in the polynomial $(x-1)(x-3)\cdots(x-(2j-1))$. Note that $e_n' = e_n = \frac{\sigma_n}{2^n n!}$. We next define two different kinds of characters of S. For any non-negative integer i, define $\chi_i(S_a) = a^i$ for $a \geq 1$. Also define $\chi_i'(S_{2a}) = 0$ for $a \geq 1$ and $\chi_i'(S_{2a+1}) = (2a + 1)^i$ for $a \geq 0$. Now observe that for any $s \in S$

$$s = \sum_{i=1}^{n} \chi_i(s) e_i + \sum_{i=0}^{n} \chi_i'(s)(e_i' - e_i).$$

This can be checked separately for $s = S_{2a}, S_{2a+1}$ using equations (8) and (9). The second summation actually goes only till $n - 1$, since $e_n' = e_n$. Now apply Proposition 1 to S and A along with the $2n$ characters χ_i for $i = 1, \ldots, n$ and χ_i' for $i = 0, 1, \ldots, n - 1$ to conclude that $A \cong k^{2n}$.

The shuffle algebras.

It is clear that $k[\sigma_1, \ldots, \sigma_n] \xrightarrow{\cong} k^{n+1}$, where the $n + 1$ factors correspond to the idempotents $e_0', e_1, e_2, \ldots, e_n$. The isomorphism maps the shuffles S_{2a} to $(0, (2a)^1, (2a)^2, \ldots, (2a)^n)$. This shows that $k[\sigma_1, \ldots, \sigma_n] = k[\sigma_1]$ is a shuffle algebra.

To see that $k[\sigma_1', \ldots, \sigma_n'] \cong k^{n+1}$, first note that $\chi_i'(s) = \chi_i(s)$ for $s \in \{S_{2a+1} : a \geq 0\}$. Hence we need to lump together the factors corresponding to e_i and $e_i' - e_i$. Thus we get $k[\sigma_1'] \xrightarrow{\cong} k^{n+1}$, with $e_0', e_1', e_2', \ldots, e_n'$ as the primitive idempotents. The isomorphism maps S_{2a+1} to $((2a + 1)^0, (2a + 1)^1, (2a + 1)^2, \ldots, (2a + 1)^n)$. Hence $k[\sigma_1', \ldots, \sigma_n'] = k[\sigma_1']$ is also a shuffle algebra.

It also follows that $A = k[\sigma_1, \sigma_1']$. It is not a shuffle algebra because the first three conditions are each violated by a factor of two. For example, in condition (3), instead of a single generator σ_1, we have two generators σ_1 and σ_1'. Hence we refer to it as a double shuffle algebra.

5. Examples of type D_n

We consider two examples, the side shuffle and the riffle shuffle. In this section, the term partition always refers to a partition of type D_n; see Appendix B.6. Elements of the label set I will be written as $s_1, s_2, \ldots, s_{n-2}, u, v$; see Figure 4 in Appendix B.3. We also recall that the random walk operates on an almost signed deck of cards or a deck of type D_n. It is a deck in which every card, except the bottommost, is signed. Also, one of the two basic algebras of Section 5.2 will not strictly satisfy condition (2) in the definition of a shuffle algebra. This is relevant to the remark after Theorem 1.

5.1. **The side shuffle.** The element of interest is $\sigma = \sigma_{s_1}$. In terms of partitions, σ_{s_1} is the sum of all three block partitions such that the first (and hence the last) block is a singleton. The associated random walk on an almost signed deck of n cards consists of removing a card at random and replacing it on top with either the same sign or its reverse both with equal probability. If the bottommost unsigned card is chosen then we give it a sign and then erase the sign on the new bottommost card.

In addition to $\sigma = \sigma_1$, we define $\sigma_j = \sigma_{J_j}$ where $J_j = \{s_1, \ldots, s_j\} \subseteq I$ for $j = 1, \ldots, n - 2$. Note that the choice for σ_{n-1} is not immediately obvious. We set $\sigma_{n-1} = \sigma_{\{s_1, \ldots, s_{n-1}, u, v\}}$. The motivation becomes clearer from the following. As partitions, σ_j is the sum of all $(2j + 1)$ block partitions such that the first (and hence the last) j blocks are singletons. Observe that for $j = n - 1$, this does give us the above definition of σ_{n-1}. We also set $\sigma_n = 2\sigma_{n-1}$. Define a-shuffles for $a \geq 0$ by

$$S_a = \sum_{j=0}^{n} S(a, j)\sigma_j,$$

where $S(a, j)$ are the signed Stirling numbers defined in Section 3.2. The associated random walks on a deck of type D_n are as follows.

σ_j : Choose j cards at random. Move them to the top in a random order. Now with equal probability either flip or do not flip the sign of the chosen cards. If the bottommost unsigned card is one of the chosen cards then we give it a sign and then erase the sign on the new bottommost card.

S_a : Assign the sign $+$ to the bottommost card. Now do the side shuffle S_a of type B_n. Drop the sign of the bottommost card.

Observe that $\sigma_n = 2\sigma_{n-1}$. We did not have any such relation for the side shuffle of type B_n. It is interesting how this relation emerges by just making the bottommost card unsigned. Also observe that $S_a S_b = S_{a+b}$. At this point, it is clear that the rest of the analysis is identical to the two sided shuffle for type A_{n-1} and we omit it.

5.2. **The riffle shuffle.** The shuffles S_a in this example can be described in the same way as the a-shuffles of type B_n with a minor modification. This is because we are now operating on a deck of type D_n rather than of type B_n. The difference between the two is that in the former the bottommost card is unsigned. For the a-shuffle of type D_n, we first assign a sign to the bottommost card of the deck. Then we do an a-shuffle of type B_n. And finally, we drop the sign of the bottommost card. We now describe S_a, the inverse a-shuffle of type D_n.

S_a : Assign the sign $+$ to the bottommost card. Now do an inverse a-shuffle of type B_n. Drop the sign of the bottommost card.

Note that the two descriptions are inverses of each other. However the normalization factors (Section 1.2) differ by a factor of 2. We mention that the geometric description that will follow matches the factor computed from the second description. We now give another equivalent description of the inverse shuffles using the analogy with the type B situation rather than using it directly. For that, we first define the inverse riffle shuffle of type D_n.

For every signed card, we either flip or do not flip its sign with equal probability. The cards whose signs were flipped move below the unsigned card (in the reverse relative order). The rest stay on top.

Note that after this shuffle, we may not have a deck of type D_n, since the unsigned card is not necessarily at the bottom. Hence we define a *correction* operation which assigns a sign to the unsigned card and then erases the sign of the bottommost card. We now give a description of our shuffles as operators on a deck of type D_n.

S_{2a}: We do a usual inverse a-shuffle with labels $1, 2, \ldots, a$. Then within each block with a fixed label we do an inverse riffle shuffle of type D_n or B_n depending on whether the block contains the unsigned card or not. Lastly, we do the correction.

S_{2a+1}: We do a usual inverse $(a+1)$-shuffle with labels $0, 1, 2, \ldots, a$. Then we repeat the above on each block except the one labeled 0 and then do the correction.

Note that S_2 is just an inverse riffle shuffle of type D_n followed by the correction.

The even part.

The element of interest is $\sigma_1 = \sigma_u + \sigma_v$. In terms of partitions, σ_1 is the sum of all three block partitions such that the central block is empty.

Put $\sigma_j = \sum_{|J|=j, u \in J} \sigma_J + \sum_{|J|=j, v \in J} \sigma_J$ for $j = 1, 2, \ldots, n$. Explicitly for $n = 3$, we have $\sigma_1 = \sigma_u + \sigma_v, \sigma_2 = \sigma_{\{s,u\}} + \sigma_{\{s,v\}} + 2\sigma_{\{u,v\}}, \sigma_3 = 2\sigma_{\{s,u,v\}}$. Note that we consider only those faces whose type contains either u or v. If it contains both u and v then we put a coefficient of 2 in front. In terms of partitions, σ_j is the sum of all $(2j + 1)$ block partitions such that the central block is empty. In this

description, we follow the convention that the partitions $(\{1,\overline{2}\},\{3\},\{\},\{\overline{3}\},\{2,\overline{1}\})$ and $(\{1,\overline{2}\},\{\overline{3}\},\{\},\{3\},\{2,\overline{1}\})$ are counted separately, even though they are both equal to $(\{1,\overline{2}\},\{3,\overline{3}\},\{2,\overline{1}\})$. This explains the coefficient of 2 in front of faces whose type contains both u and v.

The odd part.

The element of interest is $\sigma_1' = (\sigma_u + \sigma_v) + (\sigma_{s_1} + \ldots + \sigma_{s_{n-2}}) + \sigma_{\{u,v\}}$. Note that in contrast to all our examples so far, σ_1' is a combination of elements of $k\Sigma_1$ and $k\Sigma_2$; that is, it contains elements of different ranks. Next we define $\sigma_j' = \sigma_j + \sum_{|J|=j, u \notin J, v \notin J} \sigma_J + \sum_{|J|=j+1, \{u,v\} \subset J} \sigma_J$ for $j = 1, 2, \ldots, n$. Here σ_j is as defined in the even case. Thus we see that σ_j' is a combination of elements of $k\Sigma_j$ and $k\Sigma_{j+1}$. This is the only example where the geometric description of the objects of interest is somewhat complicated. For better motivation, we now consider partitions. We have already interpreted σ_j in terms of partitions. Note that the two summation terms that we added give the sum of all $(2j + 1)$ block partitions with a non-empty central block. Hence we see that σ_j' is the sum of all $(2j + 1)$ block partitions. While counting partitions, we adopt the same convention as in the even case.

The shuffle algebras.

We define shuffles S_{2a} and S_{2a+1} by equations (6) and (7) and the analysis is identical word for word to the case of B_n. We only point out that partition now means partition of type D_n and σ_j and σ_j' refer to the definitions that we made above. The shuffles can be described using weak partitions and this leads to the two random walk interpretations that we gave earlier. They are slightly complicated now because of the more involved nature of the partitions of type D_n. We also conclude that $k[\sigma_1]$ and $k[\sigma_1']$, apart for condition (2), are shuffle algebras. And we may refer to $k[\sigma_1, \sigma_1']$ as a double shuffle algebra.

A clear conceptual explanation of the close connection between the riffle shuffles of type B_n and D_n is given in the next section.

Remark. The riffle shuffle examples in Sections 3.3 and 4.2(odd part) show that for types A_{n-1} and B_n, $\sigma_i \sigma_j = \sigma_j \sigma_i$, where σ_i is the sum of all the simplices of rank i. This is because σ_i and σ_j are elements of a commutative (shuffle) algebra. The fact that $\sigma_i \sigma_j = \sigma_j \sigma_i$ can also be proved by a direct geometric argument. The key property is that the simplicial complex induced on the support of any face of the Coxeter complex of type A_{n-1} or B_n is again of the same type. This property fails for D_n. It is incomplete in this geometric sense. Note that the σ_i's as defined above did not play any role in the riffle shuffle for D_n. In fact, their role for type D_n in this theory is far from being clear.

6. Maps between Coxeter complexes

Recall that the reflection arrangements of type B_n, D_n and A_{n-1} are given by the hyperplanes $x_i = \pm x_j, x_i = 0$; $x_i = \pm x_j$ and $x_i = x_j$ $(1 \leq i < j \leq n)$ respectively. Let $\Sigma(B_n)$, $\Sigma(D_n)$ and $\Sigma(A_{n-1})$ be the corresponding Coxeter complexes. Observe that the arrangement for D_n (resp. A_{n-1}) is obtained from the one for B_n (resp. D_n) by deleting some hyperplanes. Following the discussion in Appendix A.5, we have forgetful maps

$$\Sigma(B_n) \to \Sigma(D_n) \to \Sigma(A_{n-1}).$$

These are semigroup homomorphisms and they induce algebra homomorphisms $k\Sigma(B_n) \to k\Sigma(D_n) \to k\Sigma(A_{n-1})$. Further we will see that the maps restrict to the corresponding descent algebras (see Section 1.2). We will use the language of partitions to make these maps more explicit. A generalization of the first map to buildings will be given in Section 8.3. A different set of maps will be discussed in Section 7.
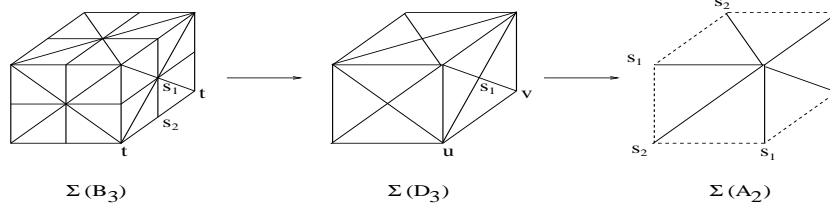

FIGURE 1. The case $n = 3$.

Figure 1 shows the intersection of the hyperplane arrangements for B_3, D_3 and A_2 with the boundary of a cube centered at the origin. For B_3 and D_3, this gives us the corresponding Coxeter complex. For A_2, we do not quite get $\Sigma(A_2)$ because the braid arrangement is not essential. So we cut the braid arrangement by the hyperplane $x_1 + x_2 + x_3 = 0$. The complex $\Sigma(A_2)$ can be seen in Figure 1 as the dotted hexagon with three vertices each of type s_1 and s_2. Also in Figure 1, we have labeled only some of the vertices since they uniquely determine the remaining labels.

6.1. **The map $\Sigma(B_n) \to \Sigma(D_n)$.** For the case B_n (resp. D_n), we split the partition description for a face F into 4 (resp. 3) cases; see Appendix B.5 and B.6. From those descriptions, it is straightforward to describe the map. Let $K \subset \{s_1, \ldots, s_{n-2}\}$.

   (i) A face of type K in $\Sigma(B_n)$ maps to a face of type K in $\Sigma(D_n)$.
  (ii) A face of type $K \cup t$ in $\Sigma(B_n)$ maps to a face either of type $K \cup u$ or $K \cup v$ depending on a parity condition.
 (iii) A face of type $K \cup s_{n-1}$ in $\Sigma(B_n)$ maps to a face of one higher dimension whose type is $K \cup \{u, v\}$. For $n = 3$, this is also clear from Figure 1, where we see that $\{s_2\} \mapsto \{u, v\}$ and $\{s_1, s_2\} \mapsto \{s_1, u, v\}$.
  (iv) A face of type $K \cup \{s_{n-1}, t\}$ in $\Sigma(B_n)$ maps to a face of type $K \cup \{u, v\}$.

The maps in the first three cases are bijective. More precisely, the faces of type K in $\Sigma(B_n)$ are in bijection with the faces of type K in $\Sigma(D_n)$ and so on. However in the last case, the map is two to one. This is again illustrated in Figure 1, where we see that there are two faces of type $\{s_1, s_2, t\}$ which map to a face of type $\{s_1, u, v\}$.

Recall that σ_J is the sum of all faces of type J. From the above discussion it follows that for the map $k\Sigma(B_n) \to k\Sigma(D_n)$, we have $\sigma_K \mapsto \sigma_K$, $\sigma_{K \cup s_{n-1}} \mapsto \sigma_{K \cup \{u,v\}}$, $\sigma_{K \cup t} \mapsto \sigma_{K \cup u} + \sigma_{K \cup v}$ and $\sigma_{K \cup \{s_{n-1}, t\}} \mapsto 2\sigma_{K \cup \{u,v\}}$.

6.2. **The map $\Sigma(B_n) \to \Sigma(A_{n-1})$.** It is easier to describe this composite map than the map from $\Sigma(D_n)$ to $\Sigma(A_{n-1})$. In contrast to the previous map, this composite map is hard to describe using face types but easy to describe using partitions. To obtain an ordered partition of $[n]$, starting with an anti-symmetric ordered partition of $[n, \overline{n}]$, we simply forget the set $[\overline{n}]$. For example, $(\{2\}, \{\overline{3}\}, \{1, \overline{1}\}, \{3\}, \{\overline{2}\})$ maps to $(\{2\}, \{1\}, \{3\})$. We explain some special cases.

Observe that a face of type s_1 maps to a face either of type s_1 or s_{n-1} depending on whether the element in the singleton first block has a plus sign or minus. It follows that for the map $k\Sigma(B_n) \to k\Sigma(A_{n-1})$, σ_{s_1} maps to $\sigma_{s_1} + \sigma_{s_{n-1}}$. This is the primary reason why the side shuffle of type B_n is more closely related to the two-sided shuffle (rather than the side shuffle) of type A_{n-1}.

Next we show that inverse a-shuffles map to inverse a-shuffles. For that observe that weak $(2a+1)$ block partitions of type B_n correspond to weak $(2a+1)$ block partitions of type A_{n-1}. For example, $(\{\overline{2}\}, \{\}, \{1, \overline{1}\}, \{\}, \{2\}) \leftrightarrow (\{\}, \{\}, \{1\}, \{\}, \{2\})$. Similarly, weak $(2a + 1)$ block partitions of type B_n with an empty zero block correspond to weak $2a$ block partitions of type A_{n-1}. For example, when $a = 1$, we get that σ_t maps to $\sigma_{s_1} + \ldots + \sigma_{s_{n-1}} + 2\sigma_0$. In other words, the inverse riffle shuffle of type B_n maps to the inverse riffle shuffle of type A_{n-1}.

6.3. Maps between side shuffles.
We recall the definitions of σ_j for the side shuffles of type B_n and D_n and the two-sided shuffle of type A_{n-1}.

$B_n\ : \sigma_j = \sigma_{J_j}$ where $J_j = \{s_1, \ldots, s_j\} \subseteq I$ for $j = 1, \ldots, n$.

$D_n\ : \sigma_j = \sigma_{J_j}$ where $J_j = \{s_1, \ldots, s_j\} \subseteq I$ for $j = 1, \ldots, n-2$ and $J_{n-1} = \{s_1, \ldots, s_{n-1}, u, v\}$. Also $\sigma_n = 2\sigma_{n-1}$.

$A_{n-1}\ : \sigma_j = \sum_{k=0}^{j} \binom{j}{k} \sigma_{J_{j,k}}$ where $J_{j,k} = \{s_1, \ldots, s_k\} \cup \{s_{n-(j-k)}, \ldots, s_{n-1}\} \subseteq I$ for $j = 1, \ldots, n-1$ and $\sigma_n = 2\sigma_{n-1}$.

We claim that $\sigma_j \mapsto \sigma_j \mapsto \sigma_j$ under the maps $k\Sigma(B_n) \to k\Sigma(D_n) \to k\Sigma(A_{n-1})$. From the discussion in Section 6.1, it is easy to see the claim for the first map. A point worth noting is that $\sigma_n \mapsto 2\sigma_{n-1}$ and the claim holds for $j = n$ because of the relation $\sigma_n = 2\sigma_{n-1}$ for D_n. Similarly, using the discussion in Section 6.2, one can see the claim for the composite map. In earlier sections, the introduction of σ_n and the relation $\sigma_n = 2\sigma_{n-1}$ for types D_n and A_{n-1} was somewhat artificial and justified only from the random walk perspective. However we now also see it from a geometric perspective.

Recall that the shuffles are given by the same formula $S_a = \sum_{j=0}^{n} S(a, j)\sigma_j$ in all three cases, with $S(a, j)$ being the signed Stirling numbers. Hence we get $S_a \mapsto S_a \mapsto S_a$. To summarize, we have:

$$\left\{ \begin{array}{c} \text{side shuffle} \\ \text{of type } B_n \end{array} \right\} \quad \twoheadrightarrow \quad \left\{ \begin{array}{c} \text{side shuffle} \\ \text{of type } D_n \end{array} \right\} \quad \overset{\cong}{\longrightarrow} \quad \left\{ \begin{array}{c} \text{two sided shuffle} \\ \text{of type } A_{n-1} \end{array} \right\}$$
$$\sigma_j \qquad \mapsto \qquad \sigma_j \qquad \mapsto \qquad \sigma_j$$
$$S_a \qquad \mapsto \qquad S_a \qquad \mapsto \qquad S_a$$

6.4. Maps between riffle shuffles.
Recall the definitions of σ_j and σ_j' for the riffle shuffles of type B_n and D_n from Sections 4.2 and 5.2. Following the discussion in Section 6.1, it is clear that $\Sigma(B_n) \to \Sigma(D_n)$ maps σ_j to σ_j and σ_j' to σ_j'. So the map restricted to the algebras $A = k[\sigma_1, \sigma_1']$ is an algebra map that maps a basis to a basis. Hence it is an isomorphism. It also clearly maps S_a to S_a. Again we see that the somewhat unmotivated definitions of σ_j and σ_j' for D_n have a more natural meaning as the images of the corresponding elements in $\Sigma(B_n)$.

For the map $\Sigma(B_n) \to \Sigma(A_{n-1})$, the discussion at the end of Section 6.2 and the interpretation of the inverse a-shuffles S_a as weak partitions shows that $S_a \mapsto S_a$. This gives us a surjective map $A = k[\sigma_1, \sigma_1'] = k[S_2, S_3] \to A = k[S_2]$. However we do not know a good way to describe the images of σ_j and σ_j' under this map. To

summarize, we have:

$$
\left\{ \begin{array}{c} \text{riffle shuffle} \\ \text{of type } B_n \end{array} \right\} \xrightarrow{\cong} \left\{ \begin{array}{c} \text{riffle shuffle} \\ \text{of type } D_n \end{array} \right\} \twoheadrightarrow \left\{ \begin{array}{c} \text{riffle shuffle} \\ \text{of type } A_{n-1} \end{array} \right\}
$$

$$
\begin{array}{ccccc}
\sigma_j & \mapsto & \sigma_j & \mapsto & ? \\
\sigma'_j & \mapsto & \sigma'_j & \mapsto & ? \\
S_a & \mapsto & S_a & \mapsto & S_a
\end{array}
$$

Remark. The Coxeter complex of type B_n is a good place to look for shuffle algebras. The images of these shuffle algebras under our maps would then yield shuffle algebras (in a suitably generalized sense) of type D_n and A_{n-1}. Among the examples in this paper, the only one that we failed to describe in this manner was the side shuffle of type A_{n-1}.

7. Shuffles on cards with an involution and a joker

The previous section shows that the D_n shuffles could have been found from the B_n shuffles had we not known them previously. Though we did not obtain new shuffles in Section 6, we can apply the same principle to obtain some new shuffles of type A_n, A_{2n-1} and A_{2n}.

7.1. **The maps.** The maps which give rise to these shuffles can be assembled as follows.

(10)
$$
\begin{array}{ccccc}
\Sigma(B_n) & \longrightarrow & \Sigma(A_n) & \longrightarrow & \Sigma(A_{n-1}) \\
\| & & \uparrow & & \uparrow \\
\Sigma(B_n) & \longrightarrow & \Sigma(A_{2n}) & \longrightarrow & \Sigma(A_{2n-1})
\end{array}
$$

The composite map $\Sigma(B_n) \hookrightarrow \Sigma(A_{2n-1})$ and its generalization to buildings (Section 8.2) was pointed out by Ken Brown.

7.1.1. *The maps $\Sigma(B_n) \to \Sigma(A_n) \to \Sigma(A_{n-1})$.* Comparing with the maps in Section 6, we have replaced the middle object $\Sigma(D_n)$ by $\Sigma(A_n)$. The composite map, however, will be the same as before. To define the two intermediate maps, we view $\Sigma(A_n)$ a little differently.

Consider the braid arrangement in $\mathbb{R}^{n+1}$ which corresponds to the Coxeter complex $\Sigma(A_n)$. The arrangement is not essential because the intersection of all the hyperplanes is the line $x_1 = x_2 = \ldots = x_{n+1}$. The standard way to make the arrangement essential is to take the quotient of $\mathbb{R}^{n+1}$ by this line (Appendix A.6). Another way is to cut it by the hyperplane $x_{n+1} = 0$. The choice of this hyperplane is somewhat arbitrary and hence the action of S_{n+1} is now a little awkward. Nevertheless we obtain an essential arrangement given by the hyperplanes $x_i = x_j$ and $x_i = 0$, where $1 \leq i < j \leq n$. We study $\Sigma(A_n)$ using this arrangement. It is most natural to identify the chambers in $\Sigma(A_n)$ with a deck with $n + 1$ cards, one of which we regard as special and call the joker. And refer to the remaining cards as playing cards.

First note that deleting the hyperplanes $x_i = -x_j$ from the arrangement for B_n gives this arrangement. Also deleting the hyperplanes $x_i = 0$ from this one gives the usual braid arrangement for A_{n-1}. Hence following Appendix A.5, we get forgetful maps $\Sigma(B_n) \to \Sigma(A_n) \to \Sigma(A_{n-1})$. They can be described using the language of partitions.

The map $\Sigma(B_n) \to \Sigma(A_n)$. Starting with an anti-symmetric ordered partition of $[n, \overline{n}]$, we simply forget the set $[\overline{n}]$ and add the element $n + 1$ to the zero block. This gives us an ordered partition of $[n + 1]$. The map is non-decreasing on ranks.

The map $\Sigma(A_n) \to \Sigma(A_{n-1})$. Given an ordered partition of $[n + 1]$, we omit the element $n + 1$ and obtain an ordered partition of $[n]$. If the element $n + 1$ occurred as a singleton block then we also delete the block. The map either preserves rank or decreases it by 1.

We give two examples illustrating the maps $\Sigma(B_3) \to \Sigma(A_3) \to \Sigma(A_2)$.

$$(\{2\}, \{\overline{3}\}, \{1, \overline{1}\}, \{3\}, \{\overline{2}\}) \;\mapsto\; (\{2\}, \{1, 4\}, \{3\}) \;\mapsto\; (\{2\}, \{1\}, \{3\}).$$
$$(\{2, \overline{3}\}, \{1, \overline{1}\}, \{3, \overline{2}\}) \;\mapsto\; (\{2\}, \{1, 4\}, \{3\}) \;\mapsto\; (\{2\}, \{1\}, \{3\}).$$

In the first example, the rank is preserved by both maps; whereas in the second, the first map increases rank by one. This is because we have a block, namely $\{2, \overline{3}\}$ containing both positive and negative numbers.

7.1.2. *The maps $\Sigma(B_n) \to \Sigma(A_{2n}) \to \Sigma(A_{2n-1})$.* Consider $\mathbb{R}^{2n}$ with coordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. Such paired coordinates arise from vector spaces with involution. Using this setup, the faces of $\Sigma(A_{2n-1})$, which correspond to the braid arrangement in $\mathbb{R}^{2n}$, can be described as ordered partitions on the set $[\overline{n}, \overline{1}] \cup [1, n]$. The chambers are a deck of cards with an involution.

Similarly, by considering coordinates $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, x_{n+1} = y_{n+1}$, the faces of $\Sigma(A_{2n-1})$, can be described as ordered partitions on the set $[\overline{n}, \overline{1}] \cup [1, n] \cup \{n + 1\}$. The chambers are a deck of cards with an involution and a joker.

The map $\Sigma(B_n) \to \Sigma(A_{2n})$ replaces 0 by $n+1$ and the map $\Sigma(A_{2n}) \to \Sigma(A_{2n-1})$ deletes $n + 1$. The vertical maps in diagram (10) are defined similarly. It follows directly from the definition that they are all semigroup homomorphisms.

7.2. **The shuffles.** We describe in detail the joker shuffles that arise from the maps $\Sigma(B_n) \to \Sigma(A_n) \to \Sigma(A_{n-1})$.

7.2.1. *The two sided joker shuffle.* Using Section 6 as a guide, we may draw the following picture.

$$\left\{ \begin{array}{c} \text{side shuffle} \\ \text{of type } B_n \end{array} \right\} \quad \xrightarrow{\;\cong\;} \quad \left\{ \begin{array}{c} \text{two sided joker} \\ \text{shuffle of type } A_n \end{array} \right\} \quad \twoheadrightarrow \quad \left\{ \begin{array}{c} \text{two sided shuffle} \\ \text{of type } A_{n-1} \end{array} \right\}$$
$$\begin{array}{ccccc} \sigma_j & \mapsto & \sigma_j & \mapsto & \sigma_j \\ S_a & \mapsto & S_a & \mapsto & S_a \end{array}$$

Note that the central column is being defined here. Observe that for the side shuffle of type B_n, any partition that plays a role consists of singleton blocks (except the zero block). Due to this special property of the partitions involved, the first map is rank preserving. This implies that it is an isomorphism. It is also clear that we have a shuffle algebra— the two sided joker shuffle algebra $k[\sigma_1]$.

In terms of ordered partitions, σ_1 is the sum of all ordered two block partitions of $[n + 1]$ such that one of the blocks is a singleton. Also the element $n + 1$ must always be in the larger of the two blocks. For example, for $n = 2$,

$$\sigma_1 = (\{1\}, \{2, 3\}) + (\{2\}, \{1, 3\}) + (\{2, 3\}, \{1\}) + (\{1, 3\}, \{2\}).$$

Observe that this element is not invariant under the action of S_3. It follows in general that the shuffle algebra $k[\sigma_1]$ is not a subalgebra of $(k\Sigma)^W$, where $\Sigma =$

$\Sigma(A_n)$ and $W = S_{n+1}$. This is the first example of this kind. Due to this, there is no nice geometric definition for σ_j.

The random walk associated to σ_1 has a similar description to the two sided shuffle, namely, we remove a playing card at random and replace it either on top or at the bottom. Note that we are not permitted to pull out the joker. The descriptions for σ_i and S_a follow the same pattern as the two sided shuffle and we omit them.

Recall that we had the relation $\sigma_n = 2\sigma_{n-1}$ for the two sided shuffle of type A_{n-1}. We no longer have such a relation. As already explained, this is clear from the geometry because the ranks of the two sides do not match. However we urge you to look at this relation from the viewpoint of probability and see how the presence of the joker breaks it down.

7.2.2. *The joker riffle shuffle.* We start with the following picture.

$$
\left\{ \begin{array}{c} \text{riffle shuffle} \\ \text{of type } B_n \end{array} \right\} \quad \overset{\cong}{\longrightarrow} \quad \left\{ \begin{array}{c} \text{joker riffle shuffle} \\ \text{of type } A_n \end{array} \right\} \quad \twoheadrightarrow \quad \left\{ \begin{array}{c} \text{riffle shuffle} \\ \text{of type } A_{n-1} \end{array} \right\}
$$

$$
\begin{array}{ccccc}
\sigma_j & \mapsto & \sigma_j & \mapsto & ? \\
\sigma'_j & \mapsto & \sigma'_j & \mapsto & ? \\
S_a & \mapsto & S_a & \mapsto & S_a
\end{array}
$$

Again the central column is being defined here. As an example, for the joker riffle shuffle of type A_2,

$$
\sigma_1 = (\{1,2\},\{3\}) + (\{3\},\{1,2\}) + (\{1\},\{3\},\{2\}) + (\{2\},\{3\},\{1\}).
$$
$$
\sigma'_1 = \sigma_1 + (\{1\},\{2,3\}) + (\{2,3\},\{1\}) + (\{2\},\{1,3\}) + (\{1,3\},\{2\}).
$$

It is not hard to see that in general σ_i and σ'_i consist of some terms of rank exactly i and other higher rank terms. Also note that in σ_i, the element $n + 1$ always appears as a singleton, whereas that is not the case for σ'_i. This shows that $\sigma_0 = 1, \sigma_1, \sigma'_1, \ldots, \sigma_{n-1}, \sigma'_{n-1}, \sigma_n = \sigma'_n$ are linearly independent in $k\Sigma(A_n)$. Hence the first map is an isomorphism. Thus we get a double shuffle algebra of type A_n. We first express the shuffles using partitions.

S_{2a}: It is the sum of all weak $2a + 1$ block partitions of $[n + 1]$ such that the middle block is always the singleton $\{n + 1\}$.

S_{2a+1}: It is the sum of all weak $2a + 1$ block partitions of $[n + 1]$ such that the element $n + 1$ always lies in the middle block.

We now discuss the shuffle descriptions. They are similar to the inverse a-shuffles except that the joker is given special treatment.

S_{2a}: Label the joker by the integer $a+1$. Then label the playing cards randomly with an integer from 1 to $2a + 1$ except $a + 1$.

S_{2a+1}: Label the joker by the integer $a + 1$. Then label the playing cards randomly with an integer from 1 to $2a + 1$.

After the labeling is done, the rest of the shuffle is same as the inverse a-shuffle.

7.2.3. *Shuffles on a deck with an involution.* The map $\Sigma(B_n) \hookrightarrow \Sigma(A_{2n-1})$ gives rise to two sided and riffle shuffles on a deck with an involution. To give some flavor, the generator σ_1 for the two sided shuffle has the following description.

Remove a card at random and replace it on top. And remove its partner (image under the involution) and replace it at the bottom.

Note that the dimension of the algebra $k[\sigma_1]$ is roughly half the dimension of $\Sigma(A_{2n-1})$. Similarly we obtain joker analogues of these shuffles on a deck with an involution and a joker. We leave the details to the reader.

8. Generalization to buildings

We first give a brief review of buildings. For the general theory, see [9, 21, 24]. Let W be a Coxeter group and $\Sigma(W)$ its Coxeter complex. Roughly, a building Δ of type W is a union of subcomplexes Σ (called apartments) which fit together nicely. Each apartment Σ is isomorphic to $\Sigma(W)$. As a simplicial complex, Δ is pure and labeled. The term *pure* means that all maximal simplices (chambers) have the same dimension. For any two simplices in Δ, there is an apartment Σ containing both of them. Using this fact, we can define a product on Δ as follows.

For $x, y \in \Delta$, we choose an apartment Σ containing x and y and define xy to be their product in Σ. Since Σ is a Coxeter complex, we know how to do this. Furthermore, it can be shown that the product does not depend on the choice of Σ. So this defines a product on Δ with the set of chambers $\mathcal{C}$ contained as an ideal. As in Section 1.1, we can now define a random walk on $\mathcal{C}$ starting with a probability distribution on Δ. Unfortunately, the product on Δ is not associative, that is, Δ is no longer a semigroup. As a result, $k\mathcal{C}$ not a module over $k\Delta$ and so our algebraic methods break down. However, we know three examples where they do work. This is because in these cases, the subalgebra of interest, namely $k[\sigma]$, turns out to be associative.

A theorem of Tits [24, Theorem 11.4] says that if Δ is a finite, irreducible Moufang building then it is the building of an absolutely simple algebraic group G over a finite field $\mathbb{F}_q$. The buildings we consider satisfy this hypothesis; hence in each example we will also specify the algebraic group G. The group G acts by simplicial type-preserving automorphisms on Δ. To avoid getting into details, we just mention that the action of G is very closely related to the geometry of the building.

In this section, we consider three examples, which are q-analogues of the side shuffles of type A_{n-1}, B_n and D_n. The example of type A_{n-1} was first studied by Brown and Diaconis (unpublished) and later by Brown [10]. The examples of type B_n and D_n are new. We use the same definitions for the σ_j's as before, except that we now apply them to the building Δ rather than the Coxeter complex Σ. We will use the notation $[j] = 1 + q + \ldots + q^{j-1}$ to denote the q-numbers. They will show up a lot in our analysis.

8.1. **The q-side shuffle for A_{n-1}.** We first briefly describe the building of type A_{n-1} associated to the algebraic group $GL_n(\mathbb{F}_q)$ (also $SL_n(\mathbb{F}_q)$). Let V be the n-dimensional vector space over $\mathbb{F}_q$ and $\mathcal{L}_n$ be the lattice of subspaces of V. The building of type A_{n-1} associated to $GL_n(\mathbb{F}_q)$, which we denote $\Delta(A_{n-1})$, is simply the flag (order) complex $\Delta(\mathcal{L}_n)$. It is a labeled simplicial complex. A vertex of $\Delta(\mathcal{L}_n)$ is by definition a proper subspace of V and we label it by its dimension. To be consistent with earlier notation, we label the vertices $s_1, \ldots, s_{n-1}$ instead of just $1, \ldots, n-1$. In particular, vertices of type s_1 are the one dimensional subspaces of V.

Let $\mathcal{B}_n$ be the Boolean lattice of rank n consisting of all subsets of an n-set ordered under inclusion. Note that the order complex $\Delta(\mathcal{B}_n)$ is the Coxeter complex of type A_{n-1}, which we earlier denoted by $\Sigma(A_{n-1})$. Also note that a choice of

a basis for V gives an embedding of $\mathcal{B}_n$ into $\mathcal{L}_n$. The subcomplexes $\Delta(\mathcal{B}_n)$, for various embeddings $\mathcal{B}_n \hookrightarrow \mathcal{L}_n$, play the role of apartments. Alternatively, it is also useful to think of $\mathcal{B}_n$ as a specialization of $\mathcal{L}_n$ for the degenerate case $q = 1$. In this sense $\Delta(\mathcal{L}_n) = \Delta(A_{n-1})$ may be regarded as a q-analogue of $\Delta(\mathcal{B}_n) = \Sigma(A_{n-1})$. For the rest of the section we will denote the building $\Delta(\mathcal{L}_n)$ by Δ.

The product in Δ can be made more explicit. A face of Δ is a chain in $\mathcal{L}_n$ and a chamber is a maximal chain in $\mathcal{L}_n$. For $x, y \in \Delta$, the face xy is the chain in $\mathcal{L}_n$ obtained by refining the chain x by the chain y, using meets and joins as in a Jordan–Hölder product. This is a generalization of the product in $\Sigma(A_{n-1})$, which we defined in terms of refinement of one ordered partition by another.

We now present the q-analogue of the Tsetlin library or the side shuffle of type A_{n-1} (Section 3.1). Let σ_1 be the sum of all the vertices of type s_1, σ_2 be the sum of all the edges of type $s_1 s_2$ and so on till σ_{n-1} which is the sum of all chambers of Δ. The number of summands in σ_j, which is same as the number of faces of type $s_1 s_2 \ldots s_j$, is $[n][n-1]\ldots[n-j+1]$. It follows directly from the definition that $\sigma_j \sigma_1 = \sigma_1 \sigma_j = [j]\sigma_j + q^j \sigma_{j+1}$ if $j < n-1$ and $\sigma_{n-1}\sigma_1 = [n]\sigma_{n-1}$. As a consistency check, we can also verify that the number of terms on each side is the same. These facts imply that σ_1 is power associative. Hence we may write the following without ambiguity.

$$(11) \qquad q^{\binom{j+1}{2}}\sigma_{j+1} = \sigma_1(\sigma_1 - [1])\ldots(\sigma_1 - [j]).$$

This along with $\sigma_{n-1}\sigma_1 = [n]\sigma_{n-1}$ implies that both $\sigma_0 = 1, \sigma_1, \ldots, \sigma_{n-1}$ and $\sigma_1^0 = 1, \sigma_1^1, \ldots, \sigma_1^{n-1}$ form a basis for the associative algebra $A = k[\sigma_1]$. The basic relation satisfied by σ_1 is $\sigma_1(\sigma_1 - [1])\ldots(\sigma_1 - [n-2])(\sigma_1 - [n]) = 0$. Hence $A \xrightarrow{\cong} \frac{k[x]}{x(x-[1])\ldots(x-[n-2])(x-[n])}$, where σ_1 maps to x. Since the roots are distinct we conclude that A is split semisimple.

As before, we extend the definition of σ_j to any j using (11) to obtain $\sigma_n = \sigma_{n-1}$ and $\sigma_j = 0$ for $j > n$. Inverting equation (11) formally, we obtain

$$\sigma_1^a = \sum_{j=0}^{a} S(a,j)\sigma_j$$

where $S(a,j)$ satisfies the recursion: $S(a,j) = [j]S(a-1,j) + q^{j-1}S(a-1,j-1)$ with $S(a,1) = 1$ and $S(a,a) = q^{\binom{a}{2}}$. The numbers $S(a,j)$ give a q-analogue to the Stirling numbers of the second kind. For more information on these numbers, see [25] and the references therein.

Remark. The analysis we gave for this example was the analogue of the direct method of Section 3.1. The difficulty with the shuffle method is that we do not know of a good random walk interpretation of σ_1^a. However as the direct method shows, the algebra A satisfies all the properties of an additive shuffle algebra (except that now we are in the more general context of buildings). There are also natural ways to define the q-analogue of the two-sided shuffle and the riffle shuffle. However the method breaks down because the basic elements of interest are not power associative.

8.2. **The q-side shuffle for** B_n**.** There are two possibilities for the building of type B_n depending on whether the associated algebraic group is the symplectic group $Sp_{2n}(\mathbb{F}_q)$ or the orthogonal group $O_{2n}(\mathbb{F}_q)$. We consider them separately. They may both be regarded as q-analogues of the Coxeter complex $\Sigma(B_n)$. The

side shuffle for B_n generalizes to both cases. As expected, the computations have the same spirit as the previous section.

The symplectic case. Let V be a (even dimensional) vector space over $\mathbb{F}_q$ with a skew-symmetric non-degenerate bilinear form Q. More explicitly, V has a basis $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ such that $Q(e_i, f_i) = 1$, $Q(f_i, e_i) = -1$ and all other pairings between basis vectors are zero. The symplectic group $Sp_{2n}(\mathbb{F}_q)$ is the group of automorphisms of V that preserve the bilinear form Q.

The building of type B_n associated to the symplectic group $\Delta_1(B_n)$ is the flag (order) complex of all isotropic subspaces of V. The explicit description of the product in $\Delta_1(B_n)$ is similar to that in $\Delta(A_{n-1})$. The easiest way to describe it is via an inclusion

$$\Delta_1(B_n) \hookrightarrow \Delta(A_{2n-1}),$$

where we map a flag of isotropic spaces $0 < V_1 < \ldots < V_l < V$ to the flag $0 < V_1 < \ldots < V_l \leq V_l^\perp < \ldots < V_1^\perp < V$. This generalizes the map $\Sigma(B_n) \hookrightarrow \Sigma(A_{2n-1})$ considered in Section 7.

Let σ_1 be the sum of all the vertices of type s_1, σ_2 be the sum of all the edges of type $s_1 s_2$ and so on till σ_n which is the sum of all chambers of $\Delta_1(B_n)$. It follows from the definition that $\sigma_j \sigma_1 = \sigma_1 \sigma_j = (1 + q^{2n-j})[j]\sigma_j + q^j \sigma_{j+1}$ if $j \leq n - 1$ and $\sigma_n \sigma_1 = \sigma_1 \sigma_n = [2n]\sigma_n$. An easy way to check this is to first figure out the coefficient of σ_{j+1}. It is q^j, the same answer that we got for the A_{n-1} case. There is no difference in the two situations with respect to this coefficient. With this information, we can find the coefficient of σ_j by simply counting the number of terms involved on both sides. The number of summands in σ_j, which is same as the number of faces of type $s_1 s_2 \ldots s_j$, is $[2n][2n-2] \ldots [2n-2j+2]$. The identity is then a consequence of the simple relation $[2n] = (1 + q^{2n-j})[j] + q^j[2n - 2j]$. It is a good exercise to compute the coefficient of σ_j directly.

These facts imply that σ_1 is power associative. Hence we may write the following without ambiguity.

$$(12) \qquad q^{\binom{j+1}{2}} \sigma_{j+1} = \sigma_1 (\sigma_1 - (1 + q^{2n-1})[1]) \ldots (\sigma_1 - (1 + q^{2n-j})[j]).$$

This along with $\sigma_n \sigma_1 = [2n]\sigma_n$ implies that both $\sigma_0 = 1, \sigma_1, \ldots, \sigma_n$ and $\sigma_1^0 = 1, \sigma_1^1, \ldots, \sigma_1^n$ form a basis for the associative algebra $A = k[\sigma_1]$. The basic relation satisfied by σ_1 is $\sigma_1(\sigma_1 - (1 + q^{2n-1})[1]) \ldots (\sigma_1 - (1 + q^{n+1})[n-1])(\sigma_1 - [2n]) = 0$. Hence $A \xrightarrow{\cong} \frac{k[x]}{x(x-(1+q^{2n-1})[1]) \ldots (x-(1+q^{n+1})[n-1])(x-[2n])}$. This shows that A is split semisimple.

As before, we extend the definition of σ_j to any j using (12) to obtain $\sigma_j = 0$ for $j > n$. Inverting equation (12) formally, we obtain

$$\sigma_1^a = \sum_{j=0}^{a} S(a, j)\sigma_j$$

where $S(a, j)$ satisfies the recursion: $S(a, j) = (1 + q^{2n-j})[j]S(a-1, j) + q^{j-1}S(a-1, j-1)$ with $S(a, 1) = (1 + q^{2n-1})^{a-1}$ and $S(a, a) = q^{\binom{a}{2}}$. The numbers $S(a, j)$ give a q-analogue to the signed Stirling numbers. Note that they now depend on n. We do not know whether they have been considered before or whether they have an explicit formula.

The orthogonal case. Let V be a (even dimensional) vector space over $\mathbb{F}_q$ with a symmetric non-degenerate bilinear form Q. We also assume that V has an isotropic

subspace of dimension n. More explicitly, V has a basis $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ such that $Q(e_i, f_i) = Q(f_i, e_i) = 1$ and all other pairings between basis vectors are zero. The orthogonal group $O_{2n}(\mathbb{F}_q)$ is the group of automorphisms of V that preserve the bilinear form Q.

The building of type B_n associated to the orthogonal group $\Delta_2(B_n)$ is the flag (order) complex of all isotropic subspaces of V. The analysis parallels the symplectic case. With the same definitions of the σ_j's, we get $\sigma_j\sigma_1 = \sigma_1\sigma_j = (1 + q^{2n-j-1})[j]\sigma_j + q^j\sigma_{j+1}$ if $j \leq n - 1$ and $\sigma_n\sigma_1 = \sigma_1\sigma_n = (1 + q^{n-1})[n]\sigma_n$. The reason for the difference is that now the number of summands in σ_j is $(1 + q^{n-1})[n](1 + q^{n-2})[n-1]\ldots(1 + q^{n-j})[n - j + 1]$. The rest is similar to the symplectic case. We now obtain a different q-analogue of the signed Stirling numbers that satisfy the recursion: $S(a, j) = (1 + q^{2n-j-1})[j]S(a-1, j) + q^{j-1}S(a-1, j-1)$ with $S(a, 1) = (1 + q^{2n-2})^{a-1}$ and $S(a, a) = q^{\binom{a}{2}}$.

Remark. The building $\Delta_2(B_n)$ that we considered here is not thick. Classically, the group $O_{2n}(\mathbb{F}_q)$ is considered to be of type D_n since there is a thick building $\Delta(D_n)$ associated to it. We will discuss this next. For more information on this, see [9, pgs 123-127].

8.3. **The q-side shuffle for D_n.** Let V and Q be as in the orthogonal case of Section 8.2. The building of type D_n, which we denote by $\Delta(D_n)$, is the flag complex of the following so-called "oriflamme geometry". The vertices in $\Delta(D_n)$ are the non-zero isotropic subspaces of V of dimension $\neq n-1$. Two such subspaces are called incident if one is contained in the other or if both have dimension n and their intersection has dimension $n - 1$. A simplex in $\Delta(D_n)$ is a set of pairwise incident vertices.

There is a natural product preserving map $\Delta_2(B_n) \to \Delta(D_n)$, which generalizes the map $\Sigma(B_n) \to \Sigma(D_n)$ in Section 6.1. To describe this map, we identify the vector space V and the bilinear form Q in the two cases. A vertex in $\Delta_2(B_n)$ is a non-zero isotropic subspace of V. If it has dimension $\neq n - 1$ then we map it to itself. Otherwise there are exactly two maximal isotropic subspaces of dimension n in V that contain it. And we map it to the edge joining the two vertices in $\Delta(D_n)$ representing these two subspaces. This again explains how a vertex of type s_{n-1} maps to an edge of type uv.

The image of the shuffle algebra $k[\sigma_1]$ in $\Delta_2(B_n)$ under the above map yields a shuffle algebra in $\Delta(D_n)$. As a q-analogue of the map between side shuffles of type B_n and D_n, we obtain a map

$$
\left\{ \begin{array}{c} q\text{-side shuffle} \\ \text{of type } B_n \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} q\text{-side shuffle} \\ \text{of type } D_n \end{array} \right\}
$$
$$
\sigma_j \qquad\qquad \mapsto \qquad\qquad \sigma_j
$$

The σ_j's on the right are defined as for the side shuffle of type D_n and we again get the relation $\sigma_n = 2\sigma_{n-1}$. The basic relation satisfied by σ_1 is $\sigma_1(\sigma_1 - (1 + q^{2n-2})[1])\ldots(\sigma_1 - (1 + q^{n+1})[n - 2])(\sigma_1 - (1 + q^{n-1})[n]) = 0$. Note that the term corresponding to $n - 1$ is absent. The rest of the analysis is routine and we omit it.

Remark. In light of the discussion in Section 6, we may say the following. The failure to define a q-two sided shuffle of type A_{n-1} is the result of our failure to define a product preserving map $\Delta(D_n) \to \Delta(A_{n-1})$. Also we failed to define a q-riffle shuffle of type B_n and as a result also failed to get q-riffle shuffles for the other two types.

9. Future prospects

We conclude by suggesting some problems for future consideration.

Shuffle algebras. One problem is to classify them with minor modifications of the definition if necessary. Notice that we never talked of a two sided shuffle of type B_n or a joker analogue of the side shuffle of type A_n. Can these fit into the framework of shuffle algebras or are there genuine geometric obstructions that make them non-examples?

There are also more intricate shuffles called affine shuffles. Apparently, the affine shuffles of type B_n coincide with our inverse a-shuffles of type B_n [15, Proposition 1]. Also there is no good description available for the affine shuffles of type A_n. We ask whether affine shuffles can be understood geometrically in terms of the affine Coxeter complex and the above connection clarified.

Buildings and complex reflection groups. There is no good theory for random walks on buildings the major difficulty being the non-associativity of the product. Our methods in Section 8 suggest that one should classify the power associative elements of $k\Delta$ for a building Δ and also product preserving maps between buildings. We may also ask if $k[\sigma]$ is split-semisimple for a power associative element of $k\Delta$.

In this paper, we considered only real reflection groups. The difficulty in passing to the complex case is the absence of an analogue of the Coxeter complex. As an example, consider the complex reflection group $S_n \ltimes \mathbb{Z}_r^n$. For $r = 2$, it specializes to the Coxeter group of type B_n. The first step would be to generalize the semigroup of ordered partitions of type B_n. However, it is not clear how to do this.

Multiplicities and derangement numbers. In [10], Brown defined a random walk associated to matroids. And he related the eigenvalue multiplicities to invariants of the lattice of flats. He called these invariants the generalized derangement numbers. The motivating examples are the side and q-side shuffle of type A_{n-1} and the generic multiplicities then are the usual and q-derangement numbers. In this sense, we may think of ordinary matroids as related to the Coxeter group of type A_{n-1}.

There is a notion of a W-matroid [8] for any Coxeter group W. We ask whether it is possible to generalize the above to this setting. As a positive result in this direction, Bidigare [4, pgs 147-148] showed that for the side shuffle of type B_n, the generic multiplicities are the signed derangement numbers. Further for the side shuffle of type D_n, we have showed that the generic multiplicities are the derangement numbers of type D_n. To define these, one restricts to signed permutations with an even number of negative signs. They satisfy the recursion $d_k - 2kd_{k-1} = (-1)^k(k+1)$ with $d_0 = 1$, $d_1 = 0$ and $d_2 = 3$ as the first few values.

Appendix A. The hyperplane face semigroup

Most of this section and a part of the next is taken directly from [10]. More details concerning the material reviewed here can be found in [5, 6, 7, 9, 11, 19, 26]. Throughout this section $\mathcal{A} = \{H_i\}_{i \in I}$ denotes a finite set of affine hyperplanes in $V = \mathbb{R}^n$. Let H_i^+ and H_i^- be the two open halfspaces determined by H_i; the choice of which one to call H_i^+ is arbitrary but fixed.

A.1. **Faces and chambers.** The hyperplanes H_i induce a partition of V into convex sets called *faces* (or *relatively open faces*). These are the nonempty sets $F \subseteq V$ of the form

$$F = \bigcap_{i \in I} H_i^{\varepsilon_i},$$

where $\varepsilon_i \in \{+, -, 0\}$ and $H_i^0 = H_i$. Equivalently, if we choose for each i an affine function $f_i \colon V \to \mathbb{R}$ such that H_i is defined by $f_i = 0$, then a face is a nonempty set defined by equalities and inequalities of the form $f_i > 0$, $f_i < 0$, or $f_i = 0$, one for each $i \in I$. The sequence $\varepsilon = (\varepsilon_i)_{i \in I}$ that encodes the definition of F is called the *sign sequence* of F and is denoted $\varepsilon(F)$.

The faces such that $\varepsilon_i \neq 0$ for all i are called *chambers*. They are convex open sets that partition the complement $V - \bigcup_{i \in I} H_i$. In general, a face F is open relative to its *support*, which is defined to be the affine subspace

$$\operatorname{supp} F = \bigcap_{\varepsilon_i(F)=0} H_i.$$

Since F is open in $\operatorname{supp} F$, we can also describe $\operatorname{supp} F$ as the affine span of F.

A.2. **The face relation.** The *face poset* of $\mathcal{A}$ is the set $\mathcal{F}$ of faces, ordered as follows: $F \leq G$ if for each $i \in I$ either $\varepsilon_i(F) = 0$ or $\varepsilon_i(F) = \varepsilon_i(G)$. In other words, the description of F by linear equalities and inequalities is obtained from that of G by changing zero or more inequalities to equalities. We say that F is a face of G. Note that the chambers are precisely the maximal elements of the face poset. We denote the set of chambers by $\mathcal{C}$.

A.3. **Product.** The set $\mathcal{F}$ of faces is also a semigroup. Given $F, G \in \mathcal{F}$, their *product* FG is the face with sign sequence

$$\varepsilon_i(FG) = \begin{cases} \varepsilon_i(F) & \text{if } \varepsilon_i(F) \neq 0 \\ \varepsilon_i(G) & \text{if } \varepsilon_i(F) = 0. \end{cases}$$

This has a geometric interpretation: If we move on a straight line from a point of F toward a point of G, then FG is the face we are in after moving a small positive distance. Hence, we may think of FG as the projection of G on F. Notice that the face relation can be described in terms of the product: One has

(13) $F \leq G \iff FG = G.$

Also note that the set of chambers $\mathcal{C}$ is an ideal of $\mathcal{F}$.

A.4. **The semilattice of flats.** A second poset associated with the arrangement $\mathcal{A}$ is the *semilattice of flats*, also called the *intersection semilattice*, which we denote by L. It consists of all nonempty affine subspaces $X \subseteq V$ of the form $X = \bigcap_{H \in \mathcal{A}'} H$, where $\mathcal{A}' \subseteq \mathcal{A}$ is an arbitrary subset (possibly empty). We order L by inclusion. [Warning: Many authors order L by reverse inclusion.] Notice that any two elements X, Y have a least upper bound $X \vee Y$ in L, which is the intersection of all hyperplanes $H \in \mathcal{A}$ containing both X and Y; hence L is an *upper semilattice* (poset with least upper bounds). It is a lattice if the arrangement $\mathcal{A}$ is *central*, i.e., if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. Indeed, this intersection is then the smallest element of L, and a finite upper semilattice with a smallest element is a lattice [23, Section 3.3]. The support map gives a surjection

$$\operatorname{supp} \colon \mathcal{F} \twoheadrightarrow L,$$

which preserves order and also behaves nicely with respect to the semigroup structure. Namely, we have

$$\text{(14)} \qquad\qquad \operatorname{supp}(FG) = \operatorname{supp} F \vee \operatorname{supp} G$$

and

$$\text{(15)} \qquad\qquad FG = F \iff \operatorname{supp} G \leq \operatorname{supp} F.$$

A.5. **The forgetful map.** Let $\mathcal{A} = \{H_i\}_{i \in I}$ and $\mathcal{A}' = \{H_i\}_{i \in I'}$ be two hyperplane arrangements such that $I' \subset I$, that is, the arrangement $\mathcal{A}'$ is obtained from $\mathcal{A}$ by deleting some of the hyperplanes. Let $\mathcal{F}(A)$ and $\mathcal{F}(A')$ be the respective hyperplane face semigroups. Then there is a natural map $\mathcal{F}(A) \to \mathcal{F}(A')$. An element of $\mathcal{F}(A)$ is a face F with a sign sequence, say $\varepsilon = (\varepsilon_i)_{i \in I}$. We map it to the face of $\mathcal{A}'$ with the sign sequence $\varepsilon' = (\varepsilon_i)_{i \in I'}$. In other words, we forget the signs of the elements of I that are not in I'. It is clear that this map is a semigroup homomorphism and preserves the face relation. The second fact is implied by the first in view of equation (13).

A.6. **Spherical representation.** Suppose now that $\mathcal{A}$ is a *central* arrangement, i.e., that the hyperplanes have a nonempty intersection. We may assume that this intersection contains the origin. Suppose further that $\bigcap_{i \in I} H_i = \{0\}$, in which case $\mathcal{A}$ is said to be *essential*. (There is no loss of generality in making this assumption; for if it fails, then we can replace V by the quotient space $V/\bigcap_i H_i$.) The hyperplanes then induce a cell-decomposition of the unit sphere, the cells being the intersections with the sphere of the faces $F \in \mathcal{F}$. Thus $\mathcal{F}$, as a poset, can be identified with the poset of cells of a regular cell-complex Σ, homeomorphic to a sphere. Note that the face $F = \{0\}$, which is the identity of the semigroup $\mathcal{F}$, is not visible in the spherical picture; it corresponds to the empty cell. The cell-complex Σ plays a crucial role in [11], to which we refer for more details.

Thus the cell-complex Σ is a semigroup which we call the hyperplane face semigroup. The maximal cells (which we again call chambers and denote $\mathcal{C}$) is an ideal in Σ.

Appendix B. Reflection arrangements

We work with an arbitrary finite Coxeter group W and its associated hyperplane face semigroup Σ (the Coxeter complex of W). But we will explain everything in concrete terms for the cases $W = A_{n-1}, B_n, D_n$. This should make the discussion accessible to readers unfamiliar with Coxeter groups.

B.1. **Finite reflection groups.** We begin with a very quick review of the basic facts that we need about finite Coxeter groups and their associated simplicial complexes Σ. Details can be found in many places, such as [9, 17, 18, 24]. A *finite reflection group* on a real inner-product space V is a finite group of orthogonal transformations of V generated by reflections s_H with respect to hyperplanes H through the origin. The set of hyperplanes H such that $s_H \in W$ is the *reflection arrangement* associated with W. Its hyperplane face semigroup Σ is called the *Coxeter complex* of W. Geometrically, this complex is obtained by cutting the unit sphere in V by the hyperplanes H, as in Section A.6. (As explained there, one might have to first pass to a quotient of V.) It turns out that the Coxeter complex Σ is always a simplicial complex. Furthermore, the action of W on V induces an action of W on Σ, and this action is simply-transitive on the chambers. Thus the

set $\mathcal{C}$ of chambers can be identified with W, once a "fundamental chamber" C is chosen.

B.2. **Types of simplices.** The number r of vertices of a chamber of Σ is called the *rank* of Σ (and of W); thus the dimension of Σ as a simplicial complex is $r - 1$. It is known that one can color the vertices of Σ with r colors in such a way that vertices connected by an edge have distinct colors. The color of a vertex is also called its *label*, or its *type*, and we denote by I the set of all types. We can also define type(F) for any $F \in \Sigma$; it is the subset of I consisting of the types of the vertices of F. For example, every chamber has type I, while the empty simplex has type $\emptyset$. The action of W is type-preserving; moreover, two simplices are in the same W-orbit if and only if they have the same type.

B.3. **The Coxeter diagram.** Choose a fundamental chamber C. It is known that the reflections s_i in the facets of C generate W. In fact, W has a presentation of the form $< s_1, \ldots, s_r \mid (s_i s_j)^{m_{ij}} >$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$. This data is conveniently encoded in a picture called the Coxeter diagram of W. This diagram is a graph, with vertices and edges, defined as follows: There are r vertices, one for each generator $i = 1, 2, \ldots, r$, and the vertices corresponding to i and j are connected by an edge if and only if $m_{ij} \geq 3$. If $m_{ij} \geq 4$ then we simply label the edge with the number m_{ij}. The figures show the Coxeter diagrams which are of interest to us, namely the ones of type A_{n-1}, B_n and D_n. It is customary to use the generators of W, or the vertices of the Coxeter diagram to label the vertices of its Coxeter complex Σ. A vertex of the fundamental chamber C is labeled s_i if it is fixed by all the fundamental reflections except s_i. Since W acts transitively on $\mathcal{C}$ and the action is type-preserving, this determines the type of all the vertices of Σ.
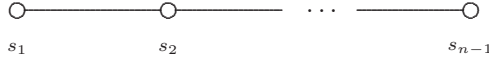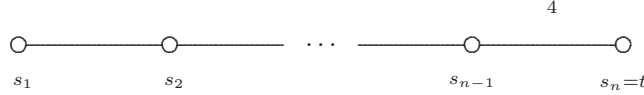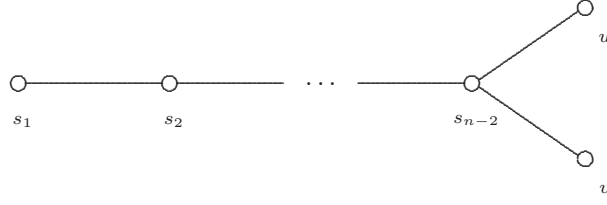

FIGURE 2. Coxeter Diagram of Type A_{n-1}

FIGURE 3. Coxeter Diagram of Type B_n

B.4. **The Coxeter group of type** A_{n-1}. The Coxeter group $W = S_n$ acts on $\mathbb{R}^n$ by permuting the coordinates. The arrangement in this case is the *braid arrangement* in $\mathbb{R}^n$. It is discussed in detail in [4, 5, 6, 11]. It consists of the $\binom{n}{2}$ hyperplanes H_{ij} defined by $x_i = x_j$, where $1 \leq i < j \leq n$. Each chamber is determined by an ordering of the coordinates, so it corresponds to a permutation. The faces of a chamber are obtained by changing to equalities some of the inequalities defining that chamber.

FIGURE 4. Coxeter Diagram of Type D_n

We fix $x_1 < x_2 < \ldots < x_n$ to be the fundamental chamber C. The supports of the facets of C are hyperplanes of the form $x_i = x_{i+1}$, where $1 \le i \le n - 1$. The reflection in the hyperplane $x_i = x_{i+1}$ corresponds to the generator s_i that interchanges the coordinates x_i and x_{i+1}. The chamber C has $n - 1$ vertices, namely

$s_1 : x_1 < x_2 = \ldots = x_n,$

$s_2 : x_1 = x_2 < x_3 = \ldots = x_n, \ldots,$

$s_{n-1} : x_1 = \ldots = x_{n-1} < x_n.$

The labels $s_1, s_2, \ldots, s_{n-1}$ are assigned by the rule mentioned in Appendix B.3. Applying the action of W we see, for example, that $x_{\pi(1)} < x_{\pi(2)} = \ldots = x_{\pi(n)}$ gives all vertices of type s_1 as π varies over all permutations of $[n] = \{1, \ldots, n\}$.

We encode the system of equalities and inequalities defining a face F by an ordered partition $(B_1, \ldots, B_k)$ of $[n]$. Here $B_1, \ldots, B_k$ are disjoint nonempty sets whose union is $[n]$, and their order counts. Thus the simplices of Σ are ordered partitions $B = (B_1, \ldots, B_l)$ of the set $[n]$. For example, the vertices of type s_1 are ordered two block partitions such that the first block is a singleton.

The product in Σ (Appendices A.3 and A.6) is also easy to describe. We multiply two ordered partitions by taking intersections and ordering them lexicographically; more precisely, if $B = (B_1, \ldots, B_l)$ and $C = (C_1, \ldots, C_m)$, then

$$BC = (B_1 \cap C_1, \ldots, B_1 \cap C_m, \ldots, B_l \cap C_1, \ldots, B_l \cap C_m)\widehat{\ },$$

where the hat means "delete empty intersections". The 1-block ordered partition is the identity. The associated lattice L (Appendix A.4) is the lattice of unordered set partitions. The support map $\mathcal{B} \twoheadrightarrow L$ forgets the ordering of the blocks.

Note that B is a face of C if and only if C consists of an ordered partition of B_1 followed by an ordered partition of B_2, and so on, that is, if and only if C is a refinement of B. The chambers are the ordered partitions into singletons, so they correspond to the permutations of $[n]$ or a deck of n cards.

B.5. **The Coxeter group of type** B_n**.** The group of signed permutations $W = S_n \ltimes \mathbb{Z}_2^n$ acts on $\mathbb{R}^n$ with the subgroup S_n permuting the coordinates and the subgroup $\mathbb{Z}_2^n$ flipping the signs of the coordinates. The reflection arrangement in this case consists of the hyperplanes defined by $x_i = \pm x_j$ and $x_i = 0$, where $1 \le i < j \le n$. A chamber is given by an ordering of the coordinates, their negatives and zero. For example, for $n = 3$, $x_2 < -x_3 < x_1 < 0 < -x_1 < x_3 < -x_2$ specifies a chamber. Note that the inequalities that appear on the left of 0 completely determine a chamber (and the same is true for any face since it is obtained by changing to equalities some of the inequalities defining a chamber). Thus we see that a chamber corresponds to a signed permutation.

We fix $x_1 < x_2 < \ldots < x_n < 0$ to be the fundamental chamber C. The supports of the facets of C are hyperplanes of the form $x_i = x_{i+1}$, where $1 \le i \le n-1$ and $x_n = 0$. The generators s_i interchange the coordinates x_i and x_{i+1} and the generator t flips the sign of x_n. The n vertices of C along with their labels are as follows.

$s_1 : x_1 < x_2 = \ldots = x_n = 0$,

$s_2 : x_1 = x_2 < x_3 = \ldots = x_n = 0$,...,

$s_{n-1} : x_1 = x_2 = \ldots = x_{n-1} < x_n = 0$,

$t : x_1 = x_2 \ldots = x_n < 0$.

Applying the group action, we see, for example, that the vertices of type t have the form $\varepsilon_1 x_1 = \varepsilon_2 x_2 \ldots = \varepsilon_n x_n < 0$ where $\varepsilon_i \in \{\pm 1\}$.

It is convenient to describe a face F by an anti-symmetric ordered partition $(B_1, \ldots, B_k, Z, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}] = \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$. We will call this a *partition of type B_n*. For example, for $n = 3$, the face $x_2 < -x_3 < x_1 = 0 = -x_1 < x_3 < -x_2$ is written $(\{2\}, \{\overline{3}\}, \{1, \overline{1}\}, \{3\}, \{\overline{2}\})$. The set Z satisfies $Z = \overline{Z}$ and is allowed to be empty. We call it the zero block. It is the only set that is allowed to contain both a number and its negative. Also, the sets $B_1, \ldots, B_k$ are necessarily non-empty. We also define a *weak partition of type B_n* to be a partition as above but where the sets $B_1, \ldots, B_k$ are allowed to be empty. Note that for a face F, the zero block of its partition is empty if and only if the type of F contains the letter t. We split the description for a face F into four cases, depending on whether the type of F contains

(i) neither s_{n-1} nor t:

An ordered partition $(B_1, \ldots, B_k, Z, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$ of type B_n with the added restriction that the zero block Z has at least 4 elements.

(ii) t but not s_{n-1}:

An ordered partition $(B_1, \ldots, B_k, Z, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$, with the restriction that the zero block is empty and B_k has at least 2 elements.

(iii) s_{n-1} but not t:

An ordered partition $(B_1, \ldots, B_k, Z, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$, with the restriction that the zero block has exactly 2 elements.

(iv) both s_{n-1} and t:

An ordered partition $(B_1, \ldots, B_k, Z, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$, with the restriction that the zero block is empty and B_k has at exactly 1 element.

The product in Σ is defined exactly as for the A_{n-1} case, where we refine the first partition by the second. Note that B is a face of C if and only if C is a refinement of B.

As mentioned earlier, the second half of the partition following Z contains no new information. Hence we may describe a $k-1$ dimensional face by a $(k+1)$ block partition $(B_1, \ldots, B_k, Z)$ of $[n]$. We think of $B_1, \ldots, B_k$ as signed sets and Z (possibly empty) as an unsigned set. The chambers are then $(n+1)$ block partitions into n singletons and an empty zero block, so they correspond to signed permutations of $[n]$ or a deck of n signed cards. A good way to describe it is as a deck where each card is either face up or face down. We will call such a deck as a *deck of type B_n*. However, the product and the face relation is not so natural now. So we will mainly use the first description.

B.6. **The Coxeter group of type** D_n**.** The group of even signed permutations $W = S_n \ltimes G$ is an index 2 subgroup of the group of signed permutations $S_n \ltimes \mathbb{Z}_2^n$. Here G is the index 2 subgroup of $\mathbb{Z}_2^n$ consisting of n-tuples that have an even number of negative signs. The group W acts on $\mathbb{R}^n$ with the subgroup S_n permuting the coordinates and the subgroup G flipping the signs of the coordinates. The reflection arrangement in this case consists of the hyperplanes defined by $x_i = \pm x_j$, where $1 \le i < j \le n$. It is obtained from the reflection arrangement of type B_n by deleting the coordinate hyperplanes $x_i = 0$ for $1 \le i \le n$. On the level of chambers, this has the effect of merging together pairs of adjacent chambers of the Coxeter complex of type B_n; see Figure 1. For example, the chamber of the Coxeter complex of type D_n, $x_1 < x_2 < \ldots < x_{n-1} < \pm x_n < -x_{n-1} < \ldots < -x_1$ (which we fix as our fundamental chamber C) is the union of the two chambers of the Coxeter complex of type B_n, namely, $x_1 < x_2 < \ldots < x_n < 0$ and $x_1 < x_2 < \ldots < -x_n < 0$. The supports of the facets of C are hyperplanes of the form $x_i = x_{i+1}$, where $1 \le i \le n-1$ and $x_{n-1} = -x_n$. The generators s_i interchange the coordinates x_i and x_{i+1}, the generator u interchanges the coordinates x_{n-1} and x_n and the generator v interchanges x_{n-1} and x_n and flips the signs of both. The n vertices of C along with their labels are as follows.

$s_1 : x_1 < x_2 = \ldots = x_n = 0$,

$s_2 : x_1 = x_2 < x_3 = \ldots = x_n = 0$,$\ldots$,

$s_{n-2} : x_1 = x_2 = \ldots = x_{n-2} < x_{n-1} = x_n = 0$,

$u : x_1 = x_2 \ldots = x_{n-1} = -x_n < x_n = -x_{n-1} = \ldots = -x_2 = -x_1$,

$v : x_1 = x_2 \ldots = x_{n-1} = x_n < -x_n = -x_{n-1} = \ldots = -x_2 = -x_1$.

Applying the group action, we see, for example, that the vertices of type u and v both have the form $\varepsilon_1 x_1 = \varepsilon_2 x_2 = \ldots = \varepsilon_n x_n < -\varepsilon_n x_n = \ldots = -\varepsilon_2 x_2 = -\varepsilon_1 x_1$, where $\varepsilon_i \in \{\pm 1\}$ and they are distinguished by the parity of the product $\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n$. Also observe that the edge of the fundamental chamber C of type $\{u, v\}$ is given by $x_1 = x_2 \ldots = x_{n-1} < \pm x_n < -x_{n-1} = \ldots = -x_2 = -x_1$.

As in the B_n case, we describe a face F by an anti-symmetric ordered partition $(B_1, \ldots, B_k, C, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$. We repeat that the sets $B_1, \ldots, B_k$ are non-empty and cannot contain both a number and its negative. We call the block $C = \overline{C}$ in the middle, the central block rather than the zero block. It always has an even number of elements. Furthermore, we impose the relation $(B_1, \ldots, B_{k-1}, C, \overline{B}_{k-1}, \ldots, \overline{B}_1) = (B_1, \ldots, B_{k-1}, B_k, C', \overline{B}_k, \overline{B}_{k-1}, \ldots, \overline{B}_1)$ where C' is empty, C has exactly two elements, namely, a number and its negative, and $C = B_k \cup C' \cup \overline{B}_k$. We split the description for a face F into three cases, depending on whether the type of F contains

  (i) neither u nor v:
     An ordered partition $(B_1, \ldots, B_k, C, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$, with the restriction that the central block C (in this case, we may think of it as the zero block) has at least 4 elements.
  (ii) exactly one of u and v:
     An ordered partition $(B_1, \ldots, B_k, C, \overline{B}_k, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$, with the restriction that C is empty and B_k has at least 2 elements.
  (iii) both u and v:
     An ordered partition $(B_1, \ldots, B_{k-1}, C, \overline{B}_{k-1}, \ldots, \overline{B}_1)$ of $[n, \overline{n}]$, with the restriction that C has exactly 2 elements, namely, a number and its negative.

An ordered partition $(B_1, \ldots, B_{k-1}, B_k, C', \overline{B}_k, \overline{B}_{k-1}, \ldots, \overline{B}_1)$ where C' is empty and B_k has exactly 1 element.

Here the number k is always the rank of the face. The first three descriptions are obtained directly from the equalities and inequalities that define a face. The reason why we call C the central block rather than the zero block should be clear now. The motivation for the second description in case (iii) is as follows. To give an example, the product of $(\{2,\overline{3}\},\{1,\overline{5},\overline{4}\},\{4,5,\overline{1}\},\{3,\overline{2}\})$ with $(\{4,1,\overline{5}\},\{2,\overline{3},3,\overline{2}\},\{5,\overline{1},\overline{4}\})$, must be $(\{2,\overline{3}\},\{1,\overline{5}\},\{\overline{4},4\},\{5,\overline{1}\},\{3,\overline{2}\})$. But if we refine the first partition by the second then we get $(\{2,\overline{3}\},\{1,\overline{5}\},\{\overline{4}\},\{4\},\{5,\overline{1}\},\{3,\overline{2}\})$. This partition has two singletons in the middle and does not fit any of the first three descriptions given above. So we allow ourselves to merge the two singletons in the middle. This identification allows us to multiply two partitions B and C in the same way as before; that is, we refine B using C.

As in the B_n case, if we throw off the (redundant) second half of the partition then we see that chambers correspond to an *almost* signed deck of cards or a *deck of type D_n*. It is a deck in which every card, except the bottommost, is signed. We also define a *weak partition of type D_n* to be a partition of type D_n where the sets $B_1, \ldots, B_k$ are allowed to be empty.

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