An introduction to algebraic geometry

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Contents

Contents iii

Chapter 1. Commutative rings 1
1.1. Commutative rings 1
1.2. Noetherian rings 6
1.3. Noetherian spaces 7
1.4. Localization 9
1.5. The category of local rings 12
1.6. The category of reduced rings 13

Chapter 2. Affine varieties 15
2.1. The category of affine varieties 15
2.2. Affine varieties and radical ideals 18
2.3. The Zariski topology 21
2.4. The coordinate ring of an affine variety 23
2.5. The sheaf of regular functions 29

Chapter 3. Projective varieties 37
3.1. Projective varieties 37
3.2. Projective varieties and homogeneous radical ideals 40
3.3. The category of projective varieties 41
3.4. Rational normal curves 43

Chapter 4. Varieties 49
4.1. Prevarieties 49
4.2. Varieties 55
4.3. Smooth manifolds 57
4.4. Rational maps 57

Chapter 5. Affine schemes 61
5.1. Spectrum of a ring 61
5.2. The category of affine schemes 68
5.3. Examples 72
5.4. Fiber product of affine schemes 75
5.5. The functor of points 76

Chapter 6. Schemes 79
6.1. Preschemes 79
6.2. Varieties and schemes 83
6.3. Fiber product in the category of preschemes 85
6.4. Quasi-coherent sheaves 88

6.5. The functor of points 90

Chapter 7. Groups and Hopf algebras 93
7.1. Sets 93
7.2. Modules and vector spaces 93
7.3. Group-like elements of a Hopf algebra 95
7.4. Problems 98

Chapter 8. Affine algebraic groups 101
8.1. Affine varieties 101
8.2. Affine algebraic groups 104
8.3. Examples 106
8.4. Subgroups 109
8.5. Modules over an affine algebraic group 110
8.6. Linearity of affine algebraic groups 112
8.7. Problems 114

Chapter 9. Affine group schemes 117
9.1. Affine group schemes 117
9.2. The functor of points 117
9.3. Examples 119
9.4. Character group 120
9.5. Diagonalizable affine group schemes 120
9.6. Problems 121

Appendix A. Category theory 123
A.1. Products and coproducts 124
A.2. Adjunctions 125
A.3. Equivalences 126
A.4. Colimits of functors 126
A.5. Problems 127

Appendix B. Closure operators 129
B.1. Closure operators 129
B.2. Topological closure operators 130
B.3. Joins and meets 130
B.4. Galois connection 130
B.5. Problems 131

Appendix C. Posets 133
C.1. Posets as categories 133
C.2. Order-preserving maps as functors 133
C.3. Problems 134

Appendix D. Sheaves 135
D.1. Sheaves 135
D.2. Stalks 136
D.3. The generic stalk 136
D.4. The category of all sheaves 137
D.5. Problems 137

Appendix E. Monoidal categories 139
E.1. Braided monoidal categories 139
E.2. Hopf monoids 141
E.3. Trivial examples 145
E.4. The diamond of categories 145
E.5. Monoidal functors 147
E.6. Cartesian monoidal categories 147
E.7. Modules and comodules 151

Appendix F. The Yoneda embedding 153
F.1. Functor categories 153
F.2. Slice categories 153
F.3. The Yoneda lemma 154

Appendix G. Functors from commutative monoids to groups 157
G.1. The functor of points of a commutative Hopf monoid 157
G.2. The functor of matrices 158
G.3. Problems 159

Bibliography 161

These notes are meant for the reader who wants a rapid as well as precise introduction to varieties, schemes, and related concepts. Corrections and suggestions for improvement are welcome.

Prerequisites.

- You should be familiar with ring theory at the level of [3, Chapter 10].
- Very little topology is required. You should be familiar with the notions of topological spaces, open sets, closed sets, closure of a set, connectedness, connected components, and the Hausdorff property.
- You should be completely comfortable with category theory [18]. To keep the material self-contained, many relevant notions are reviewed in the Appendices.

References. The material for these notes is taken from the following sources:

- For commutative algebra, I have mainly relied on Atiyah and MacDonald [4].
- For varieties, the material is taken from books by Harris [13], Hartshorne [14, Chapter 1], Humphreys [15, Chapter I], Mumford [19, Chapter I] and Smith et al [23]. (This last reference contains a lot of interesting pictures).
- For schemes, the material is taken from books by Eisenbud and Harris [11], Hartshorne [14, Chapter 2] and Mumford [19, Chapter I].
- For algebraic groups and group schemes, the material is taken from the book by Waterhouse [26], supplemented by [15] and [24].

Some other useful sources are given below.

- Dummit and Foote [9, Chapter 15] give a good introduction to basic algebraic geometry.
- For commutative algebra, the books by Eisenbud [10] and Reid [20].
- For algebraic geometry, the notes by Ravi Vakil available from his homepage.
- Wikipedia is a good source for getting a birds-eye-view of many of the concepts discussed in these notes.

Problems. A number of exercises are included in the notes (many with complete solutions). Apart from the above sources, the exercises are mainly borrowed from a variety of sources which I have not kept track of.

CHAPTER 1

Commutative rings

1.1. Commutative rings

In this section, we review some basic ring theory. A good reference is [4, Chapter 1]. Throughout this course, by a ring we mean a commutative ring with identity. Rings will be denoted usually by the letters R and S. We write Ring for the category of rings.

1.1.1. Product and coproduct in the category of rings. Let us discuss the initial object, terminal object, product and coproduct in Ring.

- Initial object is $\mathbb{Z}$.
- Terminal object is the zero ring $\{0\}$.
- Product is direct product: cartesian product with componentwise addition and multiplication.
- Coproduct is tensor product.

1.1.2. Types of ideals. An ideal M in R is *maximal* if $M \neq R$ and if there is no ideal I which is strictly between M and R.

An ideal I in R is *prime* if $I \neq R$ and if $xy \in I$ then either $x \in I$ or $y \in I$.

An ideal I in R is *primary* if $I \neq R$ and if $xy \in I$ then either $x \in I$ or $y^n \in I$ for some $n > 0$.

An ideal I in R is *radical* if

$$x^n \in I \text{ for some } n > 0 \text{ implies } x \in I.$$

An ideal I in R is *irreducible* if

$$I = J \cap K \text{ implies } I = J \text{ or } I = K,$$

for any ideals J and K.

Example 1.1. Consider the ring of integers $\mathbb{Z}$. This is the initial object in Ring. All ideals are principal, so they are of the form (n) for some $n \in \mathbb{Z}$.

- (n) is a maximal ideal iff n is a prime number.
- (n) is a prime ideal iff $n = 0$ or n is a prime number.
- (n) is a primary ideal iff $n = 0$ or n is a prime power.
- (n) is a radical ideal iff $n = 0$ or n is a product of distinct primes to the first power, that is, n is square-free.
- (n) is an irreducible ideal iff n is a primary ideal.

A ring R is *reduced* if for all $x \in R$ and each $n \in \mathbb{N}$,

$$x^n = 0 \iff x = 0.$$

In other words, a ring is reduced if it contains no nonzero nilpotent elements. Check that a ring is reduced iff the zero ideal is radical.

A ring R is *local* if it has a unique maximal ideal.

It is well-known and straightforward to verify that:

- An ideal $M \subseteq R$ is maximal iff R/M is a field.
- An ideal $P \subseteq R$ is prime iff R/P is a domain.
- An ideal $P \subseteq R$ is primary iff $R/P \neq 0$ and every zero divisor in R/P is nilpotent.
- An ideal $I \subseteq R$ is radical iff R/I is a reduced ring.
- Every maximal ideal is prime, and every prime ideal is radical. The converse is false in both cases.

Remark 1.2. For a field or domain, one requires $0 \neq 1$, so it has at least two elements. For a ring R, it may happen that $0 = 1$ in which case $R = 0$, that is, R has only one element. This is the terminal object in Ring.

Proposition 1.3. *Let $I_1, \ldots, I_n$ be ideals and let P be a prime ideal containing $\bigcap_{j=1}^n I_j$. Then $P \supseteq I_j$ for some j. If $P = \bigcap_{j=1}^n I_j$, then $P = I_j$ for some j.*

Draw picture.

Proof. The second claim immediately follows from the first. Further, it is enough to prove the first claim for two ideals: If a prime ideal P contains $I \cap J$ then it contains either I or J.

Suppose not. Then there is an $x \in I$ and $y \in J$ such that $x, y \notin P$. But $xy \in I \cap J$ and hence $xy \in P$, contradicting primeness. □

Proposition 1.4. *Let $P_1, \ldots, P_n$ be prime ideals and let I be an ideal contained in $\cup_{i=1}^n P_i$. Then $I \subseteq P_i$ for some i.*

Proof. Very good exercise. Proceed by induction on n. Solution is given in [4, Proposition 1.11]. □

1.1.3. Contraction and restriction. Let $\varphi : R \to S$ be a ring homomorphism. Let P be the poset of ideals of R and Q be the poset of ideals of S under inclusion. Define order-preserving maps

$$f : P \to Q \quad \text{and} \quad g : Q \to P$$

as follows: $f(I)$ is the smallest ideal of Q which contains $\varphi(I)$ (the latter may not be an ideal), while $g(J)$ is the inverse image of J under φ. $f(I)$ is called the *extension* of I, while $g(J)$ is called the *contraction* of J.

Observe that

$$I \leq g(J) \iff f(I) \leq J.$$

Thus f is the left adjoint to g.

By reversing the partial order on Q, we are in the setup of a Galois connection. So we have the consequences given by Proposition B.9. These are explicitly stated in [4, Proposition 1.17]. For example:

$$I \subseteq gf(I) \qquad \text{and} \qquad J \supseteq fg(J) = \text{Image}(f) \cap J.$$

It is natural to wonder whether one can describe the closed sets of P and Q explicitly. There is a simple answer to this question when φ is surjective which we now explain.

Suppose $\varphi : R \to S$ is surjective. In this case, for any ideal I of R, $\varphi(I)$ is an ideal and thus $f(I) = \varphi(I)$. It follows that $fg = \text{id}$. Thus all ideals of S are closed sets. Which ideals of R are closed sets? It is not true in general that $gf = \text{id}$.

However, it is easy to check that $gf(I) = I$ if I contains $\ker \varphi$. It follows that the closed sets of R are precisely those ideals which contain $\ker \varphi$.

Proposition 1.5. *Let $\varphi : R \to S$ be a surjective ring homomorphism. Then there is a bijection between all ideals of S and those ideals of R which contain $\ker \varphi$. Further this restricts to a bijection between all maximal (prime, primary, irreducible) ideals of S and those maximal (prime, primary, irreducible) ideals of R which contain $\ker \varphi$.*

PROOF. We have already proved the first part. The assertion about maximal (prime, primary, irreducible) ideals can be deduced from the following two facts. If J is a maximal (prime, primary, irreducible) ideal in S, then $\varphi^{-1}(J)$ is a maximal (prime, primary, irreducible) ideal in R. If $\varphi^{-1}(J)$ is a maximal (prime, primary, irreducible) ideal in R for some ideal J of S, then J is a maximal (prime, primary, irreducible) ideal. □

Radical also seems to work.

1.1.4. Radical ideals continued. The radical of an ideal $I \subseteq R$ is defined to be

$$\operatorname{rad}(I) := \{f \in R \mid f^n \in I \text{ for some } n > 0\}.$$

In particular,

$$\operatorname{rad}((0)) := \{f \in R \mid f^n = 0 \text{ for some } n > 0\}.$$

This is precisely the set of *nilpotent elements* in R. It is called the *nilradical* of R.

Observation 1.6. Let I be any ideal in R. Consider the canonical ring homomorphism $\varphi : R \to R/I$. Then the inverse image of the nilradical of R/I under φ is precisely the radical of I.

Proposition 1.7. *The radical of any ideal in R is again an ideal in R. In particular, the nilradical of R is an ideal.*

PROOF. The sum of two nilpotent elements is again nilpotent: use binomial theorem. The product of a nilpotent element with any other element is again nilpotent. It follows that the nilradical of R is an ideal.

For the general case, use Observation 1.6 and the fact that the inverse image of an ideal under a ring homomorphism is again an ideal. □

Proposition 1.8. *The nilradical of R is the intersection of all the prime ideals of R.*

We give a proof using localization discussed later. For a direct proof, see [4, Proposition 1.8]. Some other references are [22, Proposition 2, pg 3] or [16, Theorem 7.1].

PROOF. If f is nilpotent and P is a prime ideal, then $f \in P$. So the nilradical is contained in the intersection of all the prime ideals of R. To show that this is an equality, suppose f is not nilpotent. Then consider the multiplicative set $D = \{1, f, f^2, \dots\}$. Since f is not nilpotent, $0 \notin D$. So $\frac{R}{D}$ is not the zero ring. So it contains a prime ideal. Its inverse image under the ring homomorphism $R \to \frac{R}{D}$ yields a prime ideal P in R which is disjoint from D, and in particular, does not contain f. Hence f does not belong to the intersection of all the prime ideals of R. □

More generally:

Proposition 1.9. *The radical of an ideal I is the intersection of all the prime ideals which contain I.*

PROOF. One direction of this claim is trivial. The first claim then follows from the following chain of equalities.

$$\operatorname{rad}(I) = \varphi^{-1}(N) = \varphi^{-1}\Big(\bigcap P\Big) = \bigcap \varphi^{-1}(P) = \bigcap Q.$$

Here I is any ideal of R, N is the nilradical of R/I, $\varphi : R \to R/I$ is the canonical surjection, the first two intersections are over all prime ideals P of S (here we use Proposition 1.8), and the last intersection is over all prime ideals Q of R which contain I (here we use Proposition 1.5). □

Proposition 1.10. *The radical of a primary ideal P is the smallest prime ideal containing P.*

PROOF. One can check using the definitions that $\operatorname{rad}(P)$ is a prime ideal: Suppose $xy \in \operatorname{rad}(P)$. Then $(xy)^m \in P$ for some $m > 0$, and therefore either $x^m \in P$ or $y^{mn} \in P$ for some $n > 0$. Thus either $x \in \operatorname{rad}(P)$ or $y \in \operatorname{rad}(P)$ as required.

The fact that it is the smallest follows from Proposition 1.9. □

Example 1.11. Let $R = \mathbb{Z}$ be the ring of integers. Any ideal (n) in $\mathbb{Z}$ can be written as an intersection of primary ideals: Let $n = p_1^{\alpha_1} \dots p_n^{\alpha_n}$ be the prime factorization of n. Then

$$(p_1^{\alpha_1} \dots p_n^{\alpha_n}) = (p_1^{\alpha_1}) \cap \dots \cap (p_n^{\alpha_n}).$$

Note that the ideals on the right are primary. We will look at such decompositions in more detail later.

Applying the radical operator to both sides yields a decomposition of $\operatorname{rad}((n))$ into prime ideals:

$$\operatorname{rad}((n)) = (p_1 \dots p_n) = (p_1) \cap \dots \cap (p_n).$$

We note some important properties of the radical operator:

Proposition 1.12. *Let I and J be any ideals of a ring R. Then*

(1) $I \subseteq \operatorname{rad}(I)$.
(2) *If* $I \subseteq \operatorname{rad}(J)$, *then* $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$.
(3) $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) = \operatorname{rad}(I) \cap \operatorname{rad}(J)$.
(4) $\operatorname{rad}(I) = R \iff I = R$.

PROOF. Straightforward. □

Is this topological?

The first two properties show that $\operatorname{rad}(-)$ is a closure operator on the poset of ideals of R. (This uses the result of Proposition 1.7.) So intersection of radical ideals is again a radical ideal. A more precise statement is made in property (3).

The definition of a radical ideal can be restated as follows. An ideal I is called *radical* if $\operatorname{rad}(I) = I$. An example is the nilradical. Alternatively, a radical ideal is precisely a closed set of the closure operator $\operatorname{rad}(-)$.

Proposition 1.13. *Suppose $Q_1, \dots Q_n$ are primary ideals all having the same radical. Call this prime ideal P. Then the intersection $Q_1 \cap \dots \cap Q_n$ is a primary ideal and its radical is also P.*

PROOF. Let $Q = Q_1 \cap \cdots \cap Q_n$. Observe that

$$\operatorname{rad}(Q_1 \cap \cdots \cap Q_n) = \operatorname{rad}(Q_1) \cap \cdots \cap \operatorname{rad}(Q_n) = P.$$

This shows that $\operatorname{rad}(Q) = P$.

We show that Q is primary. Let $xy \in Q$ but $y \notin Q$. Then there is a i such that $xy \in Q_i$ but $y \notin Q_i$. Since Q_i is primary, some power of x belongs to Q_i. Since $\operatorname{rad}(Q_i) = P$, it follows that $x \in P$. Since $\operatorname{rad}(Q) = P$, it follows that some power of x belongs to Q as required. □

1.1.5. Problems.

(1) Give an example of a ring whose nilradical is not prime.
(2) Does a prime ideal or a maximal ideal necessarily contain all the zero divisors of the ring?
(3) Show that there cannot be any ring homomorphisms of the form $\mathbb{C} \to \mathbb{R}$, $\mathbb{C} \to \mathbb{R}[x]$, or $\mathbb{C}[x] \to \mathbb{R}[x]$.
(4) Let $f : R \to S$ be a ring homomorphism. Show that if P is a prime (primary) ideal of S, then $f^{-1}(P)$ is a prime (primary) ideal of R. In contrast, show that $f^{-1}(M)$ may not be a maximal ideal of R even if M is a maximal ideal of S.
(5) Recall that IJ is the ideal consisting of all sums of products xy, with $x \in I$ and $y \in J$. Observe that $IJ \subseteq I \cap J$. Show by example that the ideal IJ may be included properly in $I \cap J$.

    Recall that $I + J$ is the ideal whose elements are sums of elements of I and elements of J. (This construction can be extended to any family of ideals.) Show that if $I + J = R$, then $IJ = I \cap J$.
(6) Let R be a ring. Consider the closure operator on the Boolean poset 2^R of Example B.3. (Note that 2^R is a complete lattice.) It follows from Proposition B.8 that the poset of ideals of R under inclusion is a complete lattice Describe arbitrary meets and arbitrary joins in this poset and compare them with those in 2^R.
(7) Let I be an ideal of a ring R. Show that the quotient ring R/I is reduced iff I is a radical ideal.
(8) Let x be a nilpotent element of a ring R. Show that $1 + x$ is a unit of R. Deduce that the sum of a nilpotent element and a unit is a unit.
(9) Let R be a ring and let $R[x]$ be the ring of polynomials in an indeterminate x with coefficients in R. Let $f = a_0 + a_1x + \cdots + a_nx^n \in R[x]$.
    (a) Suppose R is an integral domain. Then so is $R[x]$. Further f is a unit in $R[x]$ iff a_0 is a unit in R and $a_1 = \cdots = a_n = 0$.
    (b) Show that f is a unit in $R[x]$ iff a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent.
    (c) Show that f is nilpotent iff $a_0, a_1, \ldots, a_n$ are nilpotent.
    (d) Show that f is a zero-divisor iff there exists $a \neq 0$ in R such that $af = 0$.
    (e) f is said to be primitive if $(a_0, \ldots, a_n) = R$. Show that if $f, g \in R[x]$, then fg is primitive iff f and g are primitive.
(10) Let R be a ring which is nonzero. Show that the set of prime ideals of R has a minimum element wrt inclusion.
(11) Show that $(x^2 + 1)$ is a radical ideal in the ring $\mathbb{R}[x]$.

(12) Describe all radical ideals, prime ideals and maximal ideals in the ring $\mathbb{C}[x]$. How about $\mathbb{R}[x]$ and $\mathbb{Z}[x]$?

1.2. Noetherian rings

Noetherian rings are an important class of rings with good finiteness properties. Their relevance to algebraic geometry comes from the fact that one of the basic rings in algebraic geometry, namely $\mathbb{C}[x_1, \ldots, x_n]$, is Noetherian.

A good reference for this section is [4, Chapter 6 and 7].

Definition 1.14. A ring R is said to be *Noetherian* if it satisfies the following equivalent conditions.

(1) Every ideal of R is finitely generated.
(2) Every ascending chain of ideals in R is stationary.
(3) Every nonempty set of ideals in R has a maximal element.

PROOF. (2) $\implies$ (3). If (3) is false, then there is a nonempty set of ideals in R with no maximal element, and we can construct inductively a non-terminating strictly increasing sequence in the chosen set. (Note that Zorn's lemma is not required.)

(3) $\implies$ (2). Suppose we are given an ascending chain of ideals. By (3), it has a maximal element. Then the chain stabilizes from that point onwards.

To show (1) $\iff$ (2), see [4, Propositions 6.2]. □

Example 1.15. The ring of integers $\mathbb{Z}$ is a Noetherian ring: $(n) \subseteq (m) \iff m$ divides n. We point out that $\mathbb{Z}$ does contain descending chains of ideals which are not stationary.

Every field is Noetherian, since a field has only two ideals.

Lemma 1.16. *In a Noetherian ring, every ideal is a finite intersection of irreducible ideals.*

PROOF. Suppose not; then the set of ideals for which the result is false is not empty. Using property (3), it has a maximal element I wrt inclusion. Since I is not irreducible, it can be written as $J \cap K$ in a nontrivial way. Since J and K are strictly bigger than I, they are both finite intersections of irreducible ideals, and therefore so is I. This is a contradiction. □

Lemma 1.17. *In a Noetherian ring, every irreducible ideal is primary.*

PROOF. In view of Proposition 1.5, by passing to the quotient ring, it is enough to show that is the zero ideal is irreducible, then it is primary. This can be proved in an elementary manner using the ascending chain condition [4, Lemma 7.12]. □

Theorem 1.18 (Lasker-Noether theorem). *Let R be a Noetherian ring. Then*

- *Every ideal of R can be written as a finite intersection of primary ideals. Further the radicals of these primary ideals are distinct.*
- *Every radical ideal of R is uniquely expressible as a finite intersection of prime ideals P_i with $P_i \not\subset P_j$ for $j \neq i$.*

PROOF. The first claim follows from Lemmas 1.16 and 1.17, and the second from Proposition 1.13.

For the last claim, apply $\operatorname{rad}(-)$ to the decomposition. In view of Proposition 1.10, one can deduce that any radical ideal of R can be written as a finite intersection of prime ideals. The uniqueness assertion follows from Proposition 1.3. □

It follows that in the poset of prime ideals of a Noetherian ring R which contain a given radical ideal of R, there are finitely many minimum elements.

Theorem 1.19 (Hilbert basis theorem). *If R is Noetherian, then the polynomial ring $R[x]$ is Noetherian. It follows by induction that if R is Noetherian, then the polynomial ring $R[x_1, \ldots, x_n]$ is Noetherian.*

PROOF. See [4, Theorem 7.5]. □

Corollary 1.20. *The ring $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian. So*

- *all ideals of $\mathbb{C}[x_1, \ldots, x_n]$ are finitely generated.*
- *any radical ideal of $\mathbb{C}[x_1, \ldots, x_n]$ is uniquely expressible as a finite intersection of prime ideals P_i with $P_i \not\subset P_j$ for $j \neq i$.*

1.2.1. Problems.

(1) If R is Noetherian, then so is R/I for any ideal I of R. Conclude that any finitely generated $\mathbb{C}$-algebra is Noetherian.
(2) Give an example of a ring which is not Noetherian.
(3) Describe explicitly the content of the Lasker-Noether theorem for the ring $\mathbb{C}[x]$.
(4) Show that if all the prime ideals of a ring R are finitely generated, then R is Noetherian.

1.3. Noetherian spaces

In this section, we discuss the geometric analogue of Noetherian rings. These are known as Noetherian spaces. (We will see later that the spectrum of a Noetherian ring is a Noetherian space.)

1.3.1. Irreducible topological spaces.

We define a topological notion that is similar to the notion of connectedness.

A topological space X is called *irreducible* if it cannot be written as a union of two proper nonempty closed sets. The empty set is not considered to be irreducible. A maximal irreducible subspace of X is called an *irreducible component*. A subspace $Y \subseteq X$ is called irreducible if it is irreducible as a topological space (with the induced topology).

Proposition 1.21. *Let X be a topological space and $Y \subseteq X$ be a subspace. Then*

(1) *If X is irreducible, then X is connected.*
(2) *X is irreducible iff Any two nonempty open sets in X intersect iff Any nonempty open set is dense in X.*
(3) *If Y is irreducible, then the closure of Y in X is also irreducible.*
(4) *Let $f : X \to Z$ be a continuous map. Then X irreducible implies $f(X)$ irreducible.*

In view of part (3), an irreducible component of X is always a closed set in X.

PROOF. The first two claims are straightforward. In view of these, Y is irreducible iff the intersection of two open subsets of X each meeting Y, also meets Y; and similarly for $\overline{Y}$. But an open set meets Y iff it meets $\overline{Y}$.

If U, X are open sets in Z which meet $f(X)$, we have to show that $U \cap X$ meets $f(X)$ as well. But $f^{-1}(U)$, $f^{-1}(X)$ are nonempty open sets in X, so they have a nonempty intersection (X being irreducible), whose image under f lies in $U \cap X \cap f(X)$. □

Union of two intersecting curves in $\mathbb{A}^n$ is connected, but not irreducible.

1.3.2. Noetherian spaces. A topological space is *Noetherian* if it satisfies any of the following equivalent conditions.

- The descending chain condition on closed sets: Given a descending chain of closed sets
$$V_1 \supseteq V_2 \supseteq \cdots \supseteq V_i \supseteq \ldots,$$
there is a n such that $V_i = V_n$ for all $i \geq n$.
- The minimality condition on closed sets: each nonempty collection of closed sets has a minimal element.
- The ascending chain condition on open sets: Given a ascending chain of open sets
$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_i \subseteq \ldots,$$
there is a n such that $U_i = U_n$ for all $i \geq n$.
- The maximality condition on open sets: each nonempty collection of open sets has a maximal element.

Proposition 1.22. *Let X be a Noetherian space. Then X has only finitely many irreducible components $X_1, \ldots, X_m$ and $X = \bigcup_j X_j$.*

PROOF. Consider the collection $\mathcal{A}$ of all finite unions of closed irreducible subsets of X (for example $\emptyset \in \mathcal{A}$). Suppose X itself does not belong to $\mathcal{A}$. Use the Noetherian property to find a closed subset Y of X which is minimal among the closed subsets (such as X) not belonging to $\mathcal{A}$. Evidently Y is neither empty nor irreducible; so $Y = Y_1 \cup Y_2$, both Y_1 and Y_2 being proper closed subsets of Y. The minimality of Y forces both Y_1 and Y_2 to belong to $\mathcal{A}$. But then Y also belongs to $\mathcal{A}$, which is a contradiction.

Write $X = \bigcup_j X_j$, where the X_j are irreducible closed subsets. If Y is any maximal irreducible closed subset of X, then since $Y = \bigcup_j (Y \cap X_j)$, we must have $Y \cap X_i = Y$ for some i. Thus $Y = X_i$ by maximality. □

Proposition 1.23. *Let X be a Noetherian space. Then every subspace of X is Noetherian, and X is quasi-compact.*

PROOF. Use the ascending chain condition on open sets and the definition of quasi-compactness: every cover has a finite subcover. □

Remark 1.24. Since Noetherian spaces need not be Hausdorff, it is customary to use the term quasi-compact instead of compact. If the space is not Hausdorff, one cannot use sequential convergence arguments.

1.3.3. Problems.

(1) Let X be a Noetherian space, Y a subspace having irreducible components $Y_1, \dots Y_m$. Show that the closures $\overline{Y}_j$ are the irreducible components of $\overline{Y}$.
(2) Show that the following conditions on a topological space X are equivalent.
   (a) X is Noetherian.
   (b) Every subspace of X is quasi-compact.
   (c) Every open subspace of X is quasi-compact.
(3) Describe the irreducible components of $\mathbb{V}(I) \subseteq \mathbb{C}^3$ for each of the following ideals I of $\mathbb{C}[x, y, z]$.
   (a) $(y^2 - x^4, x^2 - 2x^3 - x^2y + 2xy + y^2 - y)$
   (b) $(xy + yz + xz, xyz)$
   (c) $((x - z)(x - y)(x - 2z), x^2 - y^2z)$

1.4. Localization

In this section, we discuss the concept of localization in a ring. The notion of a multiplicative set plays the key role. For more details, see [4, Chapter 3].

1.4.1. Field of fractions of an integral domain. The field of rational numbers $\mathbb{Q}$ is constructed from the ring of integers $\mathbb{Z}$ by forming fractions m/n with $n \neq 0$, and declaring two fractions m/n and m'/n' to be equivalent whenever $mn' = nm'$. This procedure generalizes to any integral domain and produces its field of fractions as follows.

Let R be an integral domain. Let D be the set of all nonzero elements of R. Define an equivalence relation on $R \times D$:

$$(a, s) \sim (b, t) \iff at = bs.$$

This is clearly reflexive and symmetric. To check transitivity: Suppose $(a, s) \sim (b, t)$ and $(b, t) \sim (c, u)$. Then $at = bs$ and $bu = ct$. Hence $atu = bsu = bsu = cts$. Cancelling t, we have $au = cs$, or $(a, s) \sim (c, u)$ as required.

Let $\frac{R}{D}$ denote the set of equivalence classes. It is customary to denote the equivalence class of (a, s) by $\frac{a}{s}$. Define operations

$$\frac{a}{s} + \frac{b}{t} := \frac{at + bs}{st} \qquad \text{and} \qquad \big(\frac{a}{s}\big)\big(\frac{b}{t}\big) := \frac{ab}{st}.$$

Check that these operations are well-defined. This turns $\frac{R}{D}$ into a field (with the inverse of $\frac{a}{s}$ being $\frac{s}{a}$). It is called the *field of fractions* of R wrt D.

1.4.2. Ring of fractions. The above procedure generalizes further to any ring. Let R be a ring. A *multiplicative subset* of R is a subset D of R such that $1 \in D$ and whenever s and t belong to D, so does their product. Define an equivalence relation on $R \times D$:

$$(a, s) \sim (b, t) \iff (at - bs)u = 0 \text{ for some } u \in D.$$

This is clearly reflexive and symmetric. Transitivity can be checked as in the integral domain case.

Let $\frac{R}{D}$ denote the set of equivalence classes. Addition and multiplication operations as in the integral domain case turn $\frac{R}{D}$ into a ring (and not a field). It is called the *ring of fractions* of R wrt D.

Remark 1.25. If the multiplicative set contains 0, then R_D will be the zero ring. It has no prime ideals.

There is a ring homomorphism

$$\varphi : R \to \frac{R}{D} \qquad x \mapsto \frac{x}{1}. \tag{1.1}$$

It is *not* injective in general. It has the following properties:

- If $s \in D$, then $\varphi(s)$ is a unit in $\frac{R}{D}$.
- If $\varphi(a) = 0$, then $as = 0$ for some $s \in D$.
- Every element of $\frac{R}{D}$ is of the form $\varphi(a)\varphi(s)^{-1}$ for some $a \in R$ and some $s \in D$.

Proposition 1.26. *Let $\psi : R \to S$ be a ring homomorphism such that $\psi(s)$ is a unit in S for all $s \in D$. Then there exists a unique ring homomorphism $\psi' : \frac{R}{D} \to S$ such that $\psi = \psi'\varphi$.*

PROOF. Straightforward. □

1.4.3. Localization at a prime ideal. Let I be any ideal in R. Then

$$I \text{ is a prime ideal} \iff R \setminus I \text{ is a multiplicative set.}$$

For a prime ideal P in R, $D = R \setminus P$ is a multiplicative set. In this situation, it is customary to write R_P instead of $\frac{R}{D}$. The process of passing from R to R_P is called *localization* at the prime ideal P.

If R is an integral domain, then (0) is a prime ideal and $D = R \setminus \{0\}$ is a multiplicative set. In this case, observe that $R_{(0)}$ is same as the field of fractions of R, and further, $R \hookrightarrow R_P \hookrightarrow R_{(0)}$.

Example 1.27. Consider the ring of integers $\mathbb{Z}$. Then $\mathbb{Z}_{(0)}$ is the set of rational numbers $\mathbb{Q}$. Let p be a prime number. Then $\mathbb{Z}_{(p)}$ is the set of all rational numbers whose denominators are not divisible by p.

Proposition 1.28. *R_P is a local ring: The unique maximal ideal in R_P is the extension of P under (1.1).*

PROOF. The elements a/s with $a \in P$ and $s \notin P$ form an ideal in R_P. Call it M. If b/t does not belong to M, then $b \notin P$; hence b/t is a unit in R_P. It follows that if I is any ideal in R_P and $I \not\subseteq M$, then I contains a unit and hence $I = R$. Thus M is the only maximal ideal in R_P. □

1.4.4. Extended and contracted ideals. Recall that any ring homomorphism yields a Galois connection between their posets of ideals. Let us understand the extended and contracted ideals (the closed sets of the associated closure operators) for the homomorphism (1.1).

Observe that the extension of an ideal I of R is given by

$$\frac{I}{D} := \Big\{ \frac{a}{s} \in \frac{R}{D} \mid a \in I \Big\}.$$

Proposition 1.29. *Consider the ring homomorphism (1.1). Then:*

1. *Every ideal of $\frac{R}{D}$ is an extended ideal.*
2. *For any ideal I of R, the following are equivalent.*
   - *I is a contracted ideal.*
   - *$I = \{x \in R \mid xs \in I \text{ for some } s \in D\}$.*

- *No element of D is a zero-divisor in R/I.*

PROOF. This is a straightforward exercise. □

Proposition 1.30. *Let D be a multiplicative set of a ring R. Then the prime ideals of $\frac{R}{D}$ are in correspondence with the prime ideals of R which do not meet D.*

PROOF. We show that this correspondence is obtained by restricting the bijection between contracted and extended ideals.

Every prime ideal in $\frac{R}{D}$ is an extended ideal by Proposition 1.29, part (1). The contraction of a prime ideal is a prime ideal (for any ring homomorphism). And in our case, it is a prime ideal which does not meet D.

Every prime ideal of R which does not meet D is a contracted ideal by Proposition 1.29, part (2). So we need to show that if P is a prime ideal in R which does not meet D, then $\frac{P}{D}$ is a prime ideal in $\frac{R}{D}$. Accordingly, let P be such a prime ideal and suppose

$$(\frac{x}{s})(\frac{y}{t}) = \frac{z}{u} \in \frac{P}{D}$$

where $z \in P$ and $s, t, u \in D$. Then $v(xyu - zst) = 0$ for some $v \in D$. Thus $xyuv \in P$ but $uv \notin P$ by hypothesis. So either $x \in P$ or $y \in P$ as required. □

As a consequence:

Proposition 1.31. *If P is a prime ideal of R, then the prime ideals of the local ring R_P are in correspondence with the prime ideals of R contained in P.*

Proposition 1.32. *If R is an integral domain, then R is equal to the intersection (inside its quotient field) of its localizations at all maximal ideals.*

PROOF. Let K denote the quotient field of R. Suppose $h \in R_M$ for all maximal ideals M. Then consider the ideal

$$I = \{r \in R \mid rh \in R\}.$$

(I consists of all ring elements which can clear out the denominator of some fractional representation of h.) It is enough to show that $I = R$. Suppose not. Then there is a maximal ideal M' which contains I. Since $h \in R_{M'}$, it can be expressed in the form a/b where b does not belong to M'. But observe that $b \in I$, which is a contradiction. □

1.4.5. Problems.

(1) Let R be a ring. Suppose that for each prime ideal P, the local ring R_P has no nonzero nilpotent elements. Show that R has no nonzero nilpotent elements.

If each R_P is an integral domain, is R necessarily an integral domain?

(2) For any multiplicative set D not containing 0, show that there is a prime ideal which does not meet D. A more general result is proved below.

(3) Let R be a ring which is not zero, and let Σ be the set of all multiplicative subsets D of R such that $0 \notin D$. Show that Σ has maximal elements, and that $D \in \Sigma$ is maximal iff $R \setminus D$ is a minimal prime ideal of R.

Alternatively, we may also deduce this from exercise (2).

(4) A multiplicative subset D of a ring R is called saturated if $x, y \in D$ implies $x \in D$ and $y \in D$. Show that D is saturated iff $R \setminus D$ is a union of prime ideals.

(5) Show that the set of zero-divisors in a ring is a union of prime ideals.
(6) Show that if R is Noetherian and D is any multiplicative subset, then $\frac{R}{D}$ is also Noetherian.
(7) Give an example of a ring R and a multiplicative subset D such that

$$(a, s) \sim (b, t) \iff at = bs.$$

is *not* an equivalence relation on $R \times D$.

Give an example of a ring R and distinct multiplicative subsets D and E such that $\frac{R}{D} \cong \frac{R}{E}$.
(8) Consider the multiplicative set $D = \{1, 2, 4, 8, 12, 16\}$ for the ring $R = \mathbb{Z}_{20}$ of integers modulo 20. Describe the map

$$\operatorname{Spec}(\frac{R}{D}) \to \operatorname{Spec} R$$

induced by the homomorphism $R \to \frac{R}{D}$.

1.5. The category of local rings

A ring is local if it has a unique maximal ideal. Let S and R be local rings, with maximal ideals N and M. A ring homomorphism $\alpha : S \to R$ is said to be *local* if $\alpha(N) \subseteq M$, or equivalently, $\alpha^{-1}(M) = N$. This defines the category of local rings.

Definition 1.33. Suppose S is a local ring with unique maximal ideal M. Then the quotient S/M is a field and is called the *residue field* of S.

Let S and R be local rings, with maximal ideals N and M, and residue fields L and K. A local ring homomorphism $\alpha : S \to R$ induces a field homomorphism

$$\alpha : L \hookrightarrow K.$$

Thus, a local ring homomorphism allows us to compare the residue fields of the two local rings.

Example 1.34. Let $\alpha : S \to R$ be any ring homomorphism. Let P be any prime ideal in R and $\alpha^{-1}(P) = Q$ be the corresponding prime ideal in S. This yields an induced local ring homomorphism $\alpha : S_Q \to R_P$, and hence a field extension among the residue fields. We note that the residue field of R_P, denoted $\kappa(P)$ coincides with the field of fractions of the integral domain R/P: This can be deduced from the following diagram.

$$\begin{array}{ccc} R & \longrightarrow & R_P \\ \downarrow & & \downarrow \\ R/P & \hookrightarrow & \kappa(P). \end{array}$$

(see Remark at the end of [4, page 42]).

Since $\mathbb{Z}$ is an initial object in Ring, for any ring R, there is a unique ring homomorphism $\alpha : \mathbb{Z} \to R$. Let P be any prime ideal in R. Then $\alpha^{-1}(P)$ is a prime ideal of $\mathbb{Z}$. It follows that the residue field of R_P is a field extension of either $\mathbb{Q}$ or of $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

The following are simple but useful results to keep in mind.

- $f \in R$ is zero in all localizations R_P at a prime ideal P iff $f = 0$.
- $f \in R$ is zero in all residue fields $\kappa(P)$ iff f is nilpotent.

- R is a reduced ring iff R_P is a reduced ring for all prime ideals P.

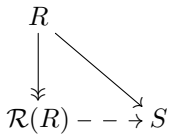
1.6. The category of reduced rings

Let $\mathsf{redRing}$ denote the category of reduced rings: objects are reduced rings and morphisms are ring homomorphisms. Define a functor

$$\mathcal{R} : \mathsf{Ring} \to \mathsf{redRing} \qquad R \mapsto R/N, \tag{1.2}$$

where N is the nilradical of R: Note that a ring homomorphism $R \to S$ induces a ring homomorphism $\mathcal{R}(R) \to \mathcal{R}(S)$ since nilpotent elements have to map to nilpotent elements.

We note that there is a natural map $R \to \mathcal{R}(R)$. If $R \to S$ is a ring homomorphism and S is reduced, then there is a unique ring homomorphism $\mathcal{R}(R) \to S$ such that the diagram



commutes. Equivalently, there is a natural bijection

$$\mathsf{Ring}(R, S) \to \mathsf{redRing}(\mathcal{R}(R), S). \tag{1.3}$$

This says that the functor $\mathcal{R}$ is the left adjoint to the inclusion functor $\mathsf{redRing} \to \mathsf{Ring}$.

CHAPTER 2

Affine varieties

2.1. The category of affine varieties

There is an axiomatic approach to doing geometry as developed by ancient greek geometers such as Euclid. The past few centuries has seen another approach to doing geometry which proposes that geometric objects can be studied using algebraic equations. This approach has turned out to be very powerful and has made seemingly complicated geometric objects accessible.

The objects that interest us in this course are those that can be defined using *polynomial* equations. There are many familiar examples such as the circle and the parabola. Objects of this type are called affine varieties. They are the most basic objects of algebraic geometry. We begin with the basic definitions in this section.

The problem of solving polynomial equations has led to many important developments in mathematics: The equation $x^2 + 1 = 0$ led to the discovery of the complex numbers. It was then discovered that any polynomial with complex coefficients can be completely factored. This is called the fundamental theorem of algebra. Thus the importance of the field or ring over which the equation is being solved was realized.

2.1.1. Affine varieties.

Definition 2.1. An *affine variety* is the common zero set of a finite set $\{f_1, \ldots, f_m\}$ of polynomials in n variables over $\mathbb{C}$, that is, $f_j \in \mathbb{C}[x_1, \ldots, x_n]$. We write

$$X = \mathbb{V}(f_1, \ldots, f_m) \subseteq \mathbb{C}^n.$$

A *subvariety* of an affine variety $X \subseteq \mathbb{C}^n$ is an affine variety $Y \subseteq \mathbb{C}^n$ that is contained in X.

An affine variety X is called *irreducible* if it cannot be decomposed as the union of two proper subvarieties. A maximal irreducible subvariety of X is called an *irreducible component* of X.

For example, $X = \mathbb{V}(x_1, x_2) \subseteq \mathbb{C}^3$ is the complex line in $\mathbb{C}^3$ consisting of the x_3-axis.

Example 2.2. We record some standard examples of affine varieties.

(1) The space $\mathbb{C}^n$, the empty set, and one-point sets, singletons, are affine varieties:

$$\begin{aligned} \mathbb{C}^n &= \mathbb{V}(0) \\ \emptyset &= \mathbb{V}(1) \\ \{(a_1, \ldots, a_n)\} &= \mathbb{V}(x_1 - a_1, \ldots, x_n - a_n) \end{aligned}$$

The space $\mathbb{C}^n$ is usually called (complex) affine n-space and denoted $\mathbb{A}^n$. The term affine as well as the notation emphasize the fact that the origin

does not play any distinguished role. The empty set is not considered to be an irreducible variety.

(2) An affine plane curve is the zero set of one polynomial in the complex plane $\mathbb{C}^2$. For example, $\mathbb{V}(y-x^2)$, $\mathbb{V}(y^2-x^2-x^3)$, $\mathbb{V}(x^2y+xy^2-x^4-y^4)$ are affine plane curve.

(3) The zero set of one polynomial in $\mathbb{C}^n$ is called a *hypersurface* in $\mathbb{C}^n$. For example, the quadratic cone $\mathbb{V}(x^2+y^2-z^2)$ is a hypersurface in $\mathbb{C}^3$. An affine plane curve is a hypersurface in $\mathbb{C}^2$.

(4) The zero set of one linear (degree-one) polynomial in $\mathbb{C}^n$ is called a *hyperplane* in $\mathbb{C}^n$. For example, $\mathbb{V}(ax+by-c)$ where a, b and c are complex scalars is a hyperplane in $\mathbb{C}^2$.

A *linear affine variety* is the zero set of a finite number of linear polynomials in $\mathbb{C}^n$. If there are k linearly independent polynomials, then the linear variety is a complex space of dimension $n-k$.

The union of two complex lines

$$\mathbb{V}(xy) = \mathbb{V}(x) \cup \mathbb{V}(y)$$

is a reducible variety. Later results imply that this is a decomposition of the variety into its irreducible components.

Warning. Some authors such as Hartshorne [14] and Mumford [19] use the term algebraic set for variety, and variety for irreducible variety.

2.1.2. Morphisms of affine varieties. We would now like to describe maps between affine varieties possibly lying in different ambient spaces. For this purpose, let us write an affine variety $X \subseteq \mathbb{C}^n$ as a pair $(X, \mathbb{C}^n)$.

A polynomial map $\mathbb{C}^n \to \mathbb{C}^m$ is a map all of whose components are given by polynomials. For example,

$$\mathbb{C}^2 \to \mathbb{C}^3 \qquad (x,y) \mapsto (x^2, xy, y^2).$$

is a polynomial map.

Definition 2.3. A morphism $(X, \mathbb{C}^n) \to (Y, \mathbb{C}^m)$ of affine varieties is a map $\varphi : X \to Y$ which is the restriction of a polynomial map $\tilde{\varphi} : \mathbb{C}^n \to \mathbb{C}^m$.

This may be visualized as a commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \overset{\tilde{\varphi}}{\dashrightarrow} & \mathbb{C}^m \\ \uparrow & & \uparrow \\ X & \xrightarrow[\varphi]{} & Y. \end{array}$$

There may be more than one choice for the polynomial map shown as a dotted arrow labeled $\tilde{\varphi}$. (Different choices represent the same morphism as long as they agree on X.)

Observe that a morphism $(\mathbb{C}^n, \mathbb{C}^n) \to (\mathbb{C}^m, \mathbb{C}^m)$ of affine varieties is same as a polynomial map $\mathbb{C}^n \to \mathbb{C}^m$.

Note that it makes sense to compose morphisms of affine varieties:

$$\begin{array}{ccccc} \mathbb{C}^n & \dashrightarrow & \mathbb{C}^m & \dashrightarrow & \mathbb{C}^p \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & Y & \longrightarrow & X. \end{array}$$

The essential observation here is that composite of two polynomial maps is again a polynomial map. This defines the category of affine varieties. We denote it by AffVariety. (Associativity of composition and existence of identity morphisms is clear.)

Convention 2.4. Since it is cumbersome to write $(X, \mathbb{C}^n)$, most of the time we will simply write X for an affine variety. It is understood that there is an ambient space in which X sits.

Example 2.5. Any affine map $\mathbb{A}^n \to \mathbb{A}^m$ is a morphism of affine varieties. For example,

$$\mathbb{A}^2 \to \mathbb{A}^3 \qquad (x, y) \mapsto (x + y - 1, x + 5, y - 2)$$

is an affine map, that is, all polynomials are of degree 1. The presence of the constants makes this an affine map rather than a linear map.

An important particular case is that of projections which delete some subset of coordinates. For example,

$$\mathbb{A}^4 \to \mathbb{A}^2 \qquad (x, y, z, w) \mapsto (y, w)$$

is an affine map which is a projection.

Example 2.6. Let $(X, \mathbb{C}^2)$ be the plane parabola defined by the vanishing of the polynomial $y - x^2$. In other words, $X = \mathbb{V}(y - x^2)$. The map

$$\mathbb{A}^1 \to X \qquad t \mapsto (t, t^2)$$

is a morphism of affine varieties (since it is the restriction of a polynomial map). In fact, it is an isomorphism in AffVariety.

The inverse map is given by the restricting the projection

$$\mathbb{A}^2 \supset X \to \mathbb{A}^1 \qquad (x, y) \mapsto x.$$

2.1.3. Product and coproduct of affine varieties. Let us discuss the initial object, terminal object, product and coproduct in AffVariety.

- Any affine variety of the form $(\emptyset, \mathbb{C}^n)$ is an initial object.
- Any affine variety of the form $(\{x\}, \mathbb{C}^n)$ is a terminal object.
- Let $(X, \mathbb{C}^n)$ and $(Y, \mathbb{C}^m)$ be two affine varieties, with the polynomials $\{f_i\}$ defining X and $\{g_j\}$ defining Y. Their categorical product is the cartesian product $(X \times Y, \mathbb{C}^{n+m})$. This is an affine variety defined by the polynomials $\{f_i, g_j\}$.
- Let $(X, \mathbb{C}^n)$ and $(Y, \mathbb{C}^m)$ be two affine varieties. Wlog, say $n \geq m$. Their categorical coproduct is $(X \amalg Y, \mathbb{C}^{n+1})$: This consists of two disjoint pieces, with X included in the first m coordinates, and Y shifted using $z = 1$ where z denotes the last coordinate. Why is this affine?

2.1.4. Problems.

(1) Show that every affine variety in $\mathbb{A}^n$ is a closed set in the Euclidean topology on $\mathbb{A}^n$.
(2) Show that every affine variety is the intersection of finitely many hypersurfaces.
(3) Consider the twisted cubic curve

$$X = \mathbb{V}(x^2 - y, x^3 - z) = \mathbb{V}(x^2 - y) \cap \mathbb{V}(x^3 - z).$$

(Try to visualize this set over $\mathbb{R}$.) Show that it consists of all points in $\mathbb{C}^3$ of the form (t, t^2, t^3), where $t \in \mathbb{C}$. Use this to show that X is isomorphic to the affine line $\mathbb{A}^1$.
(4) Show that the zero set in $\mathbb{A}^2$ of the function $y - e^x$ is *not* an affine variety.
(5) Show that every automorphism of $\mathbb{A}^1$ (isomorphism with itself) is of the form $x \mapsto ax + b$ where $a, b \in \mathbb{C}$ with $a \neq 0$.
(6) Show that the morphism $\mathbb{A}^3 \to \mathbb{A}^3$

$$x' = x,\ y' = y - x^2,\ z' = z + y + x^3.$$

is an isomorphism by explicitly computing its inverse.

Prove that the set of all (iso)morphisms $\mathbb{A}^3 \to \mathbb{A}^3$ of the form

$$x' = x,\ y' = y + p(x),\ z' = z + q(x, y)$$

with p and q polynomials, form a group under composition.
(7) Show that each component of the $\tilde{\varphi}$ of Definition 2.3 is well-defined up to an element of $\mathbb{I}(X)$.
(8) Let $f = x^2 - y^2$ and $g = x^3 + xy^2 - y^3 - x^2y - x + y$. Find the irreducible components of the affine variety $(\mathbb{V}(f, g), \mathbb{C}^2)$.
(9) Let X be the image of the map

(*) $$\mathbb{A}^1 \to \mathbb{A}^2 \qquad t \mapsto (t^2, t^n).$$

For what integral values of $n \geq 1$ is $(X, \mathbb{C}^2)$ isomorphic to $\mathbb{A}^1$ as an affine variety?
(10) Show that if V and W are irreducible affine varieties, then so is their categorical product $V \times W$.
(11) If $\varphi = (f_1, \ldots, f_n) : \mathbb{A}^n \to \mathbb{A}^n$ is an isomorphism of affine varieties, then show that the determinant of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

is a nonzero constant polynomial.

2.2. Affine varieties and radical ideals

The starting point of algebraic geometry is the Nullstellensatz of Hilbert which gives a correspondence between affine varieties which we consider to be geometric objects and radical ideals which we consider to be algebraic objects. We do not give a proof, but we explain how it arises and discuss some consequences.

TABLE 2.1. Geometry and algebra.

Noetherian space $\mathbb{C}^n$	Noetherian ring $\mathbb{C}[x_1, \ldots, x_n]$
subset	ideal
affine variety	radical ideal
irreducible affine variety	prime ideal
point	maximal ideal

2.2.1. A Galois connection. Let P be the poset whose elements are subsets of $\mathbb{C}^n$ ordered by inclusion. Let Q be the poset whose elements are ideals of $\mathbb{C}[x_1, \ldots, x_n]$ ordered by inclusion. We define two order-reversing maps

$$\mathbb{I} : P \to Q \qquad \text{and} \qquad \mathbb{V} : Q \to P$$

as follows.

$$\mathbb{V}(I) := \{x \in \mathbb{C}^n \mid f(x) = 0 \text{ for all } f \in I\}.$$
$$\mathbb{I}(V) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

$\mathbb{V}(I)$ is called the zero-locus of I, while $\mathbb{I}(X)$ is called the vanishing ideal of X.

Observe that

$$X \subseteq \mathbb{V}(I) \iff I \subseteq \mathbb{I}(X).$$

Both sides say that $f(x) = 0$ whenever $f \in I$ and $x \in X$. It follows from Proposition B.9 that

- $\mathbb{V}\mathbb{I}$ and $\mathbb{I}\mathbb{V}$ are closure operators on P and Q respectively.
- The closed sets of $\mathbb{V}\mathbb{I}$ are in bijection with the closed sets of $\mathbb{I}\mathbb{V}$.

Proposition 2.7. *The closed sets of $\mathbb{V}\mathbb{I}$ are precisely affine varieties.*

PROOF. The closed sets of $\mathbb{V}\mathbb{I}$ are precisely sets of the form $\mathbb{V}(I)$ for some ideal I. If X is an affine variety defined by polynomials $f_1, \ldots, f_m$, then X is the zero set of the ideal $I = (f_1, \ldots, f_m)$ generated by them.

Conversely, any ideal I is finitely generated, say by $f_1, \ldots, f_m$. (Here we use the Noetherian property of $\mathbb{C}[x_1, \ldots, x_n]$.) Then $\mathbb{V}(I)$ is the affine variety defined by these generators. □

Theorem 2.8 (Hilbert's Nullstellensatz). *Let I be any ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Then*

$$\mathbb{I}\mathbb{V}(I) = \operatorname{rad}(I).$$

Thus, the closed sets of $\mathbb{I}\mathbb{V}$ are precisely the radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$.

PROOF. It is easy to see that $\mathbb{I}\mathbb{V}(I) \supseteq \operatorname{rad}(I)$: Suppose $f \in \operatorname{rad}(()I)$. Then $f^n \in I$ for some $n > 0$. Therefore, for any $x \in \mathbb{V}(I)$, $f^n(x) = 0$ and hence $f(x) = 0$. This shows that $f \in \mathbb{I}\mathbb{V}(I)$ as required.

The hard part is to prove the other inclusion. A proof can be found in most books on commutative algebra, see for instance [15, page 5].

The $n = 1$ case can be proved using the fundamental theorem of algebra: All ideals in $\mathbb{C}[x]$ are principal. Suppose $I = (f)$. Let

$$f(x) = (x - a_1)^{b_1} \ldots (x - a_m)^{b_m} \quad \text{and} \quad g(x) = (x - a_1) \ldots (x - a_m).$$

Then $\mathbb{V}(I) = \{a_1, \ldots, a_m\}$ and it follows that $\mathbb{I}\mathbb{V}(I) = \operatorname{rad}(I) = (g)$. □

Corollary 2.9. *There is a bijection between affine varieties in $\mathbb{C}^n$ and radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$.*

This correspondence has some interesting consequences.

Proposition 2.10. *Irreducible affine varieties correspond to prime ideals. More precisely, an affine variety X is irreducible iff $\mathbb{I}(X)$ is a prime ideal.*

Proof. Write $I = \mathbb{I}(X)$. Suppose X is irreducible. By convention, X is nonempty, so I is a proper ideal. To show that I is prime, let $fg \in I$. Then for each $x \in X$, either $f(x) = 0$ or $g(x) = 0$. So

$$X \subseteq \mathbb{V}(f) \cup \mathbb{V}(g) \quad \text{and} \quad X = (X \cap \mathbb{V}(f) \cup (X \cap \mathbb{V}(g)).$$

Since X is irreducible, it must lie wholly in one of these sets, that is, either $f \in I$ or $g \in I$, and I is prime.

The first containment will usually be strict. As a simple illustration, take $X = \{a, b\}$ where a and b are points of $\mathbb{C}^2$, one on the x-axis and the other on the y-axis. The function xy vanishes on X; $\mathbb{V}(x) \cup \mathbb{V}(y)$ is the union of the two axes which is strictly bigger than X.

Conversely, suppose I is prime, but $X = X_1 \cup X_2$, where each X_i is a proper affine subvariety of X. Then we can find $f_i \in \mathbb{I}(X_i)$ with $f_i \notin I$. But $f_1 f_2$ vanishes on X, so $f_1 f_2 \in I$, contradicting primeness. □

Proposition 2.11. *Points correspond to maximal ideals. More precisely, the point $(a_1, \ldots, a_n)$ corresponds to the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$.*

A direct elementary proof of this result is given in [3, Section 7.6].

Proof. Let $a = (a_1, \ldots, a_n)$. Then it is easy to see that $\mathbb{I}(\{a\}) = (x_1 - a_1, \ldots, x_n - a_n)$. This is a maximal ideal because the only variety that $\{a\}$ contains is $\emptyset$.

Conversely, if M is any maximal ideal, then $\mathbb{V}(M) \neq \emptyset$ because $\mathbb{V}(R) = \emptyset$; $\mathbb{V}(M)$ cannot have more than one element, because singletons give rise to maximal ideals (as we just saw). □

2.2.2. Problems.

(1) It is true that Theorem 2.8 holds over any algebraically closed field. (Recall that $\mathbb{C}$ is algebraically closed by the fundamental theorem of algebra.) Show that Theorem 2.8 fails over $\mathbb{R}$. Are the closed sets of $\mathbb{IV}$ over $\mathbb{R}$ known?
(2) Show that a radical ideal I in $\mathbb{C}[x_1, \ldots, x_n]$ is the intersection of all the maximal ideals containing I.
(3) Show that the ideal (xy, xz) defines a reducible variety in $\mathbb{C}^3$ and is radical but not prime.
(4) Find the vanishing ideal of the curve with parametric equations $x = t + 1$, $y = t^3$ and $z = t^4 + t^2$ for $t \in \mathbb{C}$. Is this curve an affine variety in $\mathbb{C}^3$?
(5) The polynomials $f = x^2 + y^2 + 1$, $g = x^2 - y + 1$ and $h = xy - 1$ generate the unit ideal in $\mathbb{C}[x, y]$. Prove this in two ways:
   (a) by showing that they have no common zeroes, and
   (b) by writing 1 as a linear combination of f, g and h with polynomial coefficients.

2.3. The Zariski topology

In this section, we put a topology on an affine variety. It is called the Zariski topology. An affine variety with this topology is a Noetherian space reflecting the algebraic fact that the polynomial ring is a Noetherian ring. This topology will be used later for constructing general varieties out of affine varieties.

2.3.1. The Zariski topology. In this discussion, we work with a fixed ambient space $\mathbb{C}^n$ and consider affine varieties in that ambient space.

Proposition 2.12. *$\mathbb{V}\mathbb{I}$ is a topological operator on the poset of subsets of $\mathbb{C}^n$ under inclusion.*

Explicitly, $\mathbb{V}\mathbb{I}(\emptyset) = \emptyset$ and for any subsets A and B of $\mathbb{C}^n$,

$$\mathbb{V}\mathbb{I}(A \cup B) = \mathbb{V}\mathbb{I}(A) \cup \mathbb{V}\mathbb{I}(B). \tag{2.1}$$

A reformulation of this result is given below.

Proposition 2.13. *The union of two affine varieties in $\mathbb{C}^n$ is again an affine variety in $\mathbb{C}^n$. The intersection of an arbitrary number of affine varieties in $\mathbb{C}^n$ is again an affine variety in $\mathbb{C}^n$.*

Proof. The first claim follows from:

$$\mathbb{V}(IJ) = \mathbb{V}(I \cap J) = \mathbb{V}(I) \cup \mathbb{V}(J) \tag{2.2}$$

for any ideals I and J.

(Since $I \cap J$ is a subset of both I and J, so $\mathbb{V}(I \cap J) \supseteq \mathbb{V}(I) \cup \mathbb{V}(J)$. To prove the other direction, suppose $x \in \mathbb{V}(I \cap J)$. So $f(x) = 0$ whenever $x \in \mathbb{V}(I \cap J)$. Suppose $x \notin \mathbb{V}(I)$ and $x \notin \mathbb{V}(J)$. Then there is a $g \in I$ and $h \in J$ such that $g(x) \neq 0$ and $h(x) \neq 0$. Then $gh \in I \cap J$ but $gh(x) \neq 0$ which is a contradiction.)

The second claim follows from:

$$\mathbb{V}\Big(\sum_{\alpha} I_\alpha\Big) = \bigcap_{\alpha} \mathbb{V}(I_\alpha) \tag{2.3}$$

for any family of ideals $\{I_\alpha\}$.

(A point x belongs to either side precisely if $f(x) = 0$ whenever $f \in I_\alpha$ for some α.) □

We use this result to put a topology on $\mathbb{C}^n$ as follows: A subset $X \subseteq \mathbb{C}^n$ is a closed set in the topology if it is an affine variety in $\mathbb{C}^n$. This is called the *Zariski topology* on $\mathbb{C}^n$. For any subset $A \subseteq \mathbb{C}^n$, we say that $\mathbb{V}\mathbb{I}(A)$ is the Zariski closure of A.

Remark 2.14. The second claim can also be deduced directly from the fact that for any closure operator, arbitrary intersections of closed sets is again closed.

The poset of subsets of $\mathbb{C}^n$ and the poset of ideals of $\mathbb{C}[x_1, \ldots, x_n]$ are complete lattices. In view of the Galois connection between them, note that (2.3) is a consequence of Theorem C.3.

Example 2.15. Let us consider the Zariski topology on the affine line $\mathbb{A}^1$. Every ideal in $\mathbb{C}[x]$ is principal, so every affine variety is the set of zeroes of a single polynomial. Since $\mathbb{C}$ is algebraically closed, every nonzero polynomial $f(x)$ can be written as

$$f(x) = c(x - a_1)^{b_1} \ldots (x - a_m)^{b_m},$$

for $c, a_1, \ldots, a_m \in \mathbb{C}$. Then $\mathbb{V}(f) = \{a_1, \ldots, a_m\}$. Thus the affine varieties in $\mathbb{C}^1$ are just the finite subsets (including the empty set) and the whole space (corresponding to $f = 0$).

Let $(X, \mathbb{C}^n)$ be any affine variety. Put the subspace topology on X. This is called the *Zariski topology* on X. The closed sets are precisely those affine varieties in $\mathbb{C}^n$ which are contained in X.

Proposition 2.16. *Suppose $\varphi : (X, \mathbb{C}^n) \to (Y, \mathbb{C}^m)$ is a morphism of affine varieties. Then $\varphi : X \to Y$ is continuous in the Zariski topology.*

Proof. Suppose X is a variety in $\mathbb{C}^m$ which is contained in Y. Let X be the solution set of the polynomial equations

$$f_1 = 0,\ f_2 = 0, \ldots,\ f_k = 0.$$

Then $\tilde{\varphi}^{-1}(X)$ is the solution set of the polynomial equations

$$f_1 \circ \tilde{\varphi} = 0,\ f_2 \circ \tilde{\varphi} = 0 \ldots,\ f_k \circ \tilde{\varphi} = 0$$

and hence is an affine variety in $\mathbb{C}^n$. It follows that $\varphi^{-1}(X) = X \cap \tilde{\varphi}^{-1}(X)$ is an affine variety contained in X as required. □

There is a functor

$$\mathsf{AffVariety} \to \mathsf{Top}$$

which assigns to an affine variety $(X, \mathbb{C}^n)$ the set X equipped with the Zariski topology. Note that this statement uses the result of Proposition 2.16.

Example 2.17. The projection onto one of the coordinates defines a morphism $\mathbb{A}^n \to \mathbb{A}^1$ which in general fails to send closed sets to closed sets. A simple example is the projection

$$\mathbb{A}^2 \to \mathbb{A}^1 \qquad (x, y) \mapsto x.$$

This map sends the hyperbola $\mathbb{V}(xy - 1) = \{(t, t^{-1}) \mid t \neq 0\}$ which is a closed subset of $\mathbb{A}^2$, onto the set $\mathbb{A}^1 \setminus \{0\}$, which is not a closed subset of $\mathbb{A}^1$.

Remark 2.18. View affine space $\mathbb{A}^n$ as a topological space with the Zariski topology. Let us define a category $\mathsf{AffVariety}$": objects are pairs $(X, \mathbb{A}^n)$, where X is a closed set in $\mathbb{A}^n$ and morphisms $(X, \mathbb{A}^m) \to (Y, \mathbb{A}^n)$ are continuous maps $X \to Y$ which are restrictions of polynomial maps $\mathbb{A}^m \to \mathbb{A}^n$. Clearly, the categories $\mathsf{AffVariety}$ and $\mathsf{AffVariety}$" are equivalent.

2.3.2. Basic open sets. Let X be an affine variety. For each polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$, consider the set

$$X_f := \{x \in X \mid f(x) \neq 0\}.$$

This is the complement of the zero set of f. It is called the *basic open set* corresponding to f. These open subsets form a basis for the Zariski topology on $\mathbb{A}^n$. In fact, any open set in X can be written as a union of finitely many basic open sets: Take those f's which generate the vanishing ideal of the complementary closed set.

Observe that all open sets in $\mathbb{A}^1$ are basic, but that is not the case in $\mathbb{A}^2$.

2.3.3. Noetherian spaces. An affine variety with its Zariski topology is a Noetherian space: Recall that $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian. The ascending chain on the (radical) ideals of this ring translates to the descending chain on the closed sets of $\mathbb{A}^n$. This shows that affine space is Noetherian. Since any affine variety is a subset of its ambient space with the induced topology, the result follows.

Thus all results that we have proved for Noetherian spaces apply to affine varieties. For example, any affine variety is quasi-compact wrt the Zariski topology; Proposition 1.22 applied to affine varieties yields:

Proposition 2.19. *Let X be an affine variety. Then X has only finitely many irreducible components $X_1, \ldots, X_m$ and $X = \bigcup_j X_j$.*

By using the algebra-geometry correspondence in Table 2.1, this result may also be deduced from the second part of Lasker-Noether Theorem 1.18.

The notion of irreducible for affine varieties with the Zariski topology is an instance of the notion of irreducible for topological spaces. It follows that open sets in an irreducible affine variety are dense.

2.3.4. Problems.

(1) Let $X \subseteq \mathbb{C}^m$ and $Y \subseteq \mathbb{C}^n$ be any subsets. Show that $\overline{X \times Y} \cong \overline{X} \times \overline{Y}$, where bar refers to Zariski closure.

(2) Let $\{I_\alpha\}$ be a collection of ideals. Show that

$$\mathbb{V}\Big(\bigcap_\alpha I_\alpha\Big) = \bigcup_\alpha \mathbb{V}(I_\alpha)$$

does not hold in general (even if I_α are assumed to be radical).

Why does this not contradict Theorem C.4? Can you modify the above identity so that it holds?

(3) Show that a map between affine varieties which is continuous for the Zariski topologies need not be a morphism.

(4) Describe all closed sets in $\mathbb{A}^2$.

(5) Show that the Zariski topology on $\mathbb{A}^2$ is *not* the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$.

(6) Show that the graph of a morphism $\varphi : X \to Y$ of affine varieties is closed in the categorical product $X \times Y$.

(7) Let X be the affine variety in $\mathbb{C}^3$ defined by the two polynomials $x^2 - yz$ and $xz - z$. Find the irreducible components of X and the corresponding prime ideals.

2.4. The coordinate ring of an affine variety

In this section, we discuss the notion of a function on an affine variety. It turns out that functions on a variety completely characterize that variety. This takes the correspondence between algebra and geometry given in Table 2.1 to a categorical level.

2.4.1. Functions on an affine variety. For any set X, let $\mathsf{Set}(X, \mathbb{C})$ denote the set of $\mathbb{C}$-valued maps on X. It is a commutative $\mathbb{C}$-algebra under the usual pointwise operations of addition and multiplication.

Now let $(X, \mathbb{C}^n)$ be any affine variety. Any polynomial in n variables defines a function on $\mathbb{C}^n$ and its restriction to X defines a function $X \to \mathbb{C}$. These functions

form a $\mathbb{C}$-subalgebra of $\mathsf{Set}(X, \mathbb{C})$, which we denote by $\mathbb{C}[X]$. We call it the *coordinate ring* of X or the *affine algebra* associated to X. The coordinate ring of affine space $\mathbb{A}^n$ is $\mathbb{C}[x_1, \dots, x_n]$.

Observe that two polynomials in n variables define the same function on X iff their difference belongs to $\mathbb{I}(X)$. This yields an isomorphism

$$\mathbb{C}[X] \cong \mathbb{C}[x_1, \dots, x_n]/\mathbb{I}(X).$$

Since $\mathbb{I}(X)$ is a radical ideal, it follows that $\mathbb{C}[X]$ is a finitely generated reduced commutative $\mathbb{C}$-algebra.

The coordinate ring of an affine variety can be used to give alternative characterizations of a morphism between affine varieties:

Proposition 2.20. *Let $(X, \mathbb{C}^n)$ and $(Y, \mathbb{C}^m)$ be affine varieties, and let $\varphi : X \to Y$ be any map. Then the following are equivalent.*

(1) *φ is a morphism of affine varieties.*
(2) *φ has the form*

$$\varphi(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$$

where each $f_i \in \mathbb{C}[X]$.
(3) *For every $g \in \mathbb{C}[Y]$, the composite $g \circ \varphi \in \mathbb{C}[X]$.*

Proof. (1) $\iff$ (2). Consider the canonical projection $\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[X]$. Given $\tilde{\varphi}$ of Definition 2.3, the f_j's are the projections of the components of $\tilde{\varphi}$. Conversely, given the f_j's, the components of $\tilde{\varphi}$ are (some) lifts of the f_j's.

(1) $\implies$ (3). Suppose φ is a morphism of affine varieties. Let $\tilde{\varphi} : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial extension of φ. Suppose $g \in \mathbb{C}[Y]$. Then g is the restriction of a polynomial map $\tilde{g}$ on $\mathbb{C}^m$. Now observe that $g \circ \varphi$ is the restriction of $\tilde{g} \circ \tilde{\varphi}$ to $\mathbb{C}[X]$. Hence $g \circ \varphi \in \mathbb{C}[X]$.

(3) $\implies$ (1). We need to construct the polynomial extension $\tilde{\varphi}$ of φ. We do this componentwise. Take g to be the restriction of the ith coordinate function on $\mathbb{C}^m$. Then $g \circ \varphi \in \mathbb{C}[X]$ by hypothesis. So it is the restriction of some polynomial function on $\mathbb{C}^n$. Choose one and put it as the ith component of $\tilde{\varphi}$. □

To every morphism $\varphi : X \to Y$ is associated its comorphism

$$\mathbb{C}[\varphi] : \mathbb{C}[Y] \to \mathbb{C}[X] \qquad \mathbb{C}[\varphi](f) := f \circ \varphi.$$

This requires some checks: The important one is that $f \circ \varphi$ can be expressed as a polynomial in the components of φ and hence belongs to $\mathbb{C}[X]$. The other check is that $f \circ \varphi$ is an algebra homomorphism. This is generally true and does not make any use of the fact that we are working with polynomials. Further observe that

$$\mathrm{id}^* = \mathrm{id} \quad \text{and} \quad \mathbb{C}[\varphi \circ \psi] = \mathbb{C}[\psi] \circ \mathbb{C}[\varphi].$$

Thus we obtain a functor

$$\mathbb{C}[-] : \mathsf{AffVariety} \to (\mathsf{Alg}^{\mathsf{co}})^{\mathrm{op}} \qquad X \mapsto \mathbb{C}[X],\ \varphi \mapsto \mathbb{C}[\varphi], \tag{2.4}$$

where $\mathsf{Alg}^{\mathsf{co}}$ is the category of commutative $\mathbb{C}$-algebras.

Proposition 2.21. *Let $\varphi : X \to Y$ be a morphism of affine varieties. Then:*

- *$\mathbb{C}[\varphi] : \mathbb{C}[Y] \to \mathbb{C}[X]$ is injective iff $\varphi : X \to Y$ is dominant, that is, the image of φ is dense in Y.*
- *$\mathbb{C}[\varphi] : \mathbb{C}[Y] \to \mathbb{C}[X]$ is surjective iff $\varphi : X \to Y$ defines an isomorphism between X and some affine subvariety of Y.*

PROOF. This is straightforward. □

Example 2.22. Consider the morphism of affine varieties

$$\mathbb{A}^1 \to \mathbb{A}^2 \qquad x \mapsto (x, 1).$$

The induced map on the coordinate rings is

$$\mathbb{C}[x, y] \to \mathbb{C}[x] \qquad f(x, y) \mapsto f(x, 1).$$

In other words, a polynomial in two variables is mapped to the polynomial in one variable by setting the second variable to be 1.

Example 2.23. Consider the morphism of affine varieties

$$\mathbb{A}^3 \to \mathbb{A}^2 \qquad (x, y, z) \mapsto (x + y^2 - 1, xz).$$

Letting (u, v) denote the coordinates of $\mathbb{A}^2$, the induced map is

$$\mathbb{C}[u, v] \to \mathbb{C}[x, y, z] \qquad u \mapsto x + y^2 - 1,\ v \mapsto xz.$$

Since this is a morphism of algebras, specifying it on generators determines it completely.

Example 2.24. Let $X = \mathbb{V}(xy - 1) \in \mathbb{C}^2$ be the hyperbola. Its coordinate ring is

$$\mathbb{C}[X] = \mathbb{C}[x, y]/(xy - 1).$$

The function $1/x$ is well-defined on $\mathbb{V}(xy-1)$. Does it belong to the coordinate ring? At first glance, the answer may seem to be no because $1/x$ is not a polynomial. However the answer is yes because $1/x = y$ on this variety, and y is indeed a polynomial.

Example 2.25. Let $X = \mathbb{V}(y - x^2) \in \mathbb{C}^2$ be the plane parabola of Example 2.6. Its coordinate ring is

$$\mathbb{C}[X] = \mathbb{C}[x, y]/(y - x^2).$$

(This requires checking that $(y - x^2)$ is a radical ideal.) We have seen that the maps

$$\mathbb{A}^1 \to X \quad t \mapsto (t, t^2) \qquad \text{and} \qquad X \to \mathbb{A}^1 \quad (x, y) \mapsto x$$

are isomorphisms of affine varieties and inverse to each other.

Passing to the coordinate rings, one obtains isomorphisms of algebras

$$\mathbb{C}[X] \to \mathbb{C}[t] \quad x \mapsto t,\ y \mapsto t^2 \qquad \text{and} \qquad \mathbb{C}[t] \to \mathbb{C}[X] \quad t \mapsto x.$$

They are inverses of each other. (This is a manifestation of a general principle of category theory: A functor carries isomorphisms to isomorphisms.)

Example 2.26. Consider the morphism

$$\mathbb{A}^1 \to \mathbb{V}(y^2 - x^3) \in \mathbb{C}^2 \qquad t \mapsto (t^2, t^3).$$

Check that this is a bijection: Think of $t \in \mathbb{A}^1$ as the slope of a line passing through the origin; this line meets the curve at two points $(0, 0)$ and (t^2, t^3). For $t = 0$, both points coincide.

However we claim that this morphism is not an isomorphism. In other words, the set-theoretic inverse map

$$\mathbb{V}(y^2 - x^3) \to \mathbb{A}^1 \qquad (x, y) \mapsto y/x$$

is *not* a morphism of varieties. (Map the point $(0, 0)$ to 0.) This seems reasonable because y/x is not a polynomial.

This can be proved by looking at the algebra side. Passing to the coordinate rings, the first morphism yields a morphism of algebras:

$$\mathbb{C}[x,y]/(y^2-x^3) \to \mathbb{C}[t] \qquad x \mapsto t^2,\ y \mapsto t^3.$$

This is clearly not an isomorphism, since the element t is not in the image, proving our claim. The first one is not a UFD while the second one is. (This map is injective however: The map on varieties is a bijection and hence dense in the image. Now use an exercise below.)

To summarize, a bijective morphism may not be an isomorphism.

Remark 2.27. Compare the conclusion of the above example with: A bijective continuous map between topological spaces may not be a homeomorphism, and contrast with: A bijective linear map between vector spaces is an isomorphism (that is, the inverse is also linear).

2.4.2. The image of the coordinate ring functor.

Proposition 2.28. *The functor (2.4) is fully faithful, that is, for any affine varieties X and Y, the map*

$$\mathsf{AffVariety}(X,Y) \xrightarrow{\cong} \mathsf{Alg}^{\mathsf{co}}(\mathbb{C}[Y],\mathbb{C}[X])$$

is a bijection.

Proof. Let us construct the inverse map. Suppose $\mathbb{C}[Y] \to \mathbb{C}[X]$ is a $\mathbb{C}$-algebra homomorphism. Then consider the composite $\mathbb{C}$-algebra homomorphism:

$$\mathbb{C}[y_1,\dots,y_m] \twoheadrightarrow \mathbb{C}[y_1,\dots,y_m]/\mathbb{I}(Y) \to \mathbb{C}[x_1,\dots,x_n]/\mathbb{I}(X).$$

Let $f_1,\dots,f_m \in \mathbb{C}[X]$ denote the images of $y_1,\dots,y_m$. Since the above homomorphism factors through $\mathbb{I}(Y)$, the f_i have the following property: for any point $(a_1,\dots,a_n) \in X$, the point $(f_1(a_1,\dots,a_n),\dots,f_m(a_1,\dots,a_n))$ belongs to $\mathbb{V}(\mathbb{I}(Y)) = Y$. By Proposition 2.20, part (2), this defines a morphism of affine varieties $X \to Y$. □

Recall that a fully faithful functor gives rise to an equivalence between the source category and its image in the target category. In the present case, the Nullstellensatz allows us to give a good description of the image subcategory:

Theorem 2.29. *The image of the functor (2.4) is the category of finitely generated reduced commutative $\mathbb{C}$-algebras.*

We denote the latter category by $(\overline{\mathsf{Alg}^{\mathsf{co}}})^{\mathrm{op}}$.

Proof. We have already noted that for any affine variety X, $\mathbb{C}[X]$ is a finitely generated reduced commutative $\mathbb{C}$-algebra. Conversely, suppose A is a $\mathbb{C}$-algebra of this form. Pick a finite set of say m generators for A. This yields a surjective homomorphism

$$\mathbb{C}[x_1,\dots,x_m] \twoheadrightarrow A.$$

Let I be the kernel of this map. It is a radical ideal since A is reduced. So by the Nullstellensatz, $A = \mathbb{C}[X]$ where $X = \mathbb{V}(I)$. □

As a consequence:

Theorem 2.30. *There is an equivalence of categories*

$$\mathbb{C}[-] : \mathsf{AffVariety} \to (\overline{\mathsf{Alg}^{\mathsf{co}}})^{\mathrm{op}}.$$

We have in effect constructed the inverse functor

$$\mathcal{G} : (\overline{\mathsf{Alg}^{\mathsf{co}}})^{\mathrm{op}} \to \mathsf{AffVariety}. \tag{2.5}$$

Let us review it again. Let R be a finitely generated reduced commutative $\mathbb{C}$-algebra. Pick a finite set of say m generators for R. This yields a surjective homomorphism

$$\mathbb{C}[x_1, \ldots, x_m] \twoheadrightarrow R.$$

Let I be the kernel of this map. It is a radical ideal since R is reduced. Define $\mathcal{G}(R) := (\mathbb{V}(I), \mathbb{C}^m)$. (This definition can be made even if R is not reduced. The point of assuming R reduced is that the coordinate ring of $\mathcal{G}(R)$ is then isomorphic to R. This employs the Nullstellensatz.)

Now let $f : R \to S$ be a ring homomorphism. Consider the commutative digram of $\mathbb{C}$-algebras:

$$\begin{array}{ccc} \mathbb{C}[x_1, \ldots, x_m] & \overset{\tilde{f}}{\dashrightarrow} & \mathbb{C}[y_1, \ldots, y_n] \\ \downarrow & & \downarrow \\ R \cong \mathbb{C}[x_1, \ldots, x_m]/I & \xrightarrow[f]{} & \mathbb{C}[y_1, \ldots, y_n]/J \cong S. \end{array}$$

Here $\tilde{f}$ is some lift of f (which will not be unique in general).

Define $\tilde{\varphi} : \mathbb{C}^n \to \mathbb{C}^m$ as follows:

$$\tilde{\varphi}(y_1, \ldots, y_n) := (\tilde{f}(x_1), \ldots, \tilde{f}(x_m)).$$

This is a polynomial map and it sends $\mathbb{V}(J)$ to $\mathbb{V}(I)$ yielding a commutative diagram.

$$\begin{array}{ccc} \mathbb{C}^m & \overset{\tilde{\varphi}}{\dashleftarrow} & \mathbb{C}^n \\ \uparrow & & \uparrow \\ \mathbb{V}(I) & \xleftarrow[\varphi]{} & \mathbb{V}(J) \end{array}$$

Further different choices of $\tilde{f}$ yield different $\tilde{\varphi}$ but the same φ. Define $\mathcal{G}(f) := \varphi : \mathcal{G}(S) \to \mathcal{G}(R)$. By construction, this is a morphism of affine varieties.

Remark 2.31. In the construction of the functor $\mathcal{G}$, for each ring R, we made a choice of generators. This may seem ad hoc but that is alright. A different choice (even for one ring) will lead to a different functor which will work equally well.

Example 2.32. Consider the algebra homomorphism

$$\mathbb{C}[x, y, z]/(xz - y^2) \to \mathbb{C}[u, v] \qquad x \mapsto u^2,\ y \mapsto uv,\ z \mapsto v^2.$$

This gives an induced map on affine varieties

$$\mathbb{A}^2 \to \mathbb{V}(xz - y^2) \qquad (u, v) \mapsto (u^2, uv, v^2).$$

Is this an isomorphism?

Consider the algebra homomorphism

$$\mathbb{C}[x, y]/(xy - 1) \to \mathbb{C}[u, v]/(uv + 1) \qquad x \mapsto u^2,\ y \mapsto v^2.$$

Note that there are choices on how we write this map. For example, instead of $x \mapsto u^2$, we may write $x \mapsto u^2 + uv + 1$. The induced map on affine varieties is

$$\mathbb{V}(uv + 1) \to \mathbb{V}(xy - 1) \qquad (u, v) \mapsto (u^2, v^2).$$

This may also be written as $(u, v) \mapsto (u^2 + uv + 1, v^2)$.

Consider the algebra isomorphisms

$$\mathbb{C}[x, y, z]/(z) \to \mathbb{C}[x, y] \qquad x \mapsto x,\ y \mapsto y$$

and

$$\mathbb{C}[x, y]/(x - y) \to \mathbb{C}[t] \qquad x \mapsto t,\ y \mapsto t.$$

For the induced maps, in the first case, $\mathbb{A}^2$ embeds as a coordinate plane in $\mathbb{A}^3$, and in the second case, $\mathbb{A}^1$ embeds as the diagonal in $\mathbb{A}^2$.

2.4.3. The evaluation map. Proposition 2.11 says that points in affine space correspond to maximal ideals in the polynomial ring. More generally, points in an affine variety correspond to maximal ideals in its coordinate ring (see exercise below). This leads to the following result.

Proposition 2.33. *Let $(X, \mathbb{C}^n)$ be an affine variety. Then the evaluation map*

$$X \to \mathsf{Alg}(\mathbb{C}[X], \mathbb{C}) \qquad x \mapsto (f \mapsto f(x))$$

is a bijection.

Proof. Let us first understand the case $X = \mathbb{C}^n$, and $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$. Let $f : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}$ be an algebra homomorphism. It is necessarily surjective since $f(1) = 1$. Then $\ker(f)$ is a maximal ideal (because the quotient is a field). So $\ker(f) = (x_1 - a_1, \ldots, x_n - a_n)$ for a unique point $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Then f is precisely evaluation at this point (the condition $f(1) = 1$ leaves no choice for f).

For the general case, use the correspondence between points in X and maximal ideals in $\mathbb{C}[X]$ mentioned above. □

This result holds over any field $\Bbbk$, not necessarily algebraically closed. The above argument can be phrased avoiding maximal ideals, see Proposition 8.1.

Definition 2.34. We now define the category $\mathsf{AffVariety}'$. An object in $\mathsf{AffVariety}'$ is a pair (X, A) consisting of a set X and a $\mathbb{C}$-algebra A such that

- A is a finitely generated (reduced) subalgebra of $\mathsf{Set}(X, \mathbb{C})$,
- the evaluation map

$$X \to \mathsf{Alg}(A, \mathbb{C}) \qquad x \mapsto (f \mapsto f(x))$$

  is a bijection.

A morphism in $\mathsf{AffVariety}'$ between (X, A) and (Y, B) is a map $X \to Y$ such that the induced algebra homomorphism $\mathsf{Set}(Y, \mathbb{C}) \to \mathsf{Set}(X, \mathbb{C})$ restricts to an algebra homomorphism $B \to A$.

Theorem 2.35. *The category $\mathsf{AffVariety}'$ is equivalent to $\mathsf{AffVariety}$ and hence to $\overline{\mathsf{Alg}^{\mathsf{co}}}^{\mathrm{op}}$.*

This is left for you to ponder over (see exercise below). The category $\mathsf{AffVariety}'$ is interesting in the sense that it incorporates the geometric feature of $\mathsf{AffVariety}$ and the algebraic feature of $\overline{\mathsf{Alg}^{\mathsf{co}}}$.

2.4.4. Problems.

(1) Show that for any affine variety X, there is a bijection between
   - affine subvarieties of X and radical ideals of $\mathbb{C}[X]$,
   - affine irreducible subvarieties of X and prime ideals of $\mathbb{C}[X]$,
   - points of X and maximal ideals of $\mathbb{C}[X]$.

(2) Since the functor (2.4) is a contravariant equivalence, it interchanges initial objects and terminal objects, and products and coproducts. Explain what this means in explicit terms.

(3) Let $\mathsf{fgAlg}^{\mathsf{co}}$ denote the category of finitely generated commutative $\mathbb{C}$-algebras. Define a functor

$$\mathcal{G} : \mathsf{fgAlg}^{\mathsf{co}} \to \mathsf{AffVariety}$$

along the lines of (2.5). Then show that for any affine variety X and finitely generated commutative $\mathbb{C}$-algebra A, there is a natural bijection

$$\mathsf{AffVariety}(X, \mathcal{G}(A)) \cong \mathsf{fgAlg}^{\mathsf{co}}(A, \mathbb{C}[X]).$$

(4) Describe the functors involved in the equivalences in Theorem 2.35.

2.5. The sheaf of regular functions

In the previous section, we defined the notion of a function on a variety X. The goal of this section is to understand what it means to define a function locally, that is, on a Zariski-open set of X. More precisely, we define a sheaf of functions on X. It is called the structure sheaf of X. If X is irreducible, then the ring of global sections of this sheaf is precisely the coordinate ring $\mathbb{C}[X]$; the stalk at a point is the coordinate ring localized at the maximal ideal corresponding to that point; the generic stalk is the field of fractions of the coordinate ring. This further strengthens the interplay between algebra and geometry.

We know that an affine variety is determined by its coordinate ring. It then follows that the structure sheaf is determined by its ring of global sections; so the local data is not giving any new information. Nevertheless, the above interplay shows that the sheaf viewpoint is still useful. Moreover, when we go to projective or more general varieties, the global sections no longer tell the full story, so the sheaf viewpoint becomes even more crucial.

2.5.1. The sheaf of regular functions.

Definition 2.36. Let U be any open set of an affine variety X. A function $f : U \to \mathbb{C}$ is *regular* at a point $p \in U$ if in some neighbourhood V of p, it is expressible as a quotient g/h, where $g, h \in \mathbb{C}[X]$ and $h(p) \neq 0$. We say that f is regular on U if it is regular at every point of U.

The usage "f expressible as g/h in V" means that for each $x \in V$, $f(x) = g(x)/h(x)$.

In the above definition, one may equally well say $g, h \in \mathbb{C}[x_1, \ldots, x_n]$; this can be useful when dealing with more than one variety sitting in the same ambient space.

Example 2.37. Consider the basic open set $U = \mathbb{A}^1 \setminus \{0\}$ of $\mathbb{A}^1$. Observe that $1/x$ is a regular function on U. In fact, the ring of regular functions on U is precisely the ring of Laurent polynomials in x. Can you see why this is true?

Example 2.38. The slope function

$$\mathbb{A}^2 \setminus \mathbb{V}(x) \to \mathbb{C} \qquad (x, y) \mapsto \frac{y}{x}$$

is regular. This function may be interpreted as a central projection:

$$\mathbb{A}^2 \setminus \mathbb{V}(x) \to \mathbb{V}(x-1) \xrightarrow{\cong} \mathbb{C}$$

Suppose q is a point not lying on the y-axis. Then map q to the intersection of the line passing through the origin and q with the line $x = 1$.

Proposition 2.39. *Any regular function on an open set U, viewed as a map $U \to \mathbb{A}^1$ is continuous wrt the Zariski topologies.*

Proof. Let f be a regular function on U. For $\alpha \in \mathbb{C}$, we want to show that $Z := \{x \in U \mid f(x) = \alpha\}$ is a closed set in U. Since f is regular, there is a finite open cover $\{U_i\}$ of U such that $f = g_i/h_i$ on U_i. The set $Z \cap U_i$ consists of all $x \in U_i$ for which $g_i(x) - \alpha h_i(x) = 0$. So $Z \cap U_i$ is a closed set in U_i. It follows that Z is a closed set in U as required. □

We will see later that a regular function is same as a morphism to $\mathbb{A}^1$ and since morphisms are continuous, the result follows.

Proposition 2.40. *Let f and g be two regular functions on an open set U of an affine variety X. Then the subset where f and g agree is a closed set of U.*

Proof. Since f and g are regular functions, so is $f - g$. By Proposition 2.39, it is continuous in the Zariski topology. The subset where f and g agree is $(f-g)^{-1}(0)$, which is a closed set of U, as required. □

We will see later that $\mathbb{A}^1$ is a variety. The result follows.

Let X be an affine variety. For each open set U in X, define $\mathcal{O}_X(U)$ to be the $\mathbb{C}$-algebra of regular functions on U. This is a sheaf on X: Suppose we are given regular functions f_i on open sets U_i which agree on the overlaps. This then defines a unique function f on $\bigcup U_i$, and f is regular because of the local nature of regular functions.

The sheaf $\mathcal{O}_X$ is called the *structure sheaf* of X. It is a sheaf of $\mathbb{C}$-algebras over X.

Proposition 2.41. *Let f and g be two regular functions on an open set U of an irreducible affine variety X. Suppose f and g agree on a nonempty open set of U. Then they agree everywhere on U.*

Proof. This follows from Proposition 2.40 and the fact that nonempty open sets are dense in this case. □

Proposition 2.42. *Let f be a regular function on an open set U of an irreducible affine variety X. Then there is a largest open set in X to which f extends as a regular function and this extension is unique.*

We call this largest open set the *natural domain* of f.

Proof. Consider all pairs (V, g) where g is a regular function on an open set V such that $f = g$ on $U \cap V$. Suppose (V, g) and (V', g') are two such pairs. Then $g = g'$ on $V \cap V'$: Use $f = g = g'$ on $U \cap V \cap V'$ and Proposition 2.41.

The largest open set, say W, is obtained by taking union of all the V's and the extension of f to W is obtained by patching the functions given on each piece using the gluing property of a sheaf. The uniqueness of the extension is also clear. □

2.5.2. Global sections of the structure sheaf. It turns out that the ring of regular functions on the full variety is same as the coordinate ring of the variety.

Proposition 2.43. *Let X be an irreducible affine variety. Then the canonical map $\mathbb{C}[X] \to \mathcal{O}_X(X)$ is a $\mathbb{C}$-algebra isomorphism.*

A different proof of this result is given in Proposition 2.47 below.

Proof. It is clear that the canonical map above is injective. To show that it is surjective, let g be a regular function on X. Since X is quasi-compact, there is a finite cover $\{U_i\}$ of X by nonempty open sets such that on each U_i, g can be expressed in the form h_i/k_i, with k_i not vanishing on U_i. Since $\{U_i\}$ covers X, the k_i's cannot simultaneously vanish at any point of X. Thus by the Nullstellensatz applied to X, the ideal generated by the k_i's is the full ring $\mathbb{C}[X]$. Thus there exist polynomials $\ell_i \in \mathbb{C}[X]$ such that

$$1 = \sum_{i=1}^{m} \ell_i k_i.$$

(The 1 here is the unit in $\mathbb{C}[X]$, that is, the constant function on X with value 1.) We claim that

$$g = \sum_{i=1}^{m} \ell_i h_i,$$

and this will complete the proof.

To see this, first note that $h_i/k_i = h_j/k_j$ on the dense open set $U_i \cap U_j$; so $h_i k_j = h_j k_i \in \mathbb{C}[X]$. (Here we used that X is irreducible.) Thus on each U_j,

$$g = \sum_{i=1}^{m} \ell_i k_i \frac{h_j}{k_j} = \sum_{i=1}^{n} \ell_i h_i.$$

Since this holds for each j, the claim follows.

Alternatively, consider the intersection $U_1 \cap \cdots \cap U_m$. This is a dense open set in X and $g = \sum_{i=1}^{m} \ell_i h_i$ holds on this open set. Since by Proposition 2.39 regular functions are continuous, it follows that this identity holds on all of X. □

The above result holds even when the variety is not irreducible. A careful proof is given by Springer [24, page 8]. Harris [13, page 61] also claims a proof in the general case; but it is same as the proof given above with details lacking; so it does not appear to be complete.

Proposition 2.44. *Let X and Y be affine varieties, and let $\varphi : X \to Y$ be a continuous map. Then the following are equivalent.*

1. *φ is a morphism of affine varieties.*
2. *For all $f \in \mathcal{O}_Y(Y)$, we have $f \circ \varphi \in \mathcal{O}_X(X)$.*
3. *For all open $V \subseteq Y$ and $f \in \mathcal{O}_Y(V)$, we have $f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.*

Proof. (1) $\iff$ (2). This is the content of Propositions 2.20 and 2.43. (We proved the latter only in the irreducible case.)

(3) $\implies$ (2). Clear.

(1) $\implies$ (3). Let $f \in \mathcal{O}_Y(V)$. Then there is an open cover $\{V_i\}$ of V such that on each V_i, $f = g_i/h_i$ with h_i not vanishing on any point of V_i. It follows that $\{U_i := \varphi^{-1}(V_i)\}$ is an open cover of $U := \varphi^{-1}(V)$ and on each U_i, $f \circ \varphi = g_i \circ \varphi / h_i \circ \varphi$, with $h_i \circ \varphi$ not vanishing on any point of U_i. Thus $f \circ \varphi$ is regular on U as required. □

2.5.3. Regular functions on an affine subvariety.

Proposition 2.45. *Suppose X is an affine subvariety of Y. Let U be an open set in X and let g be any function on U. Then g is a regular function on U iff for each $x \in U$, there is an neighbourhood V in Y and a regular function f on V such that g is the restriction of f to $U \cap V$.*

Draw picture.

Proof. Forward implication. For any point $x \in U$, there are polynomials h and k on the ambient space such that $g(x) = h(x)/k(x)$. Since polynomials are continuous wrt the Zariski toplogy, there is a open set containing x in the ambient space where k does not vanish. Take V to be the intersection of this open set with Y, and $f := h/k$.

Backward implication. The inclusion $X \hookrightarrow Y$ is a morphism of affine varieties. Now use part (3) of Proposition 2.44, and the local nature of regular functions. □

Note that we are not saying that there is a regular function in some some neighbourhood of U whose restriction to U is g.

2.5.4. The modern affine category. We are now ready to give the modern perspective on affine varieties by defining the category ModernAffVariety. An object is a pair $(X, \mathcal{O}_X)$, where X is an affine variety and $\mathcal{O}_X$ is its structure sheaf. A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $X \to Y$ (wrt the Zariski topologies on X and Y) such that for every open set $V \subseteq Y$, and for every function $f \in \mathcal{O}_Y(V)$, the function $f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$.

Observe that there is a functor

$$\mathsf{AffVariety} \to \mathsf{ModernAffVariety}, \qquad X \mapsto (X, \mathcal{O}_X)$$

which defines an equivalence of categories; the functor in the other direction is the forgetful functor. This follows from Propostion 2.44.

We also point out that there is a functor

$$\mathsf{ModernAffVariety} \to \mathsf{Sheaf}.$$

Compare this discussion with Example D.7.

Remark 2.46. One may refer to objects of ModernAffVariety as modern affine varieties. But we will instead call them only affine varieties. The context should make it clear which objects we are refering to.

2.5.5. Stalks of the structure sheaf. The stalk at a point $x \in X$ is denoted by $(\mathcal{O}_X)_x$. It is the ring of germs of regular functions on X near x. Explicitly: An element is a pair (U, f) where U in an open set in X containing x, and f is a regular function on U, and where we identify two such pairs (U, f) and (V, f') if $f = f'$ on some open set contained in $U \cap V$ which contains x.

If X is irreducible, then the above equivalence relation can be rephrased: (U, f) and (U', f') are equivalent if $f = f'$ on $U \cap U'$. This follows from Proposition 2.41.

Now assume that X is irreducible. The generic stalk of the structure sheaf of X is denoted by $(\mathcal{O}_X)_{gen}$. Explicitly: An element is a pair (U, f) where U in a nonempty open set and f is a regular function on U, and where we identify two such pairs (U, f) and (U', f') if $f = f'$ on some nonempty open set contained in $U \cap U'$, or equivalently, if $f = f'$ on $U \cap U'$.

In view of Proposition 2.42, the generic stalk can also be defined as follows. an element of a generic stalk is a pair (U, f) where U is the natural domain of f: the largest open set to which f extends.

These descriptions of a stalk and generic stalk work in greater generality; see Appendix D. Check out this natural domain stuff in general. We point out that there we were working with sheaves of sets, whereas here we are working with sheaves of $\mathbb{C}$-algebras. So we need to check in addition that $(\mathcal{O}_X)_x$ and $(\mathcal{O}_X)_{gen}$ are canonically $\mathbb{C}$-algebras. In fact, observe that $(\mathcal{O}_X)_{gen}$ is a field. Further note that there are inclusions:

$$\mathcal{O}_X(X) \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{O}_X)_{gen}. \tag{2.6}$$

2.5.6. The coordinate ring and its localizations. Let X be an irreducible affine variety. Associated to it, we have the coordinate ring $\mathbb{C}[X]$. It is an integral domain since X is irreducible. Let $\mathbb{C}(X)$ denote its field of fractions. It is the localization of $\mathbb{C}[X]$ at the prime ideal (0). An element of $\mathbb{C}[X]$ is a fraction f/g where $f, g \in \mathbb{C}[X]$, and two fractions f/g and f'/g' are identified if $fg' = f'g$ as elements of $\mathbb{C}[X]$, that is, if $f(x)g'(x) = f'(x)g(x)$ for all $x \in \mathbb{C}[X]$.

Let $\mathbb{C}[X]_P$ denote the localization of $\mathbb{C}[X]$ at the prime ideal P. It is the subset of $\mathbb{C}(X)$ consisting of those fractions f/g where $f, g \in \mathbb{C}[X]$, and $g \notin P$. The latter condition is equivalent to saying that g does not vanish at all points of $\mathbb{V}(P)$ (though it may vanish at some points of $\mathbb{V}(P)$). If M is a maximal ideal, then $\mathbb{V}(M) =: \{x\}$ is a singleton, and $\mathbb{C}[X]_M$ consists of those fractions whose denominator does not vanish at x. Note that

$$\mathbb{C}[X] \subseteq \mathbb{C}[X]_P \subseteq \mathbb{C}(X). \tag{2.7}$$

Our goal is to show that (2.7) is the algebraic description of (2.6). The starting point is the observation that there is a map

$$\mathbb{C}(X) \to (\mathcal{O}_X)_{gen}, \qquad f/g \mapsto (X_g, f/g). \tag{2.8}$$

Recall that X_g is the open set of X consisting of points where g does not vanish; so f/g defines a regular function on X_g. We need to check that this map is well-defined: Suppose f/g and f'/g' are equivalent fractions. Then $fg' = f'g$. So we see that f/g viewed as a function on X_g and f'/g' viewed as a function on $X_{g'}$ agree on $X_g \cap X_{g'}$, so they represent the same element in the generic stalk.

Proposition 2.47. *The map (2.8) is an isomorphism of $\mathbb{C}$-algebras. Further it restricts to isomorphisms*

$$\mathbb{C}[X] \xrightarrow{\cong} \mathcal{O}_X(X)$$

and

$$\mathbb{C}[X]_M \xrightarrow{\cong} (\mathcal{O}_X)_x$$

where M is a maximal ideal of $\mathbb{C}[X]$ and $\{x\} = \mathbb{V}(M)$.

This result is also contained in [14, Theorem 3.2, page 17].

PROOF. The surjectivity of (2.8) is clear. To show that it is injective: Let $h, h' \in \mathbb{C}(X)$ map to the same element in the generic stalk. Then there is an open set U, and $f, g, f', g' \in \mathbb{C}[X]$ such that $h = f/g$ and $h' = f'/g'$, and g and g' do not vanish on any point of U, and $f(x)/g(x) = f'(x)/g'(x)$ for $x \in U$. Then $f(x)g'(x) = g(x)f'(x)$ for $x \in U$ and hence for all $x \in X$. This shows that $fg' = gf'$ are the same element of $\mathbb{C}[X]$, which implies $h = h'$ as required.

Under the bijection (2.8), it is fairly clear that $\mathbb{C}[X]_M$ corresponds to $(\mathcal{O}_X)_x$: The former consists of those h in the fraction field whose domain contains x.

It is clear that the map $\mathbb{C}[X] \to \mathcal{O}_X(X)$ is injective. Let A be the subset of $\mathbb{C}(X)$ which corresponds to $\mathcal{O}_X(X)$ under the map (2.8). Then

$$\mathbb{C}[X] \subseteq A \subseteq \mathbb{C}[X]_M$$

for all maximal ideals M. Using Proposition 1.32, it follows that $\mathbb{C}[X] = A$ as required. □

Remark 2.48. Let $h \in \mathbb{C}(X)$. Then define the natural domain of h, denoted $\mathrm{dom}(h)$ as follows. $x \in \mathrm{dom}(h)$ if h can be expressed as f/g with $f, g \in \mathbb{C}[X]$ and $g(x) \neq 0$. It is the same as the natural domain of the corresponding element of the generic stalk. In other words, $\mathrm{dom}(h)$ is the largest open set of X on which h can be interpreted as a regular function.

Now let $f \in \mathbb{C}[X]$, and let $\mathbb{C}[X](1/f)$ denote the ring of fractions of $\mathbb{C}[X]$ at the multiplicative set $\{f^n\}_{n\geq 0}$. Then

$$\mathbb{C}[X] \subseteq \mathbb{C}[X](1/f) \subseteq \mathbb{C}(X).$$

In this situation, note that for $a, b \in \mathbb{C}[X]$,

$$a/f^m = b/f^n \text{ in } \mathbb{C}[X](1/f) \iff af^n = bf^m \text{ in } \mathbb{C}[X].$$

Recall that X_f is the basic open set of X consisting of points where f does not vanish. Note that

$$\mathcal{O}_X(X) \subseteq \mathcal{O}_X(X_f) \subseteq (\mathcal{O}_X)_{gen}.$$

Proposition 2.49. *Let X be an irreducible affine variety. Then the canonical map $\mathbb{C}[X](1/f) \to \mathcal{O}_X(X_f)$ is a $\mathbb{C}$-algebra isomorphism. It is the restriction of the map (2.8).*

PROOF. Let us first show that the canonical map is injective. Let a/f^m and b/f^n have the same image in $\mathcal{O}_X(X_f)$. Then $a(x)f^n(x) = b(x)f^m(x)$ for $x \in X_f$. Since X_f is dense in X, we have $a(x)f^n(x) = b(x)f^m(x)$ for $x \in X$. Hence $af^n = bf^m$ in $\mathbb{C}[X]$, and so $a/f^m = b/f^n$ in $\mathbb{C}[X](1/f)$.

To show that the canonical map is surjective, let g be a regular function on X_f. Since X_f is quasi-compact, there is a finite open cover $\{U_i\}$ of X_f such that on each U_i, g can be expressed in the form h_i/k_i, with k_i not vanishing on U_i. Since $\{U_i\}$ covers X_f, the k_i's cannot simultaneously vanish at any point of X_f. So

$$\mathbb{V}(\dots, k_i, \dots) \subseteq \mathbb{V}(f) \quad \text{and hence} \quad f \in \mathrm{rad}\,(\dots, k_i, \dots).$$

Thus there exist polynomials $\ell_i \in \mathbb{C}[X]$ such that

$$f^p = \sum_{i=1}^{m} \ell_i k_i$$

for some p. Arguing as in the proof of Proposition 2.43, it follows that

$$gf^p = \sum_{i=1}^{n} \ell_i h_i,$$

Thus g belongs to $\mathbb{C}[X](1/f)$ as required. □

2.5.7. Problems.

(1) Let X be an affine variety and let $f \in \mathbb{C}[X]$. Show that if f does not vanish at any point of X, then f is a unit in $\mathbb{C}[X]$.
(2) Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{C}$. What is the ring of regular functions on $\mathbb{A}^1 \setminus \{p_1, \ldots, p_n\}$?
(3) Give an example of a regular function on an open set U which cannot be expressed as g/h with h nowhere zero on U.
(4) Is Proposition 2.49 true if the affine variety is not irreducible?

CHAPTER 3

Projective varieties

See exercises on page 11 of Hartshorne.

3.1. Projective varieties

Varieties are geometric objects constructed by patching together affine varieties. (A useful analogy is that of manifolds which are constructed by patching together open sets in Euclidean spaces.) In this section, we look at projective varieties. They are the most important nontrivial class of varieties.

3.1.1. Projective space. A good elementary introduction to projective spaces can be found in the book by Courant and Robbins [6]. If you want to see an axiomatic treatment of projective geometry in the spirit of Euclid, then you may look at the book of Coxeter on the real projective plane [7].

Definition 3.1. Complex projective n-space, denoted $\mathbb{P}^n$ is the set of equivalence classes of $\mathbb{C}^{n+1} \setminus \{(0,\dots,0)\}$ for the equivalence relation:

$$(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$$

iff there is a nonzero complex number λ such that $y_i = \lambda x_i$ for all i.

The notation

$$[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n$$

is commonly used to represent the equivalence class of the point $(x_0, x_1, \dots, x_n)$. They are called *homogeneous coordinates*. Let

$$U_j = \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n \mid x_j = 1\}.$$

Note that there is a canonical bijection:

$$U_j \to \mathbb{A}^n, \qquad [x_0 : x_1 : \cdots : x_n] \mapsto (\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}). \tag{3.1}$$

The latter are called the *affine coordinates* of U_j. Observe that

$$\mathbb{P}^n = \bigcup_{j=0}^{n} U_j. \tag{3.2}$$

However there is a lot of overlap between these affine patches.

The points of $\mathbb{P}^n$ that do not lie on U_j are precisely those for which $x_j = 0$. Note that there is a canonical bijection between this set and $\mathbb{P}^{n-1}$. Thus one may think of projective n-space as made of affine n-space and projective $(n-1)$-space. This may be written informally as

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}.$$

There is one such decomposition for each $0 \leq j \leq n$. More generally, one can replace the hyperplane $x_j = 1$ by any affine hyperplane not passing through the origin to get such a decomposition. The points of projective space that are not on the affine hyperplane are called the *points at infinity* wrt that hyperplane. They lie on the parallel hyperplane passing through the origin.

Example 3.2. Consider the complex projective line $\mathbb{P}^1$. Following standard convention, let us use the coordinates $[z : w]$ instead of $[x_0 : x_1]$.

$$\mathbb{P}^1 = \{[1 : z]\} \sqcup \{[0 : 1]\} = \{[z : 1]\} \sqcup \{[1 : 0]\} = \mathbb{C} \sqcup \{\infty\}.$$

This is the Riemann sphere: the complex plane $\mathbb{C}$ plus the point at infinity. It can be written as a union

$$\mathbb{P}^1 = \{[1 : w]\} \cup \{[z : 1]\}.$$

Think of the patch $\{[z : 1]\}$ as the complex plane $\mathbb{C}$ with z being the complex variable. The other patch $\{[1 : w]\}$ is $\mathbb{C}$ minus the origin plus the point at infinity. (It is another copy of the complex plane.) They intersection of the two patches is $\mathbb{C}$ minus the origin.

Remark 3.3. Since $\mathbb{P}^n$ is a quotient of $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$, it has a natural Euclidean topology: two points are close if the corresponding lines have a small angle between them. In this topology, the sets U_j are open, and (3.1) is a homeomorphism, with $\mathbb{A}^n$ given the Euclidean topology.

Definition 3.4. Let X be any nonempty subset of $\mathbb{P}^n$. The *affine cone over* X is the subset of $\mathbb{A}^{n+1}$ consisting of the origin and all points whose equivalence class is in X. If X is empty, then so is the affine cone.

3.1.2. Projective varieties. We say that a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ vanishes at a point $x \in \mathbb{P}^n$ if f vanishes on the entire equivalence class of x. In this situation, we write $f(x) = 0$.

A polynomial in $\mathbb{C}[x_0, \ldots, x_n]$ is called *homogeneous* if all its terms have the same degree. Note that if F is a homogeneous polynomial of degree d, then

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n).$$

Thus if a point belongs to the zero set of a homogeneous polynomial, then its entire equivalence class belongs to the zero set. Thus a homogeneous polynomial vanishes at a point $x \in \mathbb{P}^n$ if it vanishes on some (and hence all) representatives of x.

Observation 3.5. If $f(x) = 0$ for some $x \in \mathbb{P}^n$, then all homogeneous parts of f also vanish at x, and hence, f does not have any constant term.

PROOF. Use the fundamental theorem of algebra: A polynomial of degree n has at most n roots. □

Definition 3.6. A *projective variety* is the common zero set of a finite set $\{f_1, \ldots, f_m\}$ of homogeneous polynomiais, with each $f_j \in \mathbb{C}[x_0, \ldots, x_n]$. We write

$$X = \mathbb{V}'(f_1, \ldots, f_m) \subseteq \mathbb{P}^n.$$

We use the notation $\mathbb{V}'$ to distinguish it from $\mathbb{V}$. We will continue to use the latter for the zero locus in $\mathbb{A}^{n+1}$. The relation between $\mathbb{V}$ and $\mathbb{V}'$ is given in Observation 3.15 below.

Example 3.7. Let us discuss some standard examples of projective varieties.

(1) The space $\mathbb{P}^n$ and $\emptyset$ are projective varieties.
(2) A plane projective curve is the zero set of one homogeneous polynomial in 3 variables. It is a projective variety in $\mathbb{P}^2$. Examples include conics (degree 2 polynomial) and elliptic curves (degree 2 polynomial). The Math Department homepage shows a picture of an elliptic curve. Any line intersects such a curve in three points reflecting the fact that it is defined by a polynomial of degree 3.
(3) The zero set of one homogeneous polynomial is called a *hypersurface*. A plane projective curve is a hypersurface in $\mathbb{P}^2$.
(4) A *linear projective variety* is the zero set of a finite number of linear polynomials (all involving the same number of variables). Note that a suitable linear transformation will map any linear projective variety bijectively to a (smaller) projective space.

Remark 3.8. One would now like to define morphisms between projective varieties. However, this is not as straightforward as one would like. So we will postpone this discussion to after we discuss the Zariski topology on projective varieties.

3.1.3. Projective linear transformations. Any linear map $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ induces a map $\mathbb{P}^n \to \mathbb{P}^m$. A map on projective spaces which comes from a linear map is called a *projective linear transformation*. Note that φ and $\lambda\varphi$ induce the same map on projective spaces for any $\lambda \neq 0$. Thus the group of invertible projective linear transformations $\mathbb{P}^n \to \mathbb{P}^n$ is

$$\mathrm{PGL}(n+1, \mathbb{C}) = \mathrm{GL}(n+1, \mathbb{C})/\mathbb{C}^*.$$

Definition 3.9. Two projective varieties in $\mathbb{P}^n$ are called *projectively equivalent* if there is an invertible projective linear transformation on $\mathbb{P}^n$ which maps one variety bijectively onto the other.

Let us discuss the $n = 1$ case in more detail. A projective linear transformation $\mathbb{P}^1 \to \mathbb{P}^1$ is

$$[z : w] \mapsto [az + bw : cz + dw]$$

for complex numbers a, b, c, d. It is invertible precisely whenever $ad - bc \neq 0$. This is also called a *Mobius transformation*.

Proposition 3.10. *Given three distinct points $[z_1 : w_1]$, $[z_2 : w_2]$ and $[z_3 : w_3]$ in $\mathbb{P}^1$, there is a unique Mobius transformation which sends them to $[1 : 1]$, $[1 : 0]$ and $[0 : 1]$ in that order:*

$$[z : w] \mapsto [(z_1w_2 - z_2w_1)(z_3w - zw_3) : (z_1w_3 - z_3w_1)(z_2w - zw_2)]. \tag{3.3}$$

Proof. Observe that (3.3) maps the three points as required. Since the points are distinct, the determinant of the coefficients

$$(z_1w_2 - z_2w_1)(z_1w_3 - z_3w_1)(z_2w_3 - z_3w_2) \neq 0.$$

So (3.3) is invertible and a Mobius transformation. The fact that it is unique can be deduced from the easy fact: The only Mobius transformation which fixes $[1 : 1]$, $[1 : 0]$ and $[0 : 1]$ is the identity transformation. □

Definition 3.11. Let $([z_1, w_1], [z_2, w_2], [z_3, w_3], [z_4, w_4])$ be a 4-tuple consisting of distinct points in $\mathbb{P}^1$. Their *cross-ratio* is defined to be

$$[(z_1w_2 - z_2w_1)(z_3w_4 - z_4w_3) : (z_1w_3 - z_3w_1)(z_2w_4 - z_4w_2)].$$

Note that this is the image of $[z_4 : w_4]$ under (3.3).

Proposition 3.12. *Any two ordered subsets of 4 distinct points in $\mathbb{P}^1$ are projectively equivalent iff their cross-ratios are the same.*

PROOF. This follows from Proposition 3.10. □

Remark 3.13. Note that if the points lie in the complex plane, then so does their cross-ratio. In this situation, one may say that the cross-ratio of a 4-tuple (z_1, z_2, z_3, z_4) of distinct complex numbers is

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}.$$

In the same vein, the map (3.3) is written

$$z \mapsto \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z_3)(z_2 - z)}.$$

The understanding here is that if the denominator becomes zero, then we interpret the answer as ∞. Thus this is the map which sends z_1, z_2, and z_3 to 1, ∞, and 0 in that order.

3.1.4. Problems.

(1) Show that the affine cone over a projective variety is an affine variety.
(2) Show that any finite subset of $\mathbb{P}^n$ is a projective variety.
(3) Show that $\mathbb{P}^1$ in the Euclidean topology is homeomorphic to S^2, the two dimensional sphere.
(4) Show that $\mathbb{P}^n$ is compact in the Euclidean topology. More generally, show that every projective variety in $\mathbb{P}^n$ is compact with the induced Euclidean topology.
(5) Show that any two ordered subsets of $n+2$ points in general position in $\mathbb{P}^n$ are projectively equivalent, that is, one can map the first ordered subset to the second by an invertible projective linear transformation. (The $n = 1$ case was done in Proposition 3.10.)

3.2. Projective varieties and homogeneous radical ideals

Recall the correspondence between affine varieties and radical ideals. In this section, we establish a similar correspondence for projective varieties. The key idea is to replace ideals by homogeneous ideals.

3.2.1. Homogeneous ideals.

An ideal of $\mathbb{C}[x_0, \ldots, x_n]$ is called *homogeneous* if it contains the homogeneous parts of all its elements. For example, if $x+2z-y^2+xz$ belongs to the ideal then so do $x + 2z$ and $-y^2 + xz$.

Note that

- any homogeneous ideal is generated by finitely many homogeneous polynomials: use the Noetherian property of the polynomial algebra.
- any homogeneous ideal which is not the full algebra is contained in the homogeneous ideal $(x_0, \ldots, x_n)$.

3.2.2. A Galois connection. We can repeat the analysis of Section 2.2.1. Let P be the poset whose elements are subsets of $\mathbb{P}^n$ ordered by inclusion. Let Q be the poset whose elements are homogeneous ideals of $\mathbb{C}[x_0, \ldots, x_n]$ ordered by inclusion. We define two order-reversing maps

$$\mathbb{I}' : P \to Q \qquad \text{and} \qquad \mathbb{V}' : Q \to P$$

as follows.

$$\mathbb{V}'(I) := \{x \in \mathbb{P}^n \mid f(x) = 0 \text{ for all } f \in I\}.$$
$$\mathbb{I}'(X) := \{f \in \mathbb{C}[x_0, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

Observation 3.5 implies that $\mathbb{I}'(X)$ is a homogeneous ideal.

Note that

$$X \subseteq \mathbb{V}'(I) \iff I \subseteq \mathbb{I}'(X).$$

It follows from Proposition B.9 that

- $\mathbb{V}'\mathbb{I}'$ and $\mathbb{I}'\mathbb{V}'$ are closure operators on P and Q respectively.
- The closed sets of $\mathbb{V}'\mathbb{I}'$ are in bijection with the closed sets of $\mathbb{I}'\mathbb{V}'$.

Observe that $\mathbb{V}'(I)$ consists of those points in $\mathbb{P}^n$ where all homogeneous polynomials in I vanish, or equivalently, where some finite set of homogeneous generators of I vanish. Thus:

Proposition 3.14. *The closed sets of $\mathbb{V}'\mathbb{I}'$ are precisely projective varieties.*

Observation 3.15. Suppose $\mathbb{V}'(I) \neq \emptyset$. Then $\mathbb{V}(I) \subseteq \mathbb{A}^{n+1}$ is the affine cone over $\mathbb{V}'(I) \subseteq \mathbb{P}^n$. Further $\mathbb{I}\mathbb{V}(I) = \mathbb{I}'\mathbb{V}'(I)$.

Theorem 3.16 (Projective Nullstellensatz). *Let I be any homogeneous ideal of $R = \mathbb{C}[x_0, \ldots, x_n]$. Then*

$$\mathbb{I}'\mathbb{V}'(I) = \begin{cases} \operatorname{rad}(I) & \text{if } \mathbb{V}'(I) \neq \emptyset, \\ R & \text{if } \mathbb{V}'(I) = \emptyset. \end{cases}$$

Thus, the closed sets of $\mathbb{I}'\mathbb{V}'$ are precisely the homogeneous radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$ excluding $(x_0, \ldots, x_n)$.

Proof. The assertion when $\mathbb{V}'(I) = \emptyset$ is clear. When $\mathbb{V}'(I) \neq \emptyset$, use Observation 3.15 and the usual Nullstellensatz (Theorem 2.8). □

Corollary 3.17. *There is a bijection between projective varieties in $\mathbb{P}^n$ and homogeneous radical ideals of $\mathbb{C}[x_0, \ldots, x_n]$ excluding $(x_0, \ldots, x_n)$. (The empty set corresponds to the full algebra.)*

3.3. The category of projective varieties

Note that so far we have said nothing about morphisms between projective varieties. The reason is that morphisms between projective varieties are defined "locally" on affine patches. In this section, we will first define the Zariski topology on projective varieties and then use it to define the category of projective varieties.

3.3.1. The Zariski topology.

Proposition 3.18. *The union of two projective varieties in $\mathbb{P}^n$ is again a projective variety in $\mathbb{P}^n$. The intersection of an arbitrary number of projective varieties in $\mathbb{P}^n$ is again a projective variety in $\mathbb{P}^n$.*

PROOF. The identities (2.2) and (2.3) hold in the present setting. The latter may also be deduced by noting that the poset of subsets of $\mathbb{P}^n$ and the poset of homogeneous ideals of $\mathbb{C}[x_0, \ldots, x_n]$ are complete lattices. □

We use this result to put a topology on $\mathbb{P}^n$ as follows: A subset $X \subseteq \mathbb{P}^n$ is a closed set in the topology if it is a projective variety in $\mathbb{P}^n$. This is called the *Zariski topology* on $\mathbb{P}^n$. More generally, let X be a projective variety in $\mathbb{P}^n$. Put the subspace topology on it. This is the *Zariski topology* on X. The closed sets are precisely those projective varieties in $\mathbb{P}^n$ which are contained in X.

3.3.2. Homogenization and dehomogenization. To any polynomial $f(t_1, \ldots, t_n)$, we associate a homogeneous polynomial

$$(H_j f)(x_0, \ldots, x_n) := x_j^{\deg(f)} f\Big(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j}\Big),$$

where $\deg(f)$ is the largest degree of any monomial occurring in f. We call $H_j f$ the *homogenization* of f. Conversely, given a homogeneous polynomial g, one can set $x_j = 1$ and rename the remaining x variables by the t variables to obtain a polynomial $D_j g$ in the t_i's. We call $D_j g$ the *dehomogenization* of f. Note that $D_j H_j(f) = f$ while $H_j D_j(g)$ equals g up to a power of x_j.

For example, for $n = 2$,

$$H_1(5 + 2t_1 + t_1^2 t_2) = 5x_1^3 + 2x_0 x_1^2 + x_0^2 x_2$$

and

$$D_1(5x_1^4 + 2x_0 x_1^4 + x_0^2 x_1 x_2) = 5 + 2t_1 + t_1^2 t_2.$$

Proposition 3.19. *The bijection (3.1) is a homeomorphism between U_j (with the induced topology from $\mathbb{P}^n$) and $\mathbb{A}^n$ (with the Zariski topology).*

PROOF. Let $t_1, \ldots, t_n$ be the coordinates for $\mathbb{A}^n$ and $x_0, \ldots, x_n$ be the coordinates for $\mathbb{A}^{n+1}$ used to define $\mathbb{P}^n$. Then for any polynomials f_i in the t variables and homogeneous polynomials g_i in the x variables. Then

$$\mathbb{V}'(H_j f_1, \ldots, H_j f_k) \cap U_j \leftrightarrow \mathbb{V}(f_1, \ldots, f_k)$$

and

$$\mathbb{V}'(g_1, \ldots, g_k) \cap U_j \leftrightarrow \mathbb{V}(D_j g_1, \ldots, D_j g_k).$$

under the bijection (3.1). So closed sets in one correspond to closed sets in the other. □

3.3.3. The category of projective varieties. We now define the category of projective varieties, denoted ProjVariety:

An object is a pair $(X, \mathbb{P}^n)$ where $\mathbb{P}^n$ is projective space with the Zariski topology and X is a closed set in it. (Compare with Remark 2.18.)

A morphism $(X, \mathbb{P}^n) \to (Y, \mathbb{P}^m)$ is a map $\varphi : X \to Y$ such that: For each point $p \in X$, there is an open set $U \subseteq X$ such that the restriction of φ to the open set U is given by

$$U \to \mathbb{P}^m \qquad q \mapsto [f_0(q) : \cdots : f_m(q)]$$

for some homogeneous polynomials $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ each of the same degree. Implicit in the definition is the fact that the f_i do not simultaneously vanish at any point of U.

Example 3.20. The map

$$\mathbb{P}^1 \to \mathbb{P}^2 \qquad [s:t] \mapsto [s^2 : st : t^2].$$

is a morphism of projective varieties: s^2, st and t^2 have the same degree, and the only point at which all of them vanish is $s = t = 0$. It is globally defined in the sense that we have only one chart. Note that the image of this map belongs to the curve $C = \mathbb{V}'(xz - y^2)$. This yields a morphism $\mathbb{P}^1 \to C$ of projective varieties.

Now consider the map

$$C \to \mathbb{P}^1 \qquad [x:y:z] \mapsto \begin{cases} [x:y] & \text{if } x \neq 0, \\ [y:z] & \text{if } z \neq 0. \end{cases}$$

This is well-defined because on the overlap $[x:y] = [zx:zy] = [y^2:zy] = [y:z]$. Note that $x \neq 0$ and $z \neq 0$ are open sets in C and they cover C. The map restricted to either of them is given by polynomials, so it is a morphism of projective varieties. This illustrates the local nature of morphisms between projective varieties.

Observe that $\mathbb{P}^1 \to C$ and $C \to \mathbb{P}^1$ are inverse isomorphisms; so the conic C is isomorphic to $\mathbb{P}^1$ in the category ProjVariety.

3.3.4. Problems.

(1) Show that a morphism of projective varieties is continuous wrt the Zariski topology. Use this to deduce that a composite of morphisms is again a morphism.
(2) Describe the initial object, terminal object, product and coproduct in ProjVariety.
(3) Describe the Zariski topology on $\mathbb{P}^1$.
(4) Show that any automorphism of $\mathbb{P}^1$ is a Mobius transformation. More precisely, the group of automorphisms of $\mathbb{P}^1$ is isomorphic to the group of invertible projective linear transformations $\mathrm{PGL}(2, \mathbb{C})$.
(5) Find an example of two plane projective curves that are isomorphic but not projectively equivalent.
(6) Prove or disprove: The image of a morphism of affine varieties $\varphi : X \to Y$ is an affine subvariety of Y.

    What happens in the case when φ is a morphism of projective varieties?
(7) Consider the morphism of projective varieties:

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^2 \qquad [s:t] \mapsto [s^3 : s^2t : t^3].$$

    Show that the image of φ is a projective variety $X \subseteq \mathbb{P}^2$.

    Check that φ is injective.

    Describe the inverse map $X \to \mathbb{P}^1$ explicitly.

    Is this inverse map a morphism of projective varieties?

3.4. Rational normal curves

In this section, we discuss an important family of projective varieties called rational normal curves. Examples include conics and twisted cubic curves. We also

briefly discuss a further generalization which then includes the Veronese surface as an example.

3.4.1. Conics. An important class of plane projective curves are conics. A *conic* is the zero locus of a homogeneous polynomial of degree 2 in 3 variables. It is said to be *nondegenerate* if the polynomial is irreducible.

Example 3.21. The curves $\mathbb{V}'(xz - y^2)$ and $\mathbb{V}'(z^2 - x^2 - y^2)$ are nondegenerate conics. Let us look at the conic $X = \mathbb{V}'(xz - y^2)$ in more detail. Now consider the corresponding affine variety $\mathbb{V}(xz - y^2)$. (Visualize this variety in $\mathbb{R}^3$ and the terminology "conics" will become clear.) Cutting it by the hyperplane $y = 1$ yields the hyperbola $\mathbb{V}(xz - 1)$. It misses two points of X, namely $[1 : 0 : 0]$ and $[0 :) : 1]$. Cutting it by the hyperplane $z = 1$ yields the parabola $\mathbb{V}(x - y^2)$. It misses one point of X, namely $[1 : 0 : 0]$. Cutting it by other appropriate hyperplanes yield the circle and the ellipse. They do not miss any point of X.

A linear change of coordinates takes $\mathbb{V}'(xz - y^2)$ to $\mathbb{V}'(z^2 - x^2 - y^2)$. (Check this.) So all the features discussed above can be seen for the latter variety also. For example, cutting it by the plane $z = 1$ yields the circle $\mathbb{V}(1 - x^2 - y^2)$. Cutting it by $y = 1$ yields the hyperbola $\mathbb{V}(z^2 - x^2 - 1)$. Its asymptotes are the two missing points of infinity and lie on the plane $y = 0$. Similarly note that there are planes passing through the origin which are tangent to $\mathbb{V}(z^2 - x^2 - y^2)$. Any plane parallel to one of these planes intersects the conic in a parabola, and its missing point at infinity is precisely the tangent plane we started with.

Here is an interesting classical result about conics.

Theorem 3.22. *Given any five points in $\mathbb{P}^2$, there exists a conic passing through them. This conic is unique unless four of the points are collinear, and nondegenerate unless three of the points are collinear.*

Instead of saying "no three points are collinear", one may also say "the five points are in general position."

Proof. The space of all conics is parametrized by $\mathbb{P}^5$:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz \iff [a : b : c : d : e : f].$$

Let $[x_0 : y_0 : z_0]$ be a point in $\mathbb{P}^2$. The subspace of all conics which pass through this point is a hyperplane in $\mathbb{P}^5$.

Any five hyperplanes in general position in $\mathbb{P}^5$ intersect in exactly one point. If they are not in general position, then the intersection will be of a higher dimension. In any case, the intersection is nonempty; so there is at least one conic passing through five given points in $\mathbb{P}^2$. The rest of the proof is left as an exercise. □

The same parameter space construction can be made for elliptic curves and more generally for hypersurfaces.

Proof. We sketch an alternative proof which is more constructive. It is illustrated in Figure 3.1. Let P, Q, R, S and T be the five given points. Let L be the line passing through R and S, and let M be the line passing through R and T. Let O be the point of intersection of the line passing through P and T, and the line passing through Q and S.

Now let N be any line passing through O. It intersect L in a point and M is another point. Draw the line passing through P and the first point, and the line

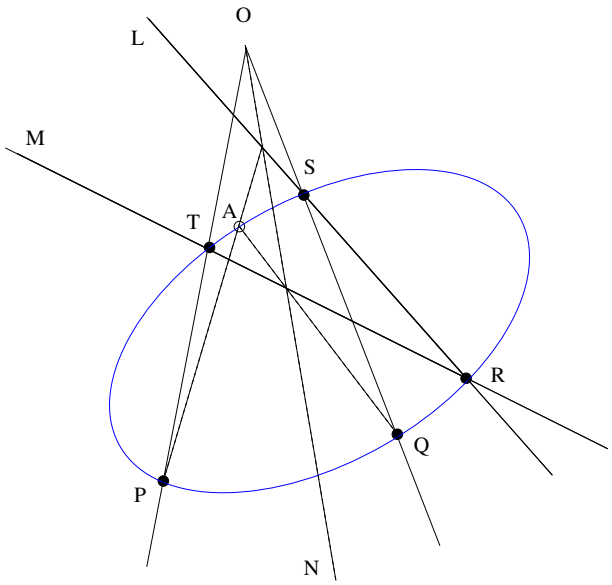


FIGURE 3.1. The conic passing through five points.

passing through Q and the second point. Call the point of intersection of these two lines A. The claim is that as one varies the line N passing through O, the resulting point O moves on a conic. This is the required conic passing through P, Q, R, S and T. □

PROOF. We provide a third proof. Suppose we are given five points in $\mathbb{P}^2$ such that no three of them are collinear. We may assume without loss of generality that the five points are

$$[1:0:0],\ [0:1:0],\ [0:0:1],\ [x_0,y_0,z_0],\ [x_1,y_1,z_1]$$

such that the x_i's and y_j's are all nonzero, and such that the points $[x_1:x_0]$, $[y_1:y_0]$, and $[z_1:z_0]$ are distinct in $\mathbb{P}^1$. (We may further assume $[x_0,y_0,z_0]=[1:1:1]$, since any four points in general position in $\mathbb{P}^2$ are projectively equivalent. However this does not cause any significant simplification; so we avoid using it.)

Now consider the map $\mathbb{P}^1 \to \mathbb{P}^2$

$$[s:t] \mapsto [(x_0s-x_1t)(y_0s-y_1t):(x_0s-x_1t)(z_0s-z_1t):(y_0s-y_1t)(z_0s-z_1t)]$$

The image of this map is a nondegenerate conic, say C: If $[x_0,y_0,z_0]=[1:1:0]$ and $[x_1,y_1,z_1]=[0:1:1]$, then this map coincides with the one considered in Example 3.20. The image in that case was the nondegenerate conic $\mathbb{V}(xz-y^2)$; so the image will be a nondegenerate conic in the present case as well.

Under this map, the five points $[x_1:x_0]$, $[y_1:y_0]$, $[z_1:z_0]$, $[1:0]$ and $[0:1]$ map to the five points that we have in $\mathbb{P}^2$. Thus C is a nondegenerate conic passing through them. □

3.4.2. Twisted cubic curves. This is the projective closure of the affine twisted cubic curve.

Consider the map of projective varieties

$$\nu_3 : \mathbb{P}^1 \to \mathbb{P}^3 \qquad [s:t] \mapsto [s^3 : s^2t : st^2 : t^3].$$

The image of this map is called the *twisted cubic curve*. Let us denote it by C.

Proposition 3.23. *The twisted cubic curve is the projective variery*

$$C = \mathbb{V}'(xw - yz, y^2 - xz, wy - z^2). \tag{3.4}$$

where $[x : y : z : w]$ denote the homogeneous coordinates of $\mathbb{P}^3$. Further ν_3 is an isomorphism of $\mathbb{P}^1$ onto C in the category ProjVariety.

Note that C is different from anything we have encountered so far: it is not a hypersurface, linear space, or finite set of points.

Proof. For simplicity of notation, let us abbreviate $\mathbb{V}'(xw - yz, y^2 - xz, wy - z^2)$ to X. Firstly, it is clear that $C \subseteq X$. Now let $U_{x=1}$ denote the affine patch of $\mathbb{P}^3$ where the first coordinate x is nonzero, and $U_{w=1}$ denote the patch where the last coordinate w is nonzero. Then

$$X = (X \cap U_{x=1}) \cup (X \cap U_{w=1}).$$

Note that $X \cap U_{x=1}$ can be identified with $\mathbb{V}(y^2 - z, y^3 - w)$. This is the (affine) twisted cubic we have met earlier, and which we know can be parametrixed as (t, t^2, t^3). Similarly, the other patch $(X \cap U_{w=1})$ can be identified with $\mathbb{V}(x - z^3, y - z^2)$ which is again the (affine) twisted cubic parametrized by (s^3, s^2, s). This shows that $X = C$ proving (3.4).

To show that the conic C is isomorphic to $\mathbb{P}^1$, consider the morphism

$$C \to \mathbb{P}^1 \qquad [x:y:z:w] \mapsto \begin{cases} [x:y] & \text{if } x \neq 0, \\ [z:w] & \text{if } z \neq 0. \end{cases}$$

It is clear that this is inverse to the morphism $\mathbb{P}^1 \to C$. □

Fix a basis (f_0, f_1, f_2, f_3) for the space of homogeneous degree 3 polynomials in the variables s and t. For example, one may take the monomial basis (s^3, s^2t, st^2, t^3). Then the image of the map

$$\mathbb{P}^1 \to \mathbb{P}^3 \qquad [s:t] \mapsto [f_0 : f_1 : f_2 : f_3]$$

is a projective variety isomorphic to $\mathbb{P}^1$. Any such curve will also be called a twisted cubic curve. Note that it is projectively equivalent to C.

Theorem 3.24. *There is a unique twisted cubic curve passing through any* 6 *points in general position in $\mathbb{P}^3$.*

Proof. Exercise. □

3.4.3. Rational normal curves. Consider the map of projective varieties

$$\nu_n : \mathbb{P}^1 \to \mathbb{P}^n \qquad [s:t] \mapsto [s^n : s^{n-1}t : \cdots : t^n].$$

The image of this map is called the *rational normal curve* of degree n. We will see below that it is a projective variety isomorphic to $\mathbb{P}^1$.

More generally, we will use the term rational normal curve of degree n to mean any curve that is obtained from this one by a linear change of coordinates in $\mathbb{P}^n$.

- ν_1 is the identity map.

- ν_2 is the map considered in Example 3.20 and its image is the conic $\mathbb{V}'(xz-y^2)$. Thus a rational normal curve of degree 2 is same as a nondegenerate conic.
- The image of ν_3 is the twisted cubic curve $\mathbb{V}'(xw-yz, y^2-xz, wy-z^2)$. Thus a rational normal curve of degree 3 is same as a twisted cubic curve.

Proposition 3.25. *The image of ν_n is a closed set in $\mathbb{P}^n$ and ν_n is an isomorphism of $\mathbb{P}^1$ onto this projective variety.*

PROOF. This can be proved in the same manner as the $n = 3$ case. We provide the outline. If C denotes the image of ν, then

$$C = \mathbb{V}'(x_i x_j - x_{i-1}x_{j+1})_{1\leq i\leq j\leq n-1},$$

where $[x_0 : \cdots : x_n]$ are the homogeneous coordinates of $\mathbb{P}^n$. Another way to express this to say that C is the zero locus of points $[x_0 : \cdots : x_n] \in \mathbb{P}^n$ such that the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdot & \cdot & x_{n-1} \\ x_1 & x_2 & \cdot & \cdot & x_{n-1} & x_n \end{pmatrix}$$

has rank 1. The projective variety C can be covered by two affine patches, and the map $C \to \mathbb{P}^1$ can be defined on these patches by restricting to the first two and to the last two coordinates respectively. □

The following result generalizes Theorems 3.22 and 3.24.

Theorem 3.26. *There is a unique rational normal curve passing through any $n+3$ points in general position in $\mathbb{P}^n$.*

PROOF. Exercise. □

3.4.4. The Veronese maps. The construction of the rational normal curve can be further generalized. For any n and d, consider the Veronese map of degree d

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^m$$

which maps $[x_0 : \cdots : x_n]$ to the point in $\mathbb{P}^m$ which lists all the degree d monomials in the x_i's in some order. The number of degree-d monomials in $n+1$ variables is the binomial coefficient $\binom{d+n}{d}$. Thus $m = \binom{d+n}{d} - 1$.

Example 3.27. The simplest instance of the Veronese map other than the case $n = 1$ of rational normal curves is the map

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5 \qquad [s : t : u] \mapsto [s^2 : t^2 : u^2 : st : su : tu].$$

The image of this map is a projective variety isomorphic to $\mathbb{P}^2$. It is called the *Veronese surface*.

The following is a generalization of Proposition 3.25 and proved in similar manner.

Proposition 3.28. *The image of ν_d is a closed set in $\mathbb{P}^m$ and ν_d is an isomorphism of $\mathbb{P}^n$ onto this projective variety.*

Note that there is a choice involved in the definition of ν_d in how the monomials are listed, and strictly speaking each choice yields a different map. This is not crucial because any two maps are related by a permutation of the variables in $\mathbb{P}^m$. It is more natural in this context to work instead of $\mathbb{P}^m$ with the projective space whose coordinates are indexed by degree-d monomials in $n+1$ variables. Let us

denote them by z_I for $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$. The image of ν_d is the projective variety

$$\mathbb{V}'(z_I z_J - z_K z_L)_{I+J=K+L}$$

where $I, J, K, L \in \mathbb{N}^{n+1}$.

3.4.5. Problems.

(1) Complete the first proof of Theorem 3.22.
(2) Check that the construction in Figure 3.1 indeed yields a conic.
(3) Let F and G be two irreducible homogeneous quadratic polynomials in three variables. Show that there is a projective linear map $\mathbb{P}^2 \to \mathbb{P}^2$ which maps $\mathbb{V}'(F)$ isomorphically onto $\mathbb{V}'(G)$. Conclusion: All nondegenerate conics are projectively equivalent.
(4) Let F be an irreducible quadratic polynomial in two variables. Show that there is a linear map $\mathbb{A}^2 \to \mathbb{A}^2$ which maps $\mathbb{V}(F)$ isomorphically onto either the parabola $\mathbb{V}(y - x^2)$ or the hyperbola $\mathbb{V}(xy - 1)$, but the parabola and hyperbola are nonisomorphic. Conclusion: there are exactly two nondegenerate affine conics up to affine isomorphism.
(5) How many conics do you expect to pass through 4 given points and tangent to one given line? Through 3 given points and tangent to 2 given lines? Through 2 given points and tangent to 3 given lines? Through 1 given point and tangent to 4 given lines? Through 5 given lines?
(6) Show that $\mathbb{V}'(f) \cap \mathbb{V}'(g)$, where f and g are any two of the three polynomials in (3.4), is the twisted cubic curve union a line.
(7) Show that $\mathbb{V}'(xz - y^2, xw^2 - 2yzw + z^3)$ is the twisted cubic curve of (3.4).
    blueReid page 91.
(8) Show that any finite set of points on a twisted cubic curve are in general position, that is, any four of them span $\mathbb{P}^3$.
(9) Show that any $n+1$ distinct points of a rational normal curve of degree n are linearly independent. In particular, no three points of a nondegenerate conic are collinear.
(10) Prove Theorems 3.24 and 3.26 by generalizing one of the proofs of Theorem 3.22.
(11) Describe the image of the lines in $\mathbb{P}^2$ on the Veronese surface (under the map ν_2).
(12) Show that the set of all nondegenerate conics form a Zariski-open set of the parameter space $\mathbb{P}^5$ of all conics. Further show that the set of double lines forms a Zariski-closed set of $\mathbb{P}^5$ isomorphic to the Veronese surface.
(13) Prove or disprove: Given four points in general position in $\mathbb{P}^2$, there exist two nondegenerate conics which intersect in precisely these four points.
(14) Show that the intersection of the quadric surfaces $\mathbb{V}'(x^2 - yw)$ and $\mathbb{V}'(xy - zw)$ is the union of a twisted cubic curve and a line. Thus, the intersection of two irreducible (projective) varieties need not be irreducible.
(15) Show that any two curves in $\mathbb{P}^2$ have a nonempty intersection.
(16) Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$ for $i \neq j$. Show that any regular function on $\mathbb{P}^n \setminus (H_i \cap H_j)$ is constant.
(17) Show that $\mathbb{P}^n \setminus H$ is an affine open set in $\mathbb{P}^n$ where H is any hypersurface.

Have a section on products (segre maps), Veronese maps, Grassmann and flag varieties.

CHAPTER 4

Varieties

4.1. Prevarieties

In this section, we introduce the basic category of classical algebraic geometry. This is the category of prevarieties, which we will denote by Prevariety. It includes the categories of affine as well as projective varieties as full subcategories.

4.1.1. Prevarieties.

Definition 4.1. A *prevariety* X is a Noetherian topological space, equipped with a sheaf $\mathcal{O}_X$ of complex-valued functions, such that X is the union of finitely many subsets U_i each isomorphic to an affine variety.

The precise meaning of isomorphic is as follows. For each i, there is an affine variety Y_i (with its structure sheaf $\mathcal{O}_{Y_i}$), and a homeomorphism $U_i \to Y_i$ which induces isomorphisms of $\mathbb{C}$-algebras $\mathcal{O}_{Y_i}(V) \to \mathcal{O}_{U_i}(U)$ for corresponding open sets U and V. Here $\mathcal{O}_{U_i}$ is the restriction of the sheaf $\mathcal{O}_X$ to the open set U_i.

We clarify that $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras. In particular, the constant functions belong to $\mathcal{O}_X(U)$ for any open set U. An element of $\mathcal{O}_X(U)$ is called a regular function on U.

Definition 4.2. A morphism $\varphi : X \to Y$ of prevarieties is a continuous map such that for every open set $V \subseteq Y$, and for every function $f \in \mathcal{O}_Y(V)$, the function $f \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V))$. In short: a morphism pulls back regular functions to regular functions.

This defines the category Prevariety of prevarieties.

Remark 4.3. In Definition 4.1,

- one may drop the requirement of "finitely many": Since the underlying topological space of a prevariety is Noetherian, it is quasi-compact.
- require that each U_i is isomorphic to an open set of an affine variety, with its restricted sheaf. This does not yield anything new: every open set is covered by basic open sets and every basic open set is isomorphic to an affine variety as shown in Example 4.7 below.

Definition 4.4. An open set U of prevariety X is called an *affine open set* if U with the restricted sheaf is isomorphic to an affine variety.

Definition 4.5. A prevariety is called *irreducible* if it is irreducible as a topological space.

Since the underlying topological space of a prevariety is Noetherian, every prevariety is the union of finitely many irreducible components. We mention that in the irreducible case, we have access to the generic stalk, whose description is exactly as given for affine varieties.

4.1.2. Affine varieties. It is clear that each affine variety with its structure sheaf is a prevariety, and morphisms of affine varieties are morphisms of prevarieties. Thus there is a functor

$$\mathsf{AffVariety} \to \mathsf{Prevariety}.$$

We now show that any basic open set in an affine variety is a prevariety (this is true of any open set as we will see later) and more surprisingly, it is isomorphic to an affine variety sitting in an ambient space which is one dimension higher.

Example 4.6. Consider the open set $U = \mathbb{A}^1 \setminus \{0\}$ in $\mathbb{A}^1$. We claim that U is isomorphic as a prevariety to the affine variety $Y = \mathbb{V}(xy-1)$ in $\mathbb{A}^2$. Here U is given the restricted structure sheaf from $\mathbb{A}^1$ and Y is given its structure sheaf. Explicitly, the maps

$$U \to Y \qquad t \mapsto (t, 1/t) \qquad \text{and} \qquad Y \to U \qquad (x, y) \mapsto x$$

are inverse isomorphisms in Prevariety: It is clear that the maps are inverse to each other. In addition, one needs to check that they are both continuous *wrt the Zariski topology*, and further the sets of regular functions over corresponding open sets are isomorphic. This is straightforward.

Example 4.7. The previous example generalizes as follows. Let

$$X = \mathbb{V}(f_1, \ldots, f_k)$$

where $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$ be any affine variety. Let $X_f = \{x \in X \mid f(x) \neq 0\}$ be the basic open in X corresponding to $f \in \mathbb{C}[X]$. Now consider the affine variety

$$Y = \mathbb{V}(f_1, \ldots, f_k, 1 - x_{n+1}f) \ \subseteq \ \mathbb{C}^{n+1}.$$

We claim that X_f is isomorphic as a prevariety to Y. The former is given the restricted sheaf from $\mathcal{O}_X$, and the latter is given its structure sheaf. Explicitly, the maps

$$X_f \to Y \qquad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1/f).$$

and

$$Y \to X_f \qquad (x_1, \ldots, x_n, x_{n+1}) \mapsto (x_1, \ldots, x_n)$$

are inverse isomorphisms in Prevariety. This is straightforward to check.

To summarize, any basic open set of an affine variety is an affine open set.

4.1.3. Subprevarieties. Let X be any prevariety and let A be any subset of X. Give A the induced topology. For any open set U of A, define $\mathcal{O}_A(U)$ to be the set of functions satisfying the condition: for each $x \in U$, there is an open set V in X and a $g \in \mathcal{O}_X(V)$ such that f is the restriction of g to $U \cap V$. This yields a sheaf $(A, \mathcal{O}_A)$.

Here are some useful remarks regarding this construction.

(1) The inclusion map $A \to X$ induces a morphism of sheaves $(A, \mathcal{O}_A) \to (X, \mathcal{O}_X)$: For any open set V in X and $g \in \mathcal{O}_X(V)$, the restriction of g to $A \cap V$ belongs to $\mathcal{O}_A(A \cap V)$.
(2) It is functorial wrt inclusion: If $B \subseteq A \subseteq X$, then the sheaf $\mathcal{O}_B$ induced from $\mathcal{O}_X$ is same as the sheaf on B induced from $\mathcal{O}_A$. [This is implicitly used in many arguments.]
(3) Given $f \in \mathcal{O}_A(U)$, it may not be possible to find a $g \in \mathcal{O}_X(V)$, with $U \subseteq V$ and $f = g$ on U. Any concrete example?

(4) If A is a singleton, then $\mathcal{O}_A(A) = \mathbb{C}$. This is different from the stalk of $\mathcal{O}_X$ at the point A. So this construction seems to be different from pullback of sheaves, the left adjoint of the pushforward.
(5) Since X is Noetherian, so is A. But $(A, \mathcal{O}_A)$ may not be a prevariety because it may fail to be locally affine.
(6) If A is an open set in X, then $\mathcal{O}_A$ is the restriction of $\mathcal{O}_X$ to A.
(7) If X is an affine variety and Y is a closed set in X, then $\mathcal{O}_A$ agrees with the structure sheaf of A. This follows from Proposition 2.45.

Definition 4.8. A subset of a topological space is called *locally closed* if it is the intersection of a closed set and an open set. In particular, open and closed sets are locally closed.

A locally closed subset of a prevariety with its induced sheaf is called a *subprevariety*.

Proposition 4.9. *The above construction turns any locally closed set of a prevariety into a prevariety. The inclusion map is a morphism of prevarieties.*

PROOF. Let X be a prevariety, and let A be a locally closed set in X. We need to show that A is a union of open affine sets. This can be done by analyzing two special cases.

Suppose A is an open set in X. Then by (6) above, $\mathcal{O}_A$ is the restriction of $\mathcal{O}_X$ to A. Further A is the union of open affine sets: Since X is a prevariety, it is the union of open affine sets. By the discussion in Example 4.7, any open set in such an affine set is also the union of open affine sets.

Suppose A is a closed set in X. We show that A has a cover by affine open sets. Let $x \in A$. Let V be an affine open set in X containing x. Then $A \cap V$ is a closed set in V, so it is an affine subvariety of V. By (7) above, the sheaf on $A \cap V$ induced by V is same as the structure sheaf of the variety $A \cap V$. Thus $A \cap V$ is an affine open set of A containing x. Since x is arbitrary, we get the desired cover.

By combining the two cases above, the result follows. □

Proposition 4.10. *Let X and Y be prevarieties and let Z be a subprevariety of Y. Let $f : X \to Z$ be any function, and let $g : X \to Y$ be the composite of f with the inclusion map. Then f is a morphism of prevarieties iff g is a morphism of prevarieties.*

PROOF. The forward implication is clear. The backward implication requires a little thought which is left as an exercise. □

4.1.4. Morphisms between prevarieties. There are many ways to recognize when a map between prevarieties is a morphism. We discuss some of them below.

Proposition 4.11. *Let X and Y be prevarieties and let $\varphi : X \to Y$ be any map. Suppose that $\{U_i\}$ is an open cover of X. Then φ is a morphism of prevarieties iff the restriction of φ to each U_i is a morphism of prevarieties.*

PROOF. The forward implication is clear. We show the backward implication: Let f be a regular function on an open set V in Y. Then $\varphi^{-1}(V) = \bigcup_i \varphi^{-1}(V) \cap U_i$, and by assumption, the restriction of $f \circ \varphi$ to each open set $\varphi^{-1}(V) \cap U_i$ is regular. The sheaf axiom then implies that $f \circ \varphi$ is regular on $\varphi^{-1}(V)$ as required. □

Proposition 4.12. *Let X be a prevariety, and let $f : X \to \mathbb{C}$ be any function. Then f is regular iff $f : X \to \mathbb{A}^1$ is a morphism of prevarieties. In other words, there is a bijection*

$$\mathcal{O}_X(X) \xrightarrow{\cong} \mathsf{Prevariety}(X, \mathbb{A}^1).$$

PROOF. The backward implication is clear. We show the forward implication: As a first step, suppose X is an affine variety. In this case, a regular function f is same as an element of $\mathbb{C}[X]$ which is a polynomial, so clearly $f : X \to \mathbb{A}^1$ is a morphism of prevarieties. For the general case, write X as a union of open affine sets U_i. Since f is regular, its restriction to each U_i is regular. Applying the previous result, each $U_i \to \mathbb{A}^1$ is a morphism of prevarieties. Proposition 4.11 now yields that $f : X \to \mathbb{A}^1$ is a morphism of prevarieties. □

Corollary 4.13. *Let f and g be regular functions on a prevariety Y. Then the subset where f and g agree (that is, they have the same value) is a closed set in Y.*

PROOF. Since f and g are regular, so is $f - g$: regular functions form a $\mathbb{C}$-algebra. Proposition 4.12 shows that $f - g$ can be viewed as a morphism to $\mathbb{A}^1$; in particular, it is continuous. So $(f - g)^{-1}(0)$ is a closed set of Y as required. □

Proposition 4.14. *Let X be a prevariety, Y be an affine variety whose ambient space is $\mathbb{C}^m$, and let $\varphi : X \to Y$ be any function. Then φ is a morphism of prevarieties iff φ has the form*

$$\varphi(x) = (f_1(x), \ldots, f_m(x))$$

where each f_i is a regular function on X.

PROOF. The forward implication follows from Proposition 4.12 by using projections on each coordinate. For the backward implication, cover X by affine open sets U_i and use Propositions 2.20 and 4.11. □

The above result shows that morphisms of prevarieties are easier to handle if the target is an affine variety. The reason is that an affine variety gives us access to coordinates; so we can subtract morphisms.

Corollary 4.15. *Let X be a prevariety, Y be an affine variety. Suppose $\varphi, \psi : X \to Y$ are morphisms of prevarieties. Then the subset where φ and ψ agree is a closed set in X.*

PROOF. This follows from Proposition 4.14 and Corollary 4.13. □

Proposition 4.16. *Let X and Y be prevarieties and let $\varphi : X \to Y$ be any map. Then φ is a morphism of prevarieties iff There is a cover of Y by affine open sets V_i and a cover of X by open sets U_i such that $\varphi(U_i) \subseteq V_i$ and $\varphi : U_i \to V_i$ is a morphism ofprevarieties.*

PROOF. Forward implication. By definition, Y has a cover by affine open sets V_i. Put $U_i := \varphi^{-1}(V_i)$. Since φ is a morphism, the second condition is satisfied.

Backward implication. Since $\varphi : U_i \to V_i$ is a morphism of prevarieties, so is the composite $\varphi : U_i \to V_i \hookrightarrow Y$. Now apply Proposition 4.11. □

4.1.5. Projective varieties. Let us look at the situation of projective varieties. We begin with the projective space $\mathbb{P}^n$. Recall from (3.1) and (3.2) and Proposition 3.19 that $\mathbb{P}^n$ can be written as a union of open sets U_j, each homeomorphic to $\mathbb{A}^n$ wrt the Zariski topologies. Call these homeomorphisms $\varphi_j : U_j \to \mathbb{A}^n$. We use these maps to define a sheaf of functions $\mathcal{O}$ on $\mathbb{P}^n$:

Let V be any open set in $\mathbb{P}^n$. Suppose V is contained in some U_j. Then a function on V is *regular* if it is of the form $f \circ \varphi_j$, where f is a regular function on $\varphi_j(V)$. Define $\mathcal{O}(V)$ to be the set of regular functions on V. The important observation is that this set is well-defined. In other words, if $V \subseteq U_i \cap U_j$, then using either of the two affine patches yields the same set of functions on V. For the general case, we say $f \in \mathcal{O}(V)$ if its restriction to $V \cap U_j$ belongs to $\mathcal{O}(V \cap U_j)$ for each j.

This turns $\mathbb{P}^n$ into a prevariety. Following the standard notation, we will write $\mathcal{O}_{\mathbb{P}^n}$ for its sheaf of functions.

Example 4.17. To understand the above construction better, consider the case of $\mathbb{P}^1$. Let us denote its homogeneous coordinates by $[z : w]$. Then

$$U_1 = \{[1 : w] \mid w \in \mathbb{C}\} \quad \text{and} \quad U_2 = \{[z : 1] \mid z \in \mathbb{C}\}$$

are the two affine patches. Suppose V is an open set of $\mathbb{P}^1$ which is contained in $U_1 \cap U_2$, If f is a regular function on V, then it may be viewed as a function of the w-coordinate only, or of the z-coordinate only. And one can pass from one situation to the other by the transformation $w = 1/z$.

Any projective variety is a closed set in its ambient projective space. So it becomes a prevariety by Proposition 4.9.

The next observation is that there is a fully faithful functor

$$\mathsf{ProjVariety} \to \mathsf{Prevariety}.$$

In more explicit terms: A morphism of projective varieties $\varphi : X \to Y$ is also a morphism $\varphi : (X, \mathcal{O}_x) \to (Y, \mathcal{O}_Y)$ of the corresponding prevarieties. Conversely, any morphism between the prevarieties corresponding to projective varieties is a morphism of the projective varieties. This requires some argument.

4.1.6. Quasi-affine and quasi-projective varieties. Most varieties that one encounters in practice are locally closed subsets of affine or projective space. These are called quasi-affine and quasi-projective varieties. By Proposition 4.9, these are prevarieties.

Definition 4.18. A *quasi-affine variety* is an open set in an affine variety. Equivalently, it is a locally closed set in affine space.

Similarly, a *quasi-projective variety* is an open set in a projective variety. Equivalently, it is a locally closed set in projective space.

It may be appropriate to denote a quasi-affine variety as a triple $(X, X', \mathbb{C}^n)$, where $(X', \mathbb{C}^n)$ is an affine variety and X in an open set in X'. However, we will usually denote it only by the open set X with the ambient affine variety being understood. If we need to specify the ambient space, we will write $X \subseteq X'$. Similar comments apply to a quasi-projective variety.

A morphism between quasi-affine (quasi-projective) varieties is defined to be a morphism between the corresponding prevarieties. This defines the category of quasi-affine (quasi-projective) varieties. We denote it by $\mathsf{QAffVariety}$ ($\mathsf{QProjVariety}$).

It is a full subcategory of Prevariety and contains the category of affine (projective) varieties.

Let X be a quasi-affine variety whose ambient affine variety X' is irreducible. Then recall that a regular function on X is represented by a unique element of $\mathbb{C}(X')$. Hence, in this case, a morphism $X \to Y$ of quasi-affine varieties is a map $\varphi : X \to Y$ of the form

$$\varphi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

where each $f_i \in \mathbb{C}(X')$.

Example 4.19. The set of all $n \times n$ matrices can be identified with the set $\mathbb{C}^{n^2}$ whose coordinates are indexed by $\{x_{ij}\}_{1 \leq i,j \leq n}$. The subset $\mathrm{SL}(n, \mathbb{C})$ of matrices of determinant 1 is an affine variety in $\mathbb{C}^{n^2}$. It is a hypersurface defined by the polynomial $\det(x_{ij}) - 1$.

The subset $\mathrm{GL}(n, \mathbb{C})$ of invertible matrices is a basic open set in $\mathbb{A}^{n^2}$ corresponding to the function $\det(x_{ij})$. It is *not* an affine variety but it is a quasi-affine variety which is isomorphic to an affine variety in the category QAffVariety. This follows from the discussion in Example 4.7.

Warning. Some authors use the term affine variety to mean any object in Prevariety which is isomorphic to an affine variety. Any basic open set of an affine variety such as $\mathrm{GL}(n, \mathbb{C})$ is then an affine variety in this sense.

4.1.7. Problems.

(1) Show that affine open sets form a base for the topology of a prevariety.
(2) Let $\varphi : X \to \mathbb{A}^1$ be a morphism of prevarieties, with X irreducible. Then φ is not dominant (that is, image is not dense) iff φ is a constant map.
(3) For morphisms $\varphi, \psi : Y \to X$ of prevarieties, show that the subset Z where φ and ψ agree is a locally closed set of Y.
(4) Let X be an affine variety and suppose $U = X \setminus \mathbb{V}(f, g)$, where $f, g \in \mathbb{C}[X]$. Show that U may not be an affine open set in X.
(5) Suppose $\varphi : X \to Y$ is a morphism of prevarieties which restricts to a map $A \to B$ where A (resp. B) is a locally closed set in X (resp. Y). Then is the restricted map a morphism?
(6) Describe the initial object, terminal object, coproduct and product in the category of prevarieties.
(7) What are the irreducible components of $X \times Y$ in terms of those of X and Y? Say both X and Y are affine.
(8) Show that $\mathbb{P}^n$ with its Zariski topology is irreducible. Show that the only regular functions on $\mathbb{P}^n$ are constant functions. What is the generic stalk of the structure sheaf of $\mathbb{P}^n$?

    This is an example of why one needs to consider sheaves: global sections are insufficient to nail the geometric object.
(9) Consider the punctured plane $X = \mathbb{A}^2 \setminus \{(0, 0)\}$. It is a quasi-affine variety.
    - Show that the ring of regular functions on X is the polynomial ring $\mathbb{C}[x, y]$.
    - Show that X is not isomorphic as a variety to any affine variety. Find an open affine cover of X.
(10) Any point in a variety is a closed set in that variety.

4.2. Varieties

We have seen that the Zariski topology on affine varieties is not Hausdorff. However note that the diagonal in $X \times X$ is a closed set in the Zariski topology since it can be defined as the zero set of a set of polynomials. For example, the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1$ is same as $\mathbb{V}(x-y)$. This is reminiscent of the Hausdorff axiom. Let us discuss this in more detail.

Proposition 4.20. *The following conditions on a topological space X are equivalent.*

(1) *X is Hausdorff.*
(2) *The diagonal $\{(x,x) \mid x \in X\}$ is a closed set in $X \times X$, the latter given the product topology.*
(3) *For any topological space Y and continuous maps $\varphi, \psi : Y \to X$, the set*
$$\{y \in Y \mid \varphi(y) = \psi(y)\}$$
is a closed set of Y.

Proof. (1) $\iff$ (2). Standard exercise.

(3) $\implies$ (2). Take $Y = X \times X$ and φ, ψ to be projections on the two coordinates. The diagonal is precisely the set where the two projections agree.

(2) $\implies$ (3). Suppose the diagonal is a closed set. A pair of continuous maps $\varphi, \psi : Y \to X$ is equivalent to one continuous map $(\varphi, \psi) : Y \to X \times X$. The given subset of Y is the inverse image of the diagonal (which is a closed set) under (φ, ψ) (which is a continuous map); so it is closed. □

The categorical product exists in Prevariety, see for example [15, Section 2.4]. Let us take this for granted. We will write $X \times Y$ for the product of X and Y. Here are some useful remarks.

- As sets, $X \times Y$ is the cartesian product of X and Y, but the topology is not the product topology.
- For affine and projective varieties, this construction agrees with the products in AffVariety and ProjVariety. More precisely, the functors AffVariety $\to$ Prevariety and ProjVariety $\to$ Prevariety preserve products.

The following result can be proved exactly as we proved the implications (2) $\iff$ (3) in Proposition 4.20.

Proposition 4.21. *The following conditions on a prevariety X are equivalent.*

(1) *The diagonal $\{(x,x) \mid x \in X\}$ is a closed set in $X \times X$, the latter being the categorical product.*
(2) *For any prevariety Y and morphisms $\varphi, \psi : Y \to X$ of prevarieties, the set*
$$\{y \in Y \mid \varphi(y) = \psi(y)\}$$
is a closed set of Y.

We will refer to either of the above two equivalent conditions as the *Hausdorff axiom*.

Definition 4.22. A prevariety X is a *variety* if it satisfies the Hausdorff axiom. A morphism $X \to Y$ of varieties is a morphism of the underlying prevarieties. This defines the category Variety of varieties.

It follows from Proposition 4.10 that any subprevariety is a variety. We will call it is *subvariety*. It is same as a locally closed subset of a variety with the induced sheaf.

Example 4.23. It follows from Corollary 4.15 that any affine variety X satisfies the Hausdorff axiom, so it is a variety. This yields a functor

$$\mathsf{AffVariety} \to \mathsf{Variety},$$

It is clear that this functor is fully faithful.

Example 4.24. Let X be the line with a doubled origin: take two copies of $\mathbb{A}^1$, say U_1 and U_2 with coordinates x_1 and x_2, patched by the map $x_1 = x_2$ on the open sets $x_1 \neq 0$ and $x_2 \neq 0$. Then X is a prevariety (with the cofinite topology). Consider the diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow[i_1]{\cong} & U_1 \\ {\scriptstyle\cong}\downarrow & {\scriptstyle i_2} & \downarrow \\ U_2 & \hookrightarrow & X. \end{array}$$

Then

$$\{y \in \mathbb{A}^1 \mid i_1(y) = i_2(y)\} = \mathbb{A}^1 \setminus \{0\}$$

is not closed in $\mathbb{A}^1$, hence X is *not* a variety.

If instead of patching by the map $x_1 = x_2$, one patches by $x_1 = \frac{1}{x_2}$, then one does obtain a variety. It is $\mathbb{P}^1$, the Riemann sphere.

Lemma 4.25. *Let X be a prevariety, and assume that each pair $x, y \in X$ lie is some affine open subset of X. Then X is a variety.*

Proof. Let $\varphi, \psi : Y \to X$ be morphisms of prevarieties, and let Z be the set on which they agree. We want to show that Z is a closed set in Y. Suppose $z_0 \in \overline{Z}$. Let $x_0 = \varphi(z_0)$ and $y_0 = \psi(z_0)$. Let V be an affine open subset of X containing x_0 and y_0. Put $U = \varphi^{-1}(V) \cap \psi^{-1}(V)$. It contains z_0. Now consider the restricted morphisms $\varphi, \psi : U \to V$. The set on which they agree is $U \cap Z$. Since V is affine, it is a variety. So $U \cap Z$ is a closed set in U. So $U \setminus (U \cap Z)$ in an open set in U, and hence an open set in Y. Further by construction, it is disjoint from Z. So it cannot contain z_0. So $z_0 \in Z$ as required. □

Example 4.26. Any projective variety satisfies the hypothesis of Lemma 4.25; so it is a variety. This yields a fully faithful functor

$$\mathsf{ProjVariety} \to \mathsf{Variety}.$$

Example 4.27. A subprevariety of a variety is again a variety. This is therefore called a *subvariety*.

Lemma 4.28. *Let Y be a variety, and let X be any prevariety. If $\varphi : X \to Y$ is a morphism, then the graph $\{(x, \varphi(x)) \mid x \in X\}$ is a closed set in $X \times Y$ and it is isomorphic as a prevariety to X.*

Proof. Consider the two morphisms $X \times Y \to Y$ which send (x, y) to y and to $\varphi(x)$ respectively. Then the subset where the two morphisms agree is precisely the graph, say Z, of φ; so it is a closed set in $X \times Y$ since Y is a variety. Thus Z is a subprevariety of $X \times Y$. Proposition 4.10 says that the map $X \to Z$ which

sends x to $(x, \varphi(x))$ is a morphism of prevarieties. It is, in fact, an isomorphism: The inverse is given by the composite morphism $Z \hookrightarrow X \times Y \twoheadrightarrow X$. □

Lemma 4.29. *Let Y be a variety, and let X be any prevariety. If $\varphi, \psi : X \to Y$ are morphisms which agree on a dense subset of X, then $\varphi = \psi$.*

Proof. Since Y is a variety, the set on which φ and ψ agree is closed set in X. If this set is dense, then it must be all of X. □

4.3. Smooth manifolds

This is a brief interlude on manifolds. Its purpose is to bring out the similarity between the way manifolds and varieties are defined. The term "affine manifold" used below is not standard, and it is only used here to make the analogy with affine varieties.

A *smooth affine manifold* is an open set U in $\mathbb{R}^n$, with the Euclidean topology on the latter. A morphism of smooth affine manifolds $U \to V$ is a smooth map. This defines the category of affine manifolds.

Alternatively, a smooth affine manifold is an open set U in $\mathbb{R}^n$ equipped with its sheaf of smooth functions $\mathcal{F}_U$. A morphism of smooth affine manifolds $(U, \mathcal{F}_U) \to (V, \mathcal{F}_V)$ is a continuous map $U \to V$ such that for any $f \in \mathcal{F}_V(V)$, the function $f \circ \varphi \in \mathcal{F}_U(U)$.

Definition 4.30. A *smooth manifold* is a Hausdorff topological space where each point has an open neighbourhood which is isomorphic as a sheaf to some smooth affine manifold.

You may have seen a definition of a smooth manifold given in terms of charts. As an exercise, you may check that that the two definitions are equivalent. Having said this, one may add that there is also an alternative definition of varieties in terms of charts. Can you say what it is? use quasi-affine varieties as models.

Riemann surfaces can be dealt in the same way. Is there any notion of a Banach manifold modeled on Banach spaces?

4.3.1. Problems.

(1) Let U and V be open sets in Euclidean space. Show that $\varphi : U \to V$ is differentiable iff For any differentiable function $f : V \to \mathbb{R}$, the composite $f \circ \varphi : U \to \mathbb{R}$ is differentiable.

4.4. Rational maps

In this section, we define the notion of a rational map. This gives rise to a category VarietyRational. We will show that VarietyRational is equivalent to fgField, the category of finitely generated field extensions of $\mathbb{C}$. This is reminiscent of the equivalence between AffVariety and $\overline{\mathsf{Alg}}^{\mathsf{co}}$.

In this section, all varieties are assumed to be irreducible. Recall that in this situation, any nonempty open set is dense. This fact will be used repeatedly.

4.4.1. The category of dominant rational maps.

Definition 4.31. Let X and Y be irreducible varieties. A *rational map* from X to Y, denoted

$$\varphi : X \dashrightarrow Y,$$

is an equivalence class of pairs (U, φ_U) where U is a dense nonempty open set in X and $\varphi_U : U \to Y$ is a morphism of varieties, with two pairs (U, φ_U) and (V, φ_V) being equivalent if φ_U and φ_V agree on $U \cap V$.

The rational map φ is *dominant* if the image of the morphism $\varphi_U : U \to Y$ is dense in Y for some choice of U.

We make a few remarks.

- A rational map is *not* a map in the usual sense. The broken arrow is used to remind ourselves of this feature. There is a largest open set which represents φ: Take union of all open sets that represent φ. This is the domain of φ beyond which it cannot be extended. Thus equivalently, a rational map *is* a map from a dense open set of X to Y, which cannot be extended further.

  Make this same comment for elts of generic stalks.
- If the image of φ is dense in Y, then so is the image of its restriction to any nonempty open set $U' \subseteq U$. Thus a rational map φ is *dominant* if the image of the morphism $\varphi_U : U \to Y$ is dense in Y for all open sets U in its domain.
- One may not be able to compose rational maps: Consider

  $$\mathbb{A}^1 \to \mathbb{A}^1 \quad x \mapsto 1, \qquad \text{and} \qquad \mathbb{A}^1 \dashrightarrow \mathbb{A}^1 \quad t \mapsto \frac{1}{t-1}.$$

  The image of the first map does not intersect the domain of the second map: The second map is defined on $\{t \neq 1\}$ and it clearly cannot be extended to all of $\mathbb{A}^1$.

  However one can compose dominant rational maps as explained below.

Suppose $(U, \varphi) : X \dashrightarrow Y$ and $(V, \psi) : Y \dashrightarrow Z$ are two dominant rational maps. Then define their composite by

$$(\varphi^{-1}(V), \psi \circ \varphi) : X \dashrightarrow Z.$$

Note that $\varphi^{-1}(V)$ is nonempty since (U, φ) is dominant, and hence is a dense open set in X as required.

This defines the category of irreducible varieties and dominant rational maps. We denote it by VarietyRational. An isomorphism in this category is called a *birational map*. Two irreducible varieties are called *birationally equivalent* if they are isomorphic in this category. Birational equivalence is of course weaker than isomorphism in Variety but stronger than other invariants such as dimension. So it is helpful in the classification of varieties.

Example 4.32. Let X be an irreducible variety. Then it follows from Proposition 4.12 that there is a bijection between elements of $(\mathcal{O}_X)_{gen}$, the generic stalk on X and rational maps from X to $\mathbb{A}^1$. More generally, $f_1, \ldots, f_m \in (\mathcal{O}_X)_{gen}$ defines a rational map

$$\varphi : X \dashrightarrow \mathbb{A}^m \qquad p \mapsto (f_1(p), \ldots, f_m(p)).$$

Let $\mathrm{dom}(f_i)$ denote the domain of f_i: it is the largest open set on which f_i is defined. Then $\mathrm{dom}(\varphi) = \bigcap_{i=1}^m \mathrm{dom}(f_i)$.

Example 4.33. In projective space, projecting from a point p onto a hyperplane H is a rational map. It is defined everywhere except at p. In other words, its domain is the complement of p. Without loss of generality, let us choose

$$p = [0 : \cdots : 0 : 1] \in \mathbb{P}^n \qquad \text{and} \qquad H = \{[x_0 : \cdots : x_{n-1} : 0]\} \subseteq \mathbb{P}^n.$$

Identifying H with $\mathbb{P}^{n-1}$ in the canonical manner, the projection is given by

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1} \qquad [x_0 : \cdots : x_{n-1} : x_n] \mapsto [x_0 : \cdots : x_{n-1}].$$

Generalizing this further, we may consider

$$\mathbb{P}^n \dashrightarrow \mathbb{P}^j \qquad [x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_j].$$

This is the projection from a $(n-j-1)$-dimensional plane onto its complementary j-dimensional plane.

4.4.2. The category of finitely generated field extension of $\mathbb{C}$. Suppose that A is an integral domain as well as a finitely generated $\mathbb{C}$-algebra. Then there is the field of fractions of A. Any field arising in this manner is called a finitely generated field extension of $\mathbb{C}$. Let $\mathsf{fgField}$ be the category whose objects are such fields, and whose morphisms are field homomorphisms. Recall that any field homomorphism, say $F \to K$, is necessarily injective. This allows us to think of K as a field extension of F.

4.4.3. Equivalence between irreducible affine varieties and their generic stalks. Recall that associated to each irreducible variety X is its generic stalk $(\mathcal{O}_X)_{gen}$. It is canonically equal to the generic stalk of any dense open set in X, and hence to the generic stalk of an affine variety. It follows that $(\mathcal{O}_X)_{gen}$ is a finitely generated field extension of $\mathbb{C}$. Further given a dominant rational map $\varphi : X \dashrightarrow Y$, there is an induced field homormorphism $(\mathcal{O}_Y)_{gen} \to (\mathcal{O}_X)_{gen}$: Suppose (V, f) represents an element in the generic stalk of Y, and (U, φ_U) represents φ, then $(U \cap \varphi_U^{-1}(V), f \circ \varphi_U)$ represents an element in the generic stalk of X. This yields a contravariant functor

$$\mathsf{VarietyRational} \to \mathsf{fgField} \qquad X \mapsto (\mathcal{O}_X)_{gen}. \tag{4.1}$$

Let us now construct a functor

$$\mathcal{G} : \mathsf{fgField} \to \mathsf{VarietyRational}. \tag{4.2}$$

Suppose K is an object in $\mathsf{fgField}$. Let R be a finitely generated $\mathbb{C}$-algebra whose field of fractions is K. Write R as $\mathbb{C}[x_1, \ldots, x_n]/I$ where I is a prime ideal. We will write $\overline{x}_1, \ldots, \overline{x}_n$ for the generators of R. Put $\mathcal{G}(K) := \mathbb{V}(I)$. Now suppose $\theta : K \to L$ is a field homomorphism (and hence necessarily injective. Define a rational map

$$\mathcal{G}(L) \dashrightarrow \mathcal{G}(K) \qquad y \mapsto (\theta(\overline{x}_1)(y), \ldots, \theta(\overline{x}_n)(y))$$

as in Example 4.32. One can deduce using Proposition 2.21 that this map is dominant.

Theorem 4.34. *The functors (4.1) and (4.2) define a contravariant equivalence between the categories* $\mathsf{VarietyRational}$ *and* $\mathsf{fgField}$.

Details in Hartshorne.

4.4.4. Problems.

(1) Find a rational parametrization for the circle. [Hint: $\sin(t)$ and $\cos(t)$ are rational functions of $\tan(t/2)$.]
(2) Must every dominant morphism $\mathbb{P}^1 \to \mathbb{P}^1$ be an isomorphism?
(3) Show that the image of a rational normal curve $C \subseteq \mathbb{P}^n$ under projection from a point $p \in C$ is a rational normal curve in $\mathbb{P}^{n-1}$.
(4) Show that $\mathbb{A}^1$ and $\mathbb{V}(xy-1) \subseteq \mathbb{A}^2$ are birationally equivalent but not isomorphic as affine varieties.
(5) Show that $\mathbb{P}^2$ and the categorical product $\mathbb{P}^1 \times \mathbb{P}^1$ are birationally equivalent but not isomorphic as projective varieties.
(6) Show that the graph of a morphism $\varphi : X \to Y$ of projective varieties is closed in the categorical product $X \times Y$.
(7) Find the equation defining the graph of the rational map
$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1 \qquad (x, y) \mapsto \frac{y}{x}$$
as an affine variety in $\mathbb{A}^3$.
(8) Prove or disprove: Let A and B be two finitely generated commutative $\mathbb{C}$-algebras which are integral domains. Suppose the field of fractions of A and B are isomorphic. Then A and B are isomorphic as $\mathbb{C}$-algebras.

CHAPTER 5

Affine schemes

5.1. Spectrum of a ring

In this section, we discuss the maximal spectrum and the prime spectrum. These are topological spaces associated to any ring. The notion of a prime spectrum gets us closer to the notion of an affine scheme.

5.1.1. Maximal spectrum. We now generalize the discussion in Section 2.2.1 by replacing $\mathbb{C}[x_1, \ldots, x_n]$ by any ring R. The question is: what should now play the role of affine n-space? Proposition 2.11 suggests a solution: use the set of maximal ideals of R. Let us see what one gets by proceeding along these lines.

Let R be a ring and let Q be the poset of ideals of R ordered by inclusion. Let $\operatorname{maxSpec}(R)$ denote the set of maximal ideals of R (called the maximal spectrum of R), and let P be its Boolean poset.

We define two order-reversing maps:

$$\mathbb{I} : P \to Q \qquad \text{and} \qquad \mathbb{V} : Q \to P.$$

$$\mathbb{V}(I) := \{M \mid M \text{ is a maximal ideal containing } I\}.$$

$$\mathbb{I}(X) := \bigcap_{M : M \in X} M.$$

Observe that

$$X \subseteq \mathbb{V}(I) \iff I \subseteq \mathbb{I}(X).$$

Both sides say: Every maximal ideal occurring in X contains I. Thus the closure operators $\mathbb{V}$ and $\mathbb{I}$ form a Galois connection. It follows from Proposition B.9 that

- $\mathbb{VI}$ and $\mathbb{IV}$ are closure operators on P and Q respectively.
- The closed sets of $\mathbb{VI}$ are in bijection with the closed sets of $\mathbb{IV}$.

Further, equations (2.2) and (2.3) hold. This implies that the closed sets of $\mathbb{VI}$ define a topology on $\operatorname{maxSpec}(R)$, or equivalently, $\mathbb{VI}$ is a topological closure operator. We call this the Zariski topology on $\operatorname{maxSpec}(R)$.

To summarize, to each ring R, we have associated a topological space $\operatorname{maxSpec}(R)$. Now suppose $R \to S$ is a ring homomorphism. Is it true that there is an induced continuous map $\operatorname{maxSpec}(S) \to \operatorname{maxSpec}(R)$? The answer is given in the discussion below on the prime spectrum.

Remark 5.1. If R is a finitely generated reduced $\mathbb{C}$-algebra, then the closed sets of $\mathbb{IV}$ will precisely be the radical ideals of R. This is the Nullstellensatz. It is natural to ask what happens for an arbitrary ring. What is the class of ideals one obtains by arbitrary intersections of maximal ideals? Intersection of all the ideals is the Jacobson radical of the ring. So an ideal will be an intersection of some maximal

ideals if it is the inverse image of the Jacobson radical of R/I under $R \twoheadrightarrow R/I$ for some ideal I. I do not know of a more direct description.

5.1.2. Prime spectrum. Let R be a ring. Let $\mathrm{Spec}(R)$ denote the set of prime ideals of R. The above discussion is also valid if one replaces $\mathrm{maxSpec}(R)$ by $\mathrm{Spec}(R)$ and maximal by prime:

$$\mathbb{V}(I) := \{P \mid P \text{ is a prime ideal containing } I\}.$$

$$\mathbb{I}(X) := \bigcap_{P : P \in X} P.$$

This yields a Zariski topology on $\mathrm{Spec}(R)$: a subset of prime ideals of R is a closed set if it is of the form $\mathbb{V}(I)$ for some ideal I. Note that the closure of a point $\{P\}$ is $\mathbb{V}(P)$, the set of all prime ideals which contain P.

Proposition 5.2. *The closed sets of $\mathbb{IV}$ are precisely the radical ideals of R. More precisely,*

$$\mathbb{IV}(I) = \mathrm{rad}\,(()I).$$

Proof. An ideal is radical iff it is the intersection of all the prime ideals containing it (Proposition 1.9). □

Remark 5.3. Suppose R is a finitely generated reduced $\mathbb{C}$-algebra. Then there is a bijection between the closed sets in $\mathrm{maxSpec}(R)$ (the affine subvarieties) and the closed sets in $\mathrm{Spec}(R)$ via their bijection with the radical ideals of R. This idea will be formalized later when we relate varieties and schemes.

Proposition 5.4. *Suppose I is an ideal in R. Then*

$$\mathbb{V}(I) \text{ is an irreducible space} \iff \mathrm{rad}\,(I) \text{ is a prime ideal.}$$

You may also see [19, page 67].

Proof. First note that $\mathbb{V}(I) = \mathbb{V}(\mathrm{rad}\,(I))$. So we may assume that I is a radical ideal.

Backward implication: Suppose I is a prime ideal. Then I is an element of $\mathbb{V}(I)$. Hence any closed set which contains I as an element contains all elements of $\mathbb{V}(I)$. This shows that $\mathbb{V}(I)$ is irreducible.

Forward implication: Suppose I is not a prime ideal. Then there exist f and g such that $fg \in I$ but neither f nor g belongs to I. Consider the ideals (I, f) and (I, g). We claim that

$$\mathbb{V}(I) = \mathbb{V}((I, f)) \cup \mathbb{V}((I, g)).$$

(If a prime ideal P contains I, then it contains fg, and hence it contains either f or g, and hence it contains either $((I, f))$ or $((I, g))$.) Further $\mathbb{V}((I, f))$ and $\mathbb{V}((I, g))$ are proper subsets of $\mathbb{V}(I)$ because I is a radical ideal. This shows that $\mathbb{V}(I)$ is not irreducible. □

Thus all closed irreducible subspaces of $\mathrm{Spec}(R)$ are of the form $\mathbb{V}(P)$, where P is a prime ideal of R. As a consequence:

Corollary 5.5. *Let R be any ring. The irreducible components of* $\mathrm{Spec}(R)$ *are the closed sets $\mathbb{V}(P)$, where P is a minimal prime ideal of R.*

Corollary 5.6. *Let R be any ring. Then the following are equivalent.*

- $\mathrm{Spec}(R)$ *is irreducible.*

- *R has a unique minimal prime ideal.*
- *the nilradical of R is a prime ideal.*

If $\mathrm{Spec}(R)$ is irreducible, then $\mathrm{Spec}(R) = \mathbb{V}(P)$ for some prime ideal P, and this is the unique minimal prime ideal of R.

Definition 5.7. Suppose Z is an irreducible closed subset of $\mathrm{Spec}(R)$. Then a point $P \in Z$ is a *generic point* of Z if $Z = \mathbb{V}(P)$ is the closure of $\{P\}$.

The following is a restatement of the above results in the language of generic points.

Proposition 5.8. *If $P \in \mathrm{Spec}(R)$, then the closure of $\{P\}$ is irreducible and P is a generic point of this set. Conversely, every irreducible closed set Z of $\mathrm{Spec}(R)$ equals $\mathbb{V}(P)$ for some prime ideal P of R, and P is its unique generic point.*

5.1.3. Basic open sets in the prime spectrum. Let R be a ring and $X = \mathrm{Spec}(R)$. Let $f \in R$. Note that $\mathbb{V}(f)$ consists of precisely those prime ideals which contain f. Let $X_f := \mathrm{Spec}(R) \setminus \mathbb{V}(f)$ consist of those prime ideals which do not contain f. They are called the *basic open sets* of $\mathrm{Spec}(R)$, since they form as a basis for the Zariski topology:

Proposition 5.9. *The sets X_f as f varies over R form a basis for the topology of $\mathrm{Spec}(R)$.*

Proof. We need to show that any open set in $\mathrm{Spec}(R)$ can be written as a union of basic open sets. For any ideal I of R,

$$\mathbb{V}(I) = \bigcap_{f \in I} \mathbb{V}((f)).$$

Now take complements. □

Proposition 5.10. (1) *$X_f = \emptyset$ iff f is nilpotent.*
(2) *$X_f = X$ iff f is a unit.*
(3) *$X_f \cap X_g = X_{fg}$.*
(4) *$X_g \subseteq X_f$ iff $\mathrm{rad}\,((g)) \subseteq \mathrm{rad}\,((f))$ iff Some power of g is a multiple of f.*
(5) *$X_f = X_g$ iff $\mathrm{rad}\,((f)) = \mathrm{rad}\,((g))$.*

It is instructive to see what these statement say for $R = \mathbb{Z}$.

Proof. For (1). Both conditions are equivalent to: Every prime ideal contains f.

For (2). Both conditions are equivalent to: No prime ideal contains f, or equivalently, no maximal ideal contains f.

For (3). Note that for any prime ideal P, $fg \in P \iff$ Either $f \in P$ or $g \in P$. This shows that $\mathbb{V}((f)) \cup \mathbb{V}((g)) = \mathbb{V}((fg))$. Now take complements.

For (4). This follows from two facts. First: The radical of any ideal I is the intersection of all prime ideals containing I. Second: For any prime ideal P,

$$f \in P \iff \mathrm{rad}\,((f)) \subseteq P.$$

(This is because prime ideals are radical and $\mathrm{rad}\,(-)$ is a closure operator.)

For (5). This follows from (4). □

Observe that

$$X = \bigcup_i X_{f_i} \iff \{f_i\} \text{ generate the unit ideal } R. \tag{5.1}$$

Both statements say that there is no prime ideal which contains all the f_j's. More generally,

$$X_f = \bigcup_i X_{f_i} \iff \operatorname{rad}((f)) = \operatorname{rad}((f_i)_i). \tag{5.2}$$

Both statements say that: a prime ideal contains f iff it contains all the f_j's.

Proposition 5.11. *For a ring R, $\operatorname{Spec}(R)$ is quasi-compact.*

PROOF. Suppose we are given an open cover of $\operatorname{Spec}(R)$. We may assume without loss of generality that the covering is by basic open sets X_{f_j} with j ranging over some index set $\mathcal{J}$. Then (5.1) implies that there is a relation of the form

$$1 = \sum_{i \in \mathcal{I}} g_i f_i, \qquad g_i \in R$$

where $\mathcal{I}$ is some finite subset of $\mathcal{J}$. Then the f_i's as i varies over $\mathcal{I}$ generate the unit ideal R and the X_{f_i}'s provide a finite subcover of $\operatorname{Spec}(R)$. □

More generally:

Proposition 5.12. *For a ring R, any basic open set in $\operatorname{Spec}(R)$ is quasi-compact.*

PROOF. Let X_f be a basic open set. Suppose without loss of generality as before that X_f is covered by basic open sets X_{f_i} with i ranging over some index set $\mathcal{I}$. Then (5.2) implies that there is a relation of the form

$$f^k = \sum_{j \in \mathcal{J}} g_j f_j \qquad g_j \in R$$

for some k and for some finite subset $\mathcal{J}$ of $\mathcal{I}$. Then

$$\operatorname{rad}((f)) = \operatorname{rad}((f_j)_{j \in \mathcal{J}})$$

and the X_{f_j}'s provide a finite subcover of X_f.

Alternatively, this result can be deduced from Proposition 5.11 using the fact that X_f is homeomorphic to $\operatorname{Spec}(R_f)$ (as discussed in Proposition 5.16 below). □

Proposition 5.13. *Let U be an open set of $\operatorname{Spec}(R)$. Then U is quasi-compact iff U is a finite union of basic open sets.*

PROOF. Since basic open sets are quasi-compact, any finite union of basic open sets is also quasi-compact. Conversely, suppose U is an open set of $\operatorname{Spec}(R)$ which is quasi-compact. Since basic open sets form a base for the topology, write U as a union of basic open sets. Since U is quasi-compact, this has a finite subcover, showing that U is a finite union of basic open sets. □

5.1.4. The prime spectrum as a functor. Suppose $\varphi : R \to S$ is a ring homomorphism. Then there is an induced map $\operatorname{Spec}(\varphi) : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ which sends a prime ideal P of S to the prime ideal $\varphi^{-1}(P)$ in R.

Proposition 5.14. *Let $X = \operatorname{Spec}(R)$ and $Y = \operatorname{Spec}(S)$. Let $\varphi : R \to S$ be a ring homomorphism. If $f \in R$, then $\operatorname{Spec}(\varphi)^{-1}(X_f) = Y_{\varphi(f)}$.*

PROOF. $\operatorname{Spec}(\varphi)^{-1}(X_f)$ consists of all prime ideals P in S such that $\varphi^{-1}(P)$ does not contain f, or equivalently, such that P does not contain $\varphi(f)$. □

This shows that $\operatorname{Spec}(\varphi)$ is continuous. This yields a contravariant functor

$$\operatorname{Spec} : \mathsf{Ring} \to \mathsf{Top}.$$

This statement requires us to check that

$$\operatorname{Spec}(\mathrm{id}) = \mathrm{id} \quad \text{and} \quad \operatorname{Spec}(\psi\varphi) = \operatorname{Spec}(\varphi)\operatorname{Spec}(\psi).$$

This is straightforward.

Remark 5.15. One cannot get such a functor from the maximum spectrum because the inverse image of a maximal ideal may not be maximal.

Proposition 5.16. *Let $Y := \operatorname{Spec}(R_f)$ and $X := \operatorname{Spec}(R)$. The spaces Y and X_f are canonically homeomorphic; the basic open sets in Y correspond to the basic open sets in X which are contained in X_f.*

PROOF. The ring homomorphism $\varphi : R \to R_f$ induces a continuous map

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(R_f) \to \operatorname{Spec}(R).$$

We know from Proposition 1.30 that this map takes the primes of R_f to the primes of R which do not contain f. In other words, there is a continuous bijection

$$\operatorname{Spec}(\varphi) : Y \to X_f. \tag{5.3}$$

In fact, this is a homeomorphism, and moreover the image (forward or inverse) of a basic open set is again a basic open set: A prime P in X does not contain fg iff the extension of P in Y does not contain $g/1$ (or $fg/1$). □

Proposition 5.17. *If $\varphi : R \to S$ is surjective, then $\operatorname{Spec}(\varphi)$ is a homeomorphism of $\operatorname{Spec}(S)$ onto the closed subset $\mathbb{V}(\ker(\varphi))$ of $\operatorname{Spec}(R)$.*

Alternatively, if I is an ideal in R, then the canonical projection $R \twoheadrightarrow R/I$ induces a homeomorphism of $\operatorname{Spec}(R/I)$ onto the closed subset $\mathbb{V}(I)$ of $\operatorname{Spec}(R)$.

PROOF. It follows from Proposition 1.5 that there is a bijection

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(S) \to \mathbb{V}(\ker(\varphi)).$$

If one puts the inclusion order on prime ideals, then it is clear that this map is a poset isomorphism. Since the closed sets in the topology of the prime spectrum are upper sets in the poset of prime ideals, it follows that the above map is a homeomorphism. □

Proposition 5.18. *If $\varphi : R \to S$ is injective, then $\operatorname{Spec}(\varphi)(\operatorname{Spec}(S))$ is dense in $\operatorname{Spec}(R)$. More precisely, $\operatorname{Spec}(\varphi)(\operatorname{Spec}(S))$ is dense in $\operatorname{Spec}(R)$ iff $\ker(\varphi)$ is contained in the nilradical of R.*

PROOF. Let $Y = \mathrm{Spec}(\varphi)(\mathrm{Spec}(S))$. Recall that the nilradical consists of all nilpotent elements. Observe that:

Y is dense in $\mathrm{Spec}(R)$ iff Y meets every nonempty basic open set iff Y meets X_f whenever f is not nilpotent.

Forward implication. Suppose f is not nilpotent. Then Y meets X_f. Thus, there is a prime Q of S such that the prime $\varphi^{-1}(Q)$ does not contain f. So Q does not contain $\varphi(f)$. This implies that $\varphi(f) \neq 0$.

Backward implication. Suppose f is not nilpotent. Then $\varphi(f)$ is also not nilpotent. (Suppose $\varphi(f)^n = 0$. Then $f^n \ker(\varphi)$. So f^n and hence f is nilpotent.) So there is a prime Q of S which does not contain $\varphi(f)$. Hence the prime $\varphi^{-1}(Q) \in Y$ does not contain f. Thus Y meets X_f as required. □

5.1.5. Boolean rings. A ring R is *Boolean* if $x^2 = x$ for all $x \in R$.

Proposition 5.19. *Let R be a Boolean ring. Then*

(1) *$2x = 0$ for all $x \in R$.*
(2) *Every prime ideal P is maximal, and R/P is a field with two elements.*
(3) *Every finitely generated ideal in R is principal.*

PROOF. Exercise. □

Let L be a Boolean lattice. Define addition and multiplication in L by

$$a + b := (a \wedge \overline{b}) \vee (\overline{a} \wedge b) \qquad \text{and} \qquad ab = a \wedge b.$$

This turns L into a Boolean ring. Conversely, starting from a Boolean ring R, define a partial order on R: $a \leq b$ if $a = ab$. This turns R into a Boolean lattice.

These constructions are inverse to each other. So there is a correspondence between isomorphism classes of Boolean rings and Boolean lattices.

Proposition 5.20. *Let R be a Boolean ring and let $X = \mathrm{Spec}(R)$. Then*

(1) *For each $f \in R$, the set X_f is both open and closed in X.*
(2) *Given $f_1, \ldots, f_n \in R$, there exists $f \in R$ such that $X_{f_1} \cup \cdots \cup X_{f_n} = X_f$.*
(3) *The sets X_f are the only subsets of X which are both open and closed.*

PROOF. Exercise. □

As a consequence:

Theorem 5.21. *Every Boolean lattice is isomorphic to the lattice of open-and-closed subsets of some compact Hausdorff topological space.*

This is known as Stone's theorem.

5.1.6. Problems.

(1) What are the irreducible components of a Hausdorff space?
(2) Let R be a ring. Show that the following are equivalent.
   (a) Every prime ideal in R is maximal.
   (b) $\mathrm{Spec}(R)$ is a T_1-space, that is, points are closed.
   (c) $\mathrm{Spec}(R)$ is Hausdorff.

   (b) $\implies$ (c). This requires some argument. Incomplete.

   If these conditions are satisfied, then show that $\mathrm{Spec}(R)$ is quasi-compact and totally disconnected, that is, the only connected subsets are singletons.

Spec(R) is quasi-compact for any ring R. For totally disconnected, we need to show that every point is also an open set. Incomplete.

(3) Let $\varphi : R \to S$ be a ring homomorphism, and $\mathrm{Spec}(\varphi) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be the induced map. Show that:
   (a) Every prime ideal of R is a contracted ideal iff $\mathrm{Spec}(\varphi)$ is surjective.
   (b) If every prime ideal of S is an extended ideal, then $\mathrm{Spec}(\varphi)$ is injective.

   Is the converse of (b) true?

(4) Let R be a ring and P be a prime ideal of R. The map $R \to R_P$ induces a map $\mathrm{Spec}(R_P) \to \mathrm{Spec}(R)$. Show that the image of this map is the intersection of all the open neighborhoods of P in $\mathrm{Spec}(R)$.

(5) If R is a Noetherian ring, then $\mathrm{Spec}(R)$ is a Noetherian space.
   Is the converse true?

(6) Let R be a ring such that $\mathrm{Spec}(R)$ is a Noetherian space. Show that the set of prime ideals of R satisfies the ascending chain condition.
   Is the converse true?

(7) Let $R = \prod_{i=1}^{n} R_i$ be the direct product of the rings R_i. Show that $\mathrm{Spec}(R)$ is the disjoint union of open (and closed) subsets X_i where X_i is canonically homeomorphic with $\mathrm{Spec}(R_i)$.
   Let R be any ring. Show that the following are equivalent.
   (a) $R \cong R_1 \times R_2$ where neither of the rings R_1, R_2 is the zero ring.
   (b) R contains an idempotent which is not 0 or 1.
   (c) $X = \mathrm{Spec}(R)$ is disconnected.

   In particular, the spectrum of a local ring is always connected (since 0 and 1 are the only idempotents in it).

(8) Let A be any set with the discrete topology. When is A equal to $\mathrm{Spec}(R)$ for some ring R.

(9) For what rings R is $\mathrm{maxSpec}(R)$ dense in $\mathrm{Spec}(R)$?

(10) Give an example of a ring with infinitely many minimal primes.

(11) Let $X = \mathrm{Spec}\, R$ for some ring R, and let U be an open subset of X. Suppose that U contains $x_1, \ldots, x_n$. Show that there exists $f \in R$ satisfying

$$\{x_1, \ldots, x_n\} \subseteq X_f \subseteq U.$$

(12) Let $R := \mathbb{C}[x, y]_{(x,y)}$, the localization of $\mathbb{C}[x, y]$ at the maximal ideal (x, y).
   (a) Is the ring R finitely generated as a $\mathbb{C}$-algebra?
   (b) Is the ring R Noetherian?
   (c) Is $\mathrm{Spec}(R)$ irreducible as a topological space?

(13) Let $\mathbb{Z}_n$ denote the ring of integers modulo n. Determine the integers $n \geq 2$ for which $\mathrm{Spec}(\mathbb{Z}_n)$ is Hausdorff.
   If R is a finite ring, then show that $\mathrm{Spec}(R)$ is Hausdorff.

(14) Let $\Bbbk$ be any field. Consider the $\Bbbk$-algebra

$$R := \Bbbk[x, y]/(x^2 = x, xy = y = y^2).$$

   (a) Find a $\Bbbk$-basis for R.
   (b) Show that $\mathrm{Spec}(R)$ has finitely many elements. Say how many. Is $\mathrm{Spec}(R)$ connected?

5.2. The category of affine schemes

In this section we begin by defining an affine scheme. The main step is to construct a sheaf of rings on the prime spectrum of a ring. A sheaf of rings is also called a *ringed space* but we do not use that terminology.

Next we define morphisms between affine schemes, the key being the notion of a local ring homomorphism. This yields the category of affine schemes. We then show that this category is contravariantly equivalent to the category of rings. This should be viewed as an extension of the contravariant equivalence between the categories of affine varieties and finitely generated reduced commutative $\mathbb{C}$-algebras.

You may also want to look at [14, Propositions 2.2 and 2.3].

5.2.1. The structure sheaf of a ring. For any $f \in R$, let R_f denote the ring of fractions of R wrt the multiplicative set $\{f^n\}_{n\geq 0}$.

Suppose $f, g \in R$ such that $X_g \subseteq X_f$. Then by Proposition 5.10, part (4), $g^k = rf$ for some $r \in R$ and some power k. Define

$$\rho_{fg} : R_f \to R_g \qquad \frac{x}{f^n} \mapsto \frac{xr^n}{g^{kn}}.$$

One can check that this is a well-defined ring homomorphism (independent of the particular choice of r and k). A special case worth pointing out is:

$$R_f \to R_{fg} \qquad \frac{x}{f^n} \mapsto \frac{xg^n}{f^n g^n}.$$

One may check that

$$\rho_{ff} = \mathrm{id} \quad \text{and} \quad \rho_{fh} = \rho_{gh}\rho_{fg}, \tag{5.4}$$

for any $f, g, h \in R$ such that $X_h \subseteq X_g \subseteq X_f$.

We first construct a $\mathcal{B}$-presheaf of rings on $X := \mathrm{Spec}(R)$. Let $\mathcal{B}$ be the set of all basic open sets in X. By general principles, it suffices to construct a $\mathcal{B}$-sheaf on X. This is done as follows.

For each basic open set U, pick a $f \in R$ such that $U = X_f$. For $U = X$, the canonical choice is 1 (though any unit will work), and for $U = \emptyset$, the canonical choice is 0 (though any nilpotent will work). Put

$$\mathcal{O}_X(X_f) := R_f.$$

Note that $\mathcal{O}_X(X) = R$, and $\mathcal{O}_X(\emptyset) = 0$. In view of (5.4), this yields a $\mathcal{B}$-presheaf on X.

Remark 5.22. The identities (5.4) show that the particular choice of f for a basic open set is not crucial: If $X_f = X_g$, then the maps ρ_{fg} and ρ_{gf} are inverses of each other and hence R_f and R_g are canonically isomorphic as rings. To avoid notational inconvenience, it is customary to treat this isomorphism as an identity.

Remark 5.23. The $\mathcal{B}$-presheaves associated to a ring R and to its ring of fractions R_f are closely related. The map (5.3) induces an isomorphism of $\mathcal{B}$-presheaves, that is, the $\mathcal{B}$-presheaf on Y is isomorphic to the restriction of the $\mathcal{B}$-presheaf on X to X_f.

Proposition 5.24. *Suppose $\{f_i\}$ generates the unit ideal R. Then the set $\{f_i^{k_i}\}$ also generates the unit ideal R, for any choice of powers k_i.*

PROOF. The set $\{f_i\}$ generates the unit ideal R iff No prime contains all the f_i's. Now note that a prime contains f iff it contains f^n for some n. The result follows. □

Proposition 5.25. *Suppose $\{f_i\}$ is a finite set which generates the unit ideal R, or equivalently, suppose $\{X_{f_i}\}$ is a finite cover of $X := \operatorname{Spec}(R)$. Then:*

1. *If $g, h \in R$ become equal in each R_{f_i}, then $g = h$.*
2. *If for each i, there is a $g_i \in R_{f_i}$ such that for each pair (i, j), the images of g_i and g_j in $R_{f_i f_j}$ are equal, then there is an element $g \in R$ whose image in R_{f_i} is g_i for all i.*

What do these statements say for $R = \mathbb{Z}$? To appreciate the proof, we should look at a ring which is not an integral domain.

PROOF. For (1). Suppose $g, h \in R$ become equal in R_{f_i}. Then $f_i^{k_i}(g - h) = 0$ for some power k_i. Note that $X_{f_i} = X_{f_i^{k_i}}$. So the $f_i^{k_i}$'s generate the unit ideal. It follows that $g - h = 0$.

For (2). Since there are only finitely many f_i's, there exists a N large enough and $x_i \in R$ such that g_i is represented by x_i/f_i^N and

$$x_i f_j^N = x_j f_i^N$$

for all i and j. Since the f_i^N generate the unit ideal, we may write

$$1 = \sum_i r_i f_i^N$$

with $r_i \in R$. We claim that

$$g := \sum_i r_i x_i$$

is the required element. This follows from the following steps. For each j,

$$f_j^N g = \sum_i r_i f_j^N x_i = \sum_i r_i f_i^N x_j = x_j.$$

So the image of g in R_{f_j} is g_j as required. □

A generalization of this result is stated below.

Proposition 5.26. *Suppose $\operatorname{rad}((f_i)_i) = \operatorname{rad}((f))$ for some $f_i, f \in R$. In other words, the radical ideal generated by the f_i's (there could be infinitely many of them) equals the radical of f. Equivalently: Let $X := \operatorname{Spec}(R)$ and suppose that X_f has a covering $\{X_{f_i}\}$ for some choice of f_i's. (There could be infinitely many of them.) Then:*

1. *If $g, h \in R_f$ become equal in each R_{f_i}, then $g = h$.*
2. *If for each i, there is a $g_i \in R_{f_i}$ such that for each pair (i, j), the images of g_i and g_j in $R_{f_i f_j}$ are equal, then there is an element $g \in R_f$ whose image in R_{f_i} is g_i for all i.*

PROOF. The case when the cover is finite and $f = 1$, $R_f = R$ and $X_f = X$ is Proposition 5.25. The general case with the cover still finite follows by applying Remark 5.23 to the first case.

Now suppose the cover is infinite. Part (1) presents no problem. So let us look at part (2). By Proposition 5.12, X_f is quasi-compact. So pick a finite subcover $\{X_{f_j}\}$ and apply the finite case to get a $g \in R$ whose image in R_{f_j} is g_j. Let X_{f_i}

be any basic open set in the given cover. Then $\{X_{f_i} \cap X_{f_j}\}$, as j varies, is a finite cover of X_{f_i}. Let g_i' be the image of g in R_{f_i}. The images of g_i' and g_i are equal in $R_{f_i f_j}$ for all j. So by part (1) of the finite case, $g_i' = g_i$. □

This result shows that $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf on X. By general principles, this extends uniquely to a sheaf on X. This is called the *structure sheaf* of X, or the sheaf of regular functions on X.

Proposition 5.27. *Let R be a ring and let $X := \mathrm{Spec}(R)$, and let $f \in R$. Then $(X_f, \mathcal{O}_X|_{X_f})$ is isomorphic to $\mathrm{Spec}(R_f)$ as sheaves of rings.*

Definition 5.28. An *affine scheme* is a sheaf of rings $(X, \mathcal{O}_X)$ associated to some ring R as explained above.

Note that R can be recovered from the affine scheme as its ring of global functions.

Proposition 5.29. *Let X be the affine scheme associated to a ring R. Then the stalk of its structure sheaf at a point P is the localization of R at the prime P:*

$$\mathcal{O}_{X,P} = R_P.$$

In particular, each stalk is a local ring.

Proof. To compute the stalk, we can take colimit over all basic open sets X_f which contain P. Note that for any f which does not belong to P (or equivalently, for which X_f contains P), there is a ring homomorphism $R_f \to R_P$, and these homomorphisms commute with the restriction maps ρ_{fg}. This shows that R_P is a cone over the R_f's. Its universality follows from the following two observations.

- Every element of R_P is the image of some $R_f \to R_P$.
- If $x \in R_f$ and $y \in R_g$ become equal in R_P, then $h(x - y) = 0$ for some h not in P, and so x and y become equal in R_{fgh}.

□

5.2.2. Morphisms between affine schemes. Let $X := \mathrm{Spec}(R)$. The stalk $\mathcal{O}_{X,P} = R_P$ at any point P is a local ring. Let $\kappa(P)$ denote its residue field. Thus, we have a field associated to each point of an affine scheme.

Suppose U is an open set in X, Then there are ring homomorphisms

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,P} \to \kappa(P).$$

An element $g \in \mathcal{O}_X(U)$ can be viewed as a "function" on U, its value at P, denoted $g(P)$ being the image of g in $\kappa(P)$. This is not a function in the usual sense because the residue field changes as the point changes. We say that g vanishes at P if $g(P) = 0$, or equivalently, if the image of g lies in the unique maximal ideal of the stalk at P. Suppose $U := X_f$ is a basic open set. Then g vanishes at $P \in X_f$ iff whenever one writes $g = x/f^n$ then $x \in P$.

Definition 5.30. A morphism $X \to Y$ between affine schemes is a pair $(\varphi, \varphi^\#)$ where $\varphi : X \to Y$ is a continuous map on the underlying topological spaces,

$$\varphi^\# : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$$

is a morphism of sheaves on Y, and whenever $\varphi(P) = Q$, the induced map

$$\mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$$

is a local ring homomorphism. This defines the category of affine schemes which we denote by AffScheme.

We refer to the condition on the stalks as the local condition. It may be rephrased in more geometric terms as follows.

Given an open set V in Y, a section $g \in \mathcal{O}_Y(V)$, a point $Q \in V$, and a point $P \in \varphi^{-1}(V)$ such that $\varphi(P) = Q$,

- g vanishes at the point Q iff $\varphi^\#(g)$ vanishes at the point P.

Note that this condition holds if it holds for each basic open set V.

If the local condition holds, then one can even say that the values $g(Q)$ and $\varphi^\#(g)(P)$ are equal, since one residue field is an extension of the other. However, without the local condition, one cannot make any such statement.

5.2.3. Equivalence with the category of rings. Suppose $\alpha : S \to R$ is a ring homomorphism. Set $Y := \operatorname{Spec}(S)$, $X := \operatorname{Spec}(R)$ and $\varphi := \operatorname{Spec}(\alpha)$. Then we have seen that $\varphi : X \to Y$ is a continuous map, and $\varphi^{-1}(Y_f) = X_{\alpha(f)}$. Note that α induces a ring homomorphism

$$S_f \to R_{\alpha(f)}.$$

This follows from Proposition 1.26. This defines a morphism $\varphi^\# : \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ of $\mathcal{B}$-sheaves and hence of sheaves. The discussion in Example 1.34 shows that the local condition holds. Hence $(\varphi, \varphi^\#)$ is a morphism of affine schemes. [The local condition may also be verified as follows: Suppose $\varphi(P) = Q$, or equivalently, $\alpha^{-1}(P) = Q$. Let $Q \in Y_f$ and $P \in X_{\alpha(f)}$. Suppose $g \in S_f$. Write $g = x/f^n$ and $\varphi^\#(g) = \alpha(x)/\alpha(f)^n$. Clearly $x \in Q$ iff $\alpha(x) \in P$.]

This yields a contravariant functor

$$\operatorname{Spec} : \mathsf{Ring} \to \mathsf{AffScheme} \tag{5.5}$$

In the other direction, suppose $(\varphi, \varphi^\#)$ is a morphism of affine schemes $X \to Y$. Put $R := \mathcal{O}_X(X)$ and $S := \mathcal{O}_Y(Y)$. Then $\varphi^\#$ induces a ring homomorphism $\alpha : S \to R$. This yields a contravariant functor

$$\mathcal{G} : \mathsf{AffScheme} \to \mathsf{Ring} \tag{5.6}$$

which we may call the functor of global sections.

Theorem 5.31. *The functors (5.5) and (5.6) define a contravariant equivalence between the category of rings and the category of affine schemes.*

Proof. Applying Spec followed by $\mathcal{G}$ to a ring homomorphism clearly gives the same ring homomorphism back.

Now let us start with a morphism of affine schemes $\varphi : X \to Y$. Applying $\mathcal{G}$ yields a ring homomorphism $\alpha : S \to R$, where $R = \mathcal{O}_X(X)$ and $S = \mathcal{O}_Y(Y)$. Any morphism of sheaves induces a ring homomorphism between corresponding stalks. This means that whenever $\varphi(P) = Q$, there is a map $\mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$. In other words, α induces a ring homomorphism $S_Q \to R_P$, or equivalently α maps the complement of Q into the complement of P. Now the local condition says that α maps Q into P. Combining we get $\alpha^{-1}(P) = Q$. This shows that applying Spec to α would recover φ. □

5.2.4. Affine varieties. Suppose R is a finitely generated reduced commutative $\mathbb{C}$-algebra. Then the entire process can be repeated with maxSpec instead of Spec. This allows a new approach to affine varieties. An affine variety is a sheaf of $\mathbb{C}$-algebras obtained as $\mathrm{maxSpec}(R)$. This construction makes no reference to any ambient space because we do not choose generators for R. Instead we directly construct the space of the affine variety as the set of maximal ideals of R.

Question 5.32. For what class of rings does the maxSpec construction work? What about $\mathbb{Z}$? There is no obvious reason to believe that finitely generated reduced commutative $\mathbb{C}$-algebras is the optimal class.

Something special that happens in this construction is that the residue field at any point is the base field $\mathbb{C}$. This allows us to think of sections over an open set U as $\mathbb{C}$-valued functions on U. And these points of view are equivalent, meaning that one can uniquely recover the section by knowing this $\mathbb{C}$-valued function.

It is natural to ask: if R is as above, then does one get anything more by considering the prime spectrum of R instead of its maximal spectrum. The answer is no. Details are given in a later section.

5.2.5. Problems.

(1) Describe the initial object, terminal object, product and coproduct in the category Ring. Do the same for the category AffScheme. Check that your results are consistent with Theorem 5.31.
(2) Problems 18 to 30 in [4, Chapter 3] contain useful information about affine schemes.

5.3. Examples

Example 5.33. Let $R := \Bbbk$ be any field. Then $X = \mathrm{Spec}(R)$ consists of one point. The only sheaf data is the ring of global sections which is $\Bbbk$.

If $R = \Bbbk^n$, then $\mathrm{Spec}(R)$ consists of n points with the discrete topology. The ring of sections over an open set U is $\Bbbk^i$, where i is the number of elements in U. In particular, the stalk at each point is $\Bbbk$.

Example 5.34. Let us consider the ring $\mathbb{Z}$. It is an initial object in the category of rings. Note that

$$\mathrm{maxSpec}(\mathbb{Z}) = \{(2), (3), (5), (7), \dots\}.$$

All points (singletons) are closed. More generally, the closed sets in $\mathrm{maxSpec}(\mathbb{Z})$ are finite subsets and the entire space.

The prime spectrum has an additional point:

$$\mathrm{Spec}(\mathbb{Z}) = \{(0), (2), (3), (5), (7), \dots\}.$$

The point (0) is not closed. In fact, note that its closure is the entire space; so it is dense. So we say that (0) is a generic point of $\mathrm{Spec}(\mathbb{Z})$. The closed sets in $\mathrm{Spec}(\mathbb{Z})$ are finite subsets not containing (0) and the entire space.

Every open set in $X := \mathrm{Spec}(\mathbb{Z})$ is basic and can be written as X_f where $f = p_1 \dots p_n$ is a product of distinct primes. The ring associated to X_f is R_f. It consists of all rationals whose denominators are powers of f. This is same as the set of rationals whose denominators are divisible by f.

The residue field at the prime (p) is $\mathbb{Z}/p\mathbb{Z}$, the field with p elements, and the residue field at the prime (0) is $\mathbb{Q}$. Now take for example, the regular function

$12 \in \mathbb{Z}$. Then its value at (5) is $2 \in \mathbb{Z}/5\mathbb{Z}$. In other words, to find the value of an integer at the prime (p), we reduce it mod p.

Many examples admit a similar analysis to the one given above for $\mathbb{Z}$. A possible general framework is: In any Dedekind domain, a prime ideal is either maximal or (0). So the prime spectrum is the set of closed points plus a generic point. The closed sets of this topology are finite subsets of maximal ideals (this includes the empty set) and the entire space.

We briefly sketch another example of this kind.

Example 5.35. For the ring $\mathbb{C}[x]$, the maximal spectrum

$$\operatorname{maxSpec}(\mathbb{C}[x]) = \{(x - \alpha) \mid \alpha \in \mathbb{C}\}.$$

The closed sets are all finite subsets and the entire space. Thus $\operatorname{maxSpec}(\mathbb{C}[x])$ is isomorphic to $\mathbb{A}^1$ (something we already know).

The prime spectrum

$$\operatorname{Spec}(\mathbb{C}[x]) = \{(x - \alpha) \mid \alpha \in \mathbb{C}\} \cup \{(0)\}.$$

The point (0) is a generic point. The closed sets are all finite subsets not containing (0) and the entire space.

Example 5.36. Let p be a prime number and let $R := \mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at the prime ideal (p). Observe that R is a local ring: $(p)R$ is the unique maximal ideal. It consists of fractions whose denominators are not divisible by p. There is only one other prime ideal in R, namely (0). So $X = \operatorname{Spec}(R)$ consists of two points. The open sets are

$$\emptyset \subset U := \{(0)\} \subset \{(0), (p)\} = X.$$

All of them are basic open sets, since $U = X_p$. The sheaf $\mathcal{O}_X$ is as follows: $\mathcal{O}_X(\emptyset) = 0$ (the zero ring), $\mathcal{O}_X(X) = R$ and $\mathcal{O}_X(U) = \mathbb{Q}$, the field of rationals. The restriction map from the first to the second is the canonical inclusion.

The same analysis works for any discrete valuation ring. For example, take $R := \Bbbk[x]_{(x)}$.

Example 5.37. For the ring $R = \mathbb{C}[x, y]$, the maximal spectrum is isomorphic to $\mathbb{A}^2$. The additional prime ideals in R are those generated by irreducible polynomials $f(x, y)$ and (0). So one can imagine the prime spectrum as the affine plane to which one adds a generic point (corresponding to (0)), and for each irreducible curve, one adds a point generic in that curve but not sticking out of it.

Example 5.38. Consider the ring $R = \mathbb{C}[x]/(x^2)$. As a set, this can expressed as

$$\{a + bx \mid a, b \in \mathbb{C}\}.$$

Addition is as usual; multiplication is as for usual polynomials with the understanding that the resulting x^2-term is ignored. It follows that as a $\mathbb{C}$-algebra, R is 2-dimensional.

The maximal as well as the prime spectrum of R is a singleton consisting of the nilradical $P := (x)$. The residue field at P is $\mathbb{C}$. The value of $a + bx \in R$ at P is a. Thus elements of R contain more information than their values in the residue field. In particular, x has value 0 at P, yet it is not zero.

Example 5.39. Consider the ring $R = \mathbb{C}[x,y]/(x^2)$. As a set, this can expressed as

$$\{a(y) + b(y)x \mid a(y), b(y) \in \mathbb{C}[y]\}.$$

Addition and multiplication works as in the previous example. From this description, one may deduce that (x) is the nilradical of R. It consists of precisely those elements for which there is no $a(y)$-term.

To understand the spectrum of R, one may work with this explicit description, or use the surjective homomorphism

$$\varphi : \mathbb{C}[x,y]/(x^2) \twoheadrightarrow \mathbb{C}[y], \qquad x \mapsto 0,\ y \mapsto y.$$

The kernel of this map is precisely the nilradical (x). Applying Proposition 1.5, we deduce that

$$\operatorname{maxSpec}(\mathbb{C}[x,y]/(x^2)) = \{\varphi^{-1}(M) \mid M \text{ maximal in } \mathbb{C}[y]\}.$$

And further this is homeomorphic to $\operatorname{maxSpec}(\mathbb{C}[y]) \cong \mathbb{A}^1$, the affine line. Similarly,

$$\operatorname{Spec}(\mathbb{C}[x,y]/(x^2)) = \{\varphi^{-1}(P) \mid P \text{ prime in } \mathbb{C}[y]\}.$$

is homeomorphic to $\operatorname{Spec}(\mathbb{C}[y])$.

Example 5.40. Let $S = \mathbb{Z}_{(2)}$ and $R = \mathbb{Q}$. Let $X = \operatorname{Spec}(R)$ and $Y = \operatorname{Spec}(S)$. Then X is a singleton and Y is a doubleton. Define $\varphi : X \to Y$ to be the Zariski continuous map which sends the ideal (0) of R to the maximal ideal (2) of S. Define $\varphi^{\#} : \mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)$ as follows. $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is the inclusion map $S \hookrightarrow R$, and for all other open sets U in Y, the map on the sections is zero. One may check that $(\varphi, \varphi^{\#})$ is a morphism of sheaves but it does not satisfy the local condition.

Note that the morphism induced from the inclusion $S \hookrightarrow R$ would send (0) to (0).

5.3.1. Problems.

(1) Describe $\operatorname{Spec}(\mathbb{Z}/(3))$, $\operatorname{Spec}(\mathbb{Z}/(6))$, $\operatorname{Spec}(\mathbb{R})$, $\operatorname{Spec}(\mathbb{R}[x])$, $\operatorname{Spec}(\mathbb{Z}[x])$.

(2) Fix a complex number $t \in \mathbb{C}$. Describe the prime spectrum of

$$R = \frac{\mathbb{C}[x,y]}{(x(x-t))}.$$

How does it vary with t? What happens at t approaches 0?

(3) Describe the points, the sheaf of functions, the stalks, the residue fields, of each of the following affine schemes.
   (a) $\operatorname{Spec} \Bbbk[x]$, where $\Bbbk$ is a field,
   (b) $X_1 = \operatorname{Spec} \mathbb{C}[x]/(x^2)$,
   (c) $X_2 = \operatorname{Spec} \mathbb{C}[x]/(x^2 - x)$,
   (d) $X_3 = \operatorname{Spec} \mathbb{C}[x]/(x^3 - x^2)$,
   (e) $\operatorname{Spec} \mathbb{R}[x]/(x^2 + 1)$.

   For part (a): If $\Bbbk$ is the field with p elements, how many points are there with a given residue field? How does the topological space of $\operatorname{Spec} \mathbb{R}[x]$ compare to the sets $\mathbb{R}$ and $\mathbb{C}$?

   The schemes $X_1 = \operatorname{Spec} \mathbb{C}[x]/(x^2)$, $X_2 = \operatorname{Spec} \mathbb{C}[x]/(x^2 - x)$ and $X_3 = \operatorname{Spec} \mathbb{C}[x]/(x^3 - x^2)$ may all be viewed as closed subschemes of $\operatorname{Spec} \mathbb{C}[x]$. Show that X_1 and X_2 are closed subschemes of X_3, but no other inclusions hold, even though the underlying sets of X_2 and X_3 coincide and the underlying set of X_1 is contained in the underlying set of X_2.

(b), (c) and (d) are schemes over $\mathbb{C}$. Describe their automorphism groups in the category of schemes over $\mathbb{C}$. Do the same for (a) and (e) in the category of schemes over $\Bbbk$ and $\mathbb{R}$.

5.4. Fiber product of affine schemes

5.4.1. Coproducts in the category of rings. Consider the category Ring. The initial object is $\mathbb{Z}$. The coproduct of S and T is $S \otimes T$. [Take the tensor product of S and T as abelian groups, and define the multiplication by $(s \otimes t)(s' \otimes t') = ss' \otimes tt'$.] There is a nice framework to understand this result, see Corollary E.18 and the preceding discussion.

Now fix a ring R, and consider the category of commutative R-algebras. Suppose S and T are commutative R-algebras. Then their coproduct is $S \otimes_R T$. We recover the previous case by setting $R = \mathbb{Z}$.

- For any commutative R-algebra S, we have $R \otimes_R S = S \otimes_R R = S$.
- If S and T are commutative R-algebras and I is an ideal in S, then

$$(S/I) \otimes_R T = (S \otimes_R T)/(I \otimes 1)(S \otimes_R T).$$

If particular, $(R/I) \otimes_R S = S/IS$.

The following is an important characterization of commutative R-algebras.

Proposition 5.41. *Let R be a ring. A commutative R-algebra is a ring S equipped with a ring homomorphism $R \to S$. In particular, a commutative $\mathbb{Z}$-algebra is same as a ring.*

A morphism of commutative R-algebras is a ring homomorphism $S \to T$ which commutes with the given homomorphisms from R.

Many authors take this as a definition, see [9, pg 323] or [4, page 30]. I believe this is also true if we drop commutative. Of course, we are then dealing with arbitrary rings.

Proof. Exercise. □

A reformulation is given below.

Proposition 5.42. *The category of commutative R-algebras is equivalent to $R \downarrow$ Ring, the slice category of* Ring *under R.*

It is clear that a coproduct in the slice category $R \downarrow$ Ring is the same as a pushout in Ring. Hence, by the above equivalence, the pushout in Ring is given by the tensor product. Let us spell this out.

For any ring homomorphisms $R \to S$ and $R \to T$, there is a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \otimes_R T \end{array} \tag{5.7}$$

such that $S \otimes_R T$ is universal wrt such diagrams.

5.4.2. Products in the category of affine schemes.

Definition 5.43. Let X be a fixed affine scheme. An *affine scheme over* X is an affine scheme Y together with a morphism $Y \to X$.

Let Y and Z be affine schemes over X. A X-morphism between Y and Z is a morphism $Y \to Z$ of affine schemes which commutes with the morphisms to X.

This defines $\mathsf{AffScheme}(X)$, the category of affine schemes over X. Observe that

$$\mathsf{AffScheme}(X) \cong \mathsf{AffScheme} \downarrow X,$$

the slice category over X.

If $X = \mathrm{Spec}(R)$, we also write $\mathsf{AffScheme}(R)$ instead and refer to this as the category of affine schemes over R.

Theorem 5.44. *The category of affine schemes over R is contravariantly equivalent to the category of commutative R-algebras.*

$$\mathsf{AffScheme}(R) \cong (\mathsf{Alg}_R^{\mathsf{co}})^{\mathrm{op}}$$

Setting $R = \mathbb{Z}$ recovers Theorem 5.31.

Proof. If two categories are equivalent, then so are their corresponding slice categories. So we deduce from Theorem 5.31 that $\mathsf{AffScheme} \downarrow X$ and $R \downarrow \mathsf{Ring}$ are contravariantly equivalent, with $X = \mathrm{Spec}(R)$. Now apply Proposition 5.42. □

The pushout diagram (5.7) translates to the pullback diagram:

$$\begin{array}{ccc} \mathrm{Spec}(S \otimes_R T) & \longrightarrow & \mathrm{Spec}(S) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(T) & \longrightarrow & \mathrm{Spec}(R). \end{array}$$

We rewrite it as follows.

$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X. \end{array} \tag{5.8}$$

Here X, Y and Z are affine schemes, and $Y \times_X Z$ is the pullback. It is usually called the *fiber product* of Y and Z over X. This is the product in the category of affine schemes over X.

5.5. The functor of points

We apply the ideas related to the Yoneda embedding (Appendix F) to the category of affine schemes.

5.5.1. The functor of points.

Consider the Yoneda embedding

$$\mathsf{AffScheme} \to \mathsf{Set}^{\mathsf{AffScheme}^{\mathrm{op}}}, \qquad X \mapsto h_X \tag{5.9}$$

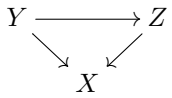
where

$$h_X(Y) := \mathsf{AffScheme}(Y, X).$$

This is called the set of Y-valued points of X. Thus a Y-valued point of X is simply a morphism $Y \to X$ of affine schemes.

If $Y = \mathrm{Spec}(R)$, then it is customary to write $h_X(R)$ and call it the set of R-valued points of X.

One may also fix an affine scheme X and consider the Yoneda embedding for the slice category $\mathsf{AffScheme}(X)$ of affine schemes over X. A Y-valued point of Z would be a commutative diagram



The Yoneda embedding

$$\mathsf{AffScheme} \to \mathsf{Set}^{\mathsf{Ring}}$$

is fully faithful and preserves products (Lemma F.2). The first statement implies that a affine scheme is determined by its R-points, as R varies over all rings. The second statement implies that the set of Y-valued points of the fiber product $X_1 \times X_2$ is the cartesian product of the sets of Y-valued points of X_1 and X_2. In contrast, the underlying set of a fiber product $X_1 \times X_2$ is in general not the cartesian product of the underlying sets of X_1 and X_2.

This and similar features make it convenient to think of a affine scheme in terms of its R-points. Analogous to thinking of a random variable in terms of its distribution.

Definition 5.45. A functor $\mathsf{Ring} \to \mathsf{Set}$ is *representable* if it is of the form h_X for some affine scheme X.

There are criteria to determine when a functor of this kind is representable.

5.5.2. Solving equations. Suppose $f_1, \ldots, f_k$ are polynomials in n variables with coefficients in some ring R. What geometric object should one associate to this data?

The naive answer is: The affine variety consisting of solutions of the equations $f_i = 0$. Do we lose information in this passage from equations to the affine variety?

- No, if R is an algebraically closed field such as $\mathbb{C}$, and the ideal $(f_1, \ldots, f_k)$ is a radical ideal. (By the Nullstellensatz, the vanishing ideal of the affine variety is precisely this radical ideal.)
- Yes, in general. You can imagine an extreme situation in which the equations have no solution.

A possibly correct answer is: The affine scheme $X = \mathrm{Spec}(S)$ where

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_k). \tag{5.10}$$

How does the affine scheme X relate to the solutions of the equations $f_i = 0$? Observe that a tuple $(a_1, \ldots, a_n)$ of elements $a_i \in R$ solves these equations iff the map $S \to R$ which sends x_i to a_i is a R-algebra homomorphism. In other words, the solutions correspond to the R-valued points of X.

More generally, instead of solving the equations over R we may solve them over any other R-algebra T. Such a solution corresponds to a R-algebra homomorphism $S \to T$. In other words, the solutions over T correspond to the T-valued points of X.

In conjunction with the Yoneda embedding, we can now improve our naive answer.

- In general, there is loss of information if we only consider the affine variety over R which is the set of solutions over R. However, there is no loss of information if we consider solutions over every R-algebra (R being one of them).

Theorem 5.46. *Let $\mathcal{F} : \mathsf{Alg}_R^{\mathsf{co}} \to \mathsf{Set}$ be any functor. If the elements of $\mathcal{F}(A)$ are the solutions over A of some family of equations (defined over R), then there is a R-algebra S such that $\mathcal{F}$ is isomorphic to h^S, where $h^S(T) = \mathsf{Alg}_R^{\mathsf{co}}(S, T)$.*

The algebra S is defined by (5.10). Also note that in this result, we do not need to assume that the number of variables or the number of equations is finite.

Example 5.47. Consider

$$S = \mathbb{Z}[x_{ij}]/(\det(x_{ij}) - 1)$$

Then the set of R-points of $X = \operatorname{Spec}(S)$ is $\mathrm{SL}(n, R)$, the special linear group with entries in the ring R. Note that the set of R-points of X has the structure of a group for each R, and these group structures are compatible: A ring homomorphism $R \to R'$ induces a group homomorphism $\mathrm{SL}(n, R) \to \mathrm{SL}(n, R')$. In other words, h_X defines a functor $\mathsf{Ring} \to \mathsf{Group}$. Due to this feature, one says that X is an affine group scheme. This can also be said more directly: An affine group scheme is a group object in the category of affine schemes. Nevertheless, it is convenient to think of X as a functor of points.

CHAPTER 6

Schemes

6.1. Preschemes

Definition 6.1. A *prescheme* X is a topological space, equipped with a sheaf $\mathcal{O}_X$ of rings, such that X is the union of finitely many subsets U_i each isomorphic to an affine scheme.

The precise meaning of isomorphic is as follows. For each i, there is an affine scheme Y_i (with its structure sheaf $\mathcal{O}_{Y_i}$), and an isomorphism $(U_i, \mathcal{O}_{U_i}) \to (Y_i, \mathcal{O}_{Y_i})$ of sheaves. Here $\mathcal{O}_{U_i}$ is the restriction of the sheaf $\mathcal{O}_X$ to the open set U_i.

We do not require a prescheme to be Noetherian since even an affine scheme may not be Noetherian.

A morphism between preschemes is defined exactly as in Definition 5.30: The stalk at any point of a prescheme is a local ring, so the local condition makes sense. This defines the category of preschemes denoted $\mathsf{Prescheme}$.

Definition 6.2. Let Z be a fixed prescheme. A *prescheme over* Z is a prescheme X together with a morphism $X \to Z$.

Let X and Y be preschemes over Z. A Z-morphism between X and Y is a morphism $X \to Y$ of preschemes which commutes with the morphisms to Z.

This defines the category of preschemes over Z which we denote by $\mathsf{Prescheme}(Z)$.

If R is a ring, then we also write $\mathsf{Prescheme}(R)$ for the category of preschemes over $\mathrm{Spec}(R)$. The same notation will be used for affine schemes. So $\mathsf{AffScheme}(R)$ is the category of affine schemes over $\mathrm{Spec}(R)$. This is contravariantly equivalent to the category whose object is a ring S equipped with a ring homomorphism $R \to S$. In particular, note that $\mathsf{AffScheme}(\mathbb{C})$ is contravariantly equivalent to the category of commutative $\mathbb{C}$-algebras.

6.1.1. Open preschemes. Any open set U in a prescheme X is again a prescheme: The sheaf on U is the restriction of the sheaf on X. To see that this is a prescheme, cover X by affine open sets. Intersecting these with U gives an open cover for U. Each open set in this cover is an open set in an affine scheme. So it can be covered by basic open sets and a basic open set is isomorphic to an affine scheme by Proposition 5.27.

6.1.2. The gluing construction. Let X_1 and X_2 be sheaves, and let $U_1 \subseteq X_1$ and $U_2 \subseteq X_2$ be open subsets, and let $\varphi : U_1 \to U_2$ be an isomorphism of sheaves. (The sheaves have been suppressed in the notation, and only the underlying topological spaces are being written.) Then define a sheaf X obtained by gluing X_1 and X_2 along U_1 and U_2 via the isomorphism φ: The topological space is the quotient of the disjoint union of X_1 and X_2 by the relation $x_1 \sim \varphi(x_1)$ for each $x_i \in U_1$

equipped with the quotient topology. The sheaf structure is defined in the obvious manner.

Alternatively, one may consider three sheaves U, X_1 and X_2, with continuous maps $i_1 : U \hookrightarrow X_1$ and $i_2 : U \hookrightarrow X_1$ such that $i_j(U)$ is open in X_j and the sheaves on U_j and $i_j(U)$ (obtained by restricting the sheaf on X_j) are isomorphic. Then there is a sheaf X along with morphisms $X_j \to X$ such that the diagram

$$\begin{array}{ccc} U & \hookrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \to & X \end{array}$$

commutes, and X is universal wrt this property. In more precise terms, X is a pushout (a special kind of colimit).

Simplest examples are the Riemann sphere and the line with doubled origin.

The discussion above goes through if we replace "sheaf" by "prescheme", the essential point being that if X_1 and X_2 are locally affine, then so is X.

As a special case, one may take $U_1 = U_2$ to be the empty set. In that case, we say that the prescheme X is the disjoint union of the preschemes X_1 and X_2 (since it is the disjoint union of the underlying topological spaces). Categorically, X is the coproduct of X_1 and X_2.

The discussion above can be extended to an arbitrary number of preschemes X_i. Try to think of the case when you have three preschemes to glue. Details of the general case are outlined in [14, Exercises 1.22 and 2.12].

Remark 6.3. It seems clear that Top is cocomplete, that is, all colimits exists: Take the disjoint union of all the topological spaces involved and then pass to a quotient space. The situation with Sheaf is expectedly more complicated. But one can ask: what is the most general kind of colimit that exists?

6.1.3. Reduced schemes. Recall from (1.2) and (1.3): To any ring R, one can associate the reduced ring $\mathcal{R}(R) := R/N$, where N is the nilradical of R. This construction is functorial in R, and the resulting functor

$$\mathcal{R} : \mathsf{Ring} \to \mathsf{redRing} \qquad R \mapsto R/N$$

is the left adjoint to the inclusion functor $\mathsf{redRing} \to \mathsf{Ring}$.

Definition 6.4. A sheaf of rings $(X, \mathcal{O}_X)$ is *reduced* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ of sections is reduced. We say a scheme is reduced if its sheaf of rings is reduced.

Let X be a sheaf of rings. Define $\mathcal{R}(X)$ to the reduced sheaf of rings which has the same underlying topological space as X and the sections are given by

$$\mathcal{O}_{\mathcal{R}(X)}(U) := \mathcal{R}(\mathcal{O}_X(U)).$$

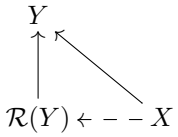
Given a morphism $X \to Y$ of sheaves, there is an induced morphism $\mathcal{R}(X) \to \mathcal{R}(Y)$. Thus we have a functor

$$\mathcal{R} : \mathsf{Sheaf}_{\mathsf{Ring}} \to \mathsf{Sheaf}_{\mathsf{redRing}},$$

the latter being the category of reduced sheaves.

We note that there is a natural map $\mathcal{R}(X) \to X$ defined by the canonical projection $\mathcal{O}_X(U) \to \mathcal{R}(\mathcal{O}_X(U))$. If $X \to Y$ is a morphism of sheaves and X is

reduced, then there is a unique morphism $X \to \mathcal{R}(Y)$ such that the diagram



commutes. Equivalently, there is a natural bijection

$$\mathsf{Sheaf}_{\mathsf{Ring}}(X, Y) \to \mathsf{Sheaf}_{\mathsf{redRing}}(X, \mathcal{R}(Y)).$$

This says that the functor $\mathcal{R}$ is the right adjoint to the inclusion functor $\mathsf{Sheaf}_{\mathsf{redRing}} \to \mathsf{Sheaf}_{\mathsf{Ring}}$.

Proposition 6.5. *Let $(X, \mathcal{O}_X)$ be a sheaf of rings. Then $(X, \mathcal{O}_X)$ is reduced iff all stalks $\mathcal{O}_{X,x}$ are reduced.*

In particular, if R is reduced, then so is $\operatorname{Spec}(R)$.

Proof. Suppose $(X, \mathcal{O}_X)$ is reduced. Suppose $f^n = 0$ for some $f \in \mathcal{O}_{X,x}$. Then this equation holds in some neighborhood U of x. Since $\mathcal{O}_X(U)$ is reduced, $f = 0$ in U and hence in $\mathcal{O}_{X,x}$. So all stalks are reduced.

Conversely, suppose f is nilpotent in $\mathcal{O}_X(U)$. Then it is nilpotent and hence 0 in all stalks $\mathcal{O}_{X,x}$ with $x \in U$. Since a section is determined by its images in the stalks, $f = 0$ as required. □

Proposition 6.6. *For any ring R,*

$$\mathcal{R}(\operatorname{Spec}(R)) = \operatorname{Spec}(\mathcal{R}(R)) = \operatorname{Spec}(R/N)$$

where N is the nilradical of R. More precisely, the following diagram of functors

$$\begin{array}{ccc} \mathsf{Ring} & \xrightarrow{\mathcal{R}} & \mathsf{redRing} \\ {\scriptstyle \operatorname{Spec}}\downarrow & & \downarrow{\scriptstyle \operatorname{Spec}} \\ \mathsf{Sheaf}_{\mathsf{Ring}} & \xrightarrow[\mathcal{R}]{} & \mathsf{Sheaf}_{\mathsf{redRing}} \end{array}$$

commutes. (The vertical functors are contravariant.)

Proof. The underlying topological space of $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R/N)$ are homeomorphic. The ring homomorphism $R \twoheadrightarrow R/N$ induces a morphism of sheaves $\operatorname{Spec}(R/N) \to \operatorname{Spec}(R)$. It is an isomorphism on the underlying topological spaces, which we identify and call X. Further, $\operatorname{Spec}(R/N)$ is reduced, so we obtain a morphism of sheaves $\operatorname{Spec}(\mathcal{R}(R)) \to \mathcal{R}(\operatorname{Spec}(R))$ on X. One can check that this induces a isomorphism on the stalks at every point of X (that is, at every prime ideal of R), hence it is an isomorphism of sheaves by Proposition D.8. □

This shows that the functor $\mathcal{R}$ takes an affine scheme to a reduced affine scheme, from which we deduce that it takes a prescheme to a reduced prescheme. So we have a functor

$$\mathcal{R} : \mathsf{Prescheme} \to \mathsf{redPrescheme},$$

and is the left adjoint to the inclusion functor.

6.1.4. Closed subpreschemes. Let Y be a closed set in a topological space X. If $\mathcal{O}_X$ is a sheaf (of rings) on X, then it does not induce a sheaf on Y in any obvious manner. This makes the notion of a closed subprescheme more complicated to define. To understand it, let us start with the affine case.

Let $X := \mathrm{Spec}(R)$ be an affine scheme, and let I be any ideal in R. Then $Y := \mathbb{V}(I)$ is a closed set in X, and it is canonically homeomorphic to $\mathrm{Spec}(R/I)$. This homeomorphism can be used to turn Y into an affine scheme.

A *closed subprescheme* of an affine scheme X is a sheaf on a closed set of Y which is isomorphic to $\mathrm{Spec}(R/I)$ for some ideal I of R with $Y = \mathbb{V}(I)$.

Remark 6.7. In general, there will be many closed subprescheme structures on the same closed set: The vanishing set of two ideals is the same iff they have the same radical.

What is not clear is whether two such distinct ideals will always yield distinct closed subpreschemes.

Example 6.8. Let $R = \mathbb{Z}$. Let $I_n = (p^n)$ for some fixed prime p. Let Y_n be the closed subscheme of $X = \mathrm{Spec}(\mathbb{Z})$ associated to I_n. Topologically, all the Y_n are identical. They consist of only one point, namely (p). But they are distinct as schemes. Y_1 is reduced while the rest are nonreduced. We call Y_n, the nth infinitesimal neighborhood of (p) in X.

How does one define a closed subprescheme of an arbitrary prescheme? A possible approach is to take the quotient of the sheaf on X by "a sheaf of ideals". However this sheaf of ideals cannot be arbitrary since after taking the quotient one must obtain a sheaf that looks locally like $\mathrm{Spec}(R/I)$. Such a sheaf is called a quasi-coherent sheaf of ideals. A precise definition is given later.

Definition 6.9. A *closed subprescheme* of X is a prescheme $(Y, \mathcal{O}_Y)$ where

- Y is a closed subset of X,
- the inclusion map $i : Y \hookrightarrow X$ is a morphism of preschemes, and
- the kernel of $i^{\#} : \mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is a quasi-coherent sheaf of ideals in X.

An alternative definition which avoids sheaves of ideals altogether and is more in the spirit of what we did for varieties is given below.

Definition 6.10. A *closed subprescheme* of X is a prescheme $(Y, \mathcal{O}_Y)$ where

- Y is a closed subset of X,
- the inclusion map $i : Y \hookrightarrow X$ is a morphism of preschemes, and
- the map $i^{\#} : \mathcal{O}_X \to i_*(\mathcal{O}_Y)$ is surjective on stalks.

Remark 6.11. The surjectivity condition on stalks is weaker than requiring that $\mathcal{O}_X(U) \to \mathcal{O}_Y(U \cap Y)$ is surjective for all U. It says that any "function" in the sheaf on Y can be locally extended to a "function" in the sheaf on X.

For the equivalence of the two definitions, see [19, page 106]. or [14, Corollary 5.10].

6.1.5. Problems.

(1) Show that the functor $\mathrm{Spec} : \mathsf{Ring} \to \mathsf{Prescheme}$ is adjoint to the global sections functor $\mathsf{Prescheme} \to \mathsf{Ring}$. More precisely, for any ring R and a prescheme X, there is a natural bijection

$$\mathsf{Prescheme}(X, \mathrm{Spec}(R)) \to \mathsf{Ring}(R, \mathcal{O}_X(X)).$$

Since the functors are contravariant, we cannot use the terms left or right adjoint.

Deduce that $\mathrm{Spec}(\mathbb{Z})$ is a terminal object in the category of preschemes.

(2) If X is a topological space and Z an irreducible closed subset of X, a generic point for Z is a point x whose closure is Z. If X is a prescheme, show that every nonempty irreducible closed subset has a unique generic point.

6.2. Varieties and schemes

In this section, we show how an affine variety can be viewed as an affine scheme, and more generally how a prevariety can be viewed as a prescheme.

6.2.1. A functor on topological spaces. Define a functor

$$t : \mathsf{Top} \to \mathsf{Top}$$

as follows. Let X be a topological space, and let $t(X)$ be the set of nonempty irreducible closed subsets of X. Note that if Y is a closed set in X, then there is a canonical inclusion $t(Y) \hookrightarrow t(X)$: The image of this map consists of those nonempty irreducible closed subsets of X which are contained in Y. It is convenient to identify this image with $t(Y)$. The closed sets of $t(X)$ are defined to be subsets of the form $t(Y)$ for Y a closed subset of X. Observe that

$$t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2) \quad \text{and} \quad t(\cap Y_i) = \cap t(Y_i).$$

(If an irreducible closed subset is contained in $Y_1 \cup Y_2$, then it is either contained in Y_1 or in Y_2.) So finite union and arbitrary intersection of closed sets is closed as required. If $f : X_1 \to X_2$ is a continuous map, then define $t(f) : t(X_1) \to t(X_2)$ by sending an irreducible closed subset to the closure of its image. This shows that t is a functor.

There is a *bijection* between the open (closed) sets of X and the open (closed) sets of $t(X)$ as follows. To a closed set Y in X, we associate the closed set $t(Y)$ in $t(X)$. If $t(Y_1) = t(Y_2)$, then $Y_1 = Y_2$ consists of precisely those points in X whose closures are elements of $t(Y_1) = t(Y_2)$.

It is instructive to understand how the bijection works on open sets. If U is an open set in X, then there is an inclusion map $t(U) \hookrightarrow t(X)$ which sends a nonempty irreducible closed set Y of U to $\overline{Y}$. All irreducible closed sets of X which meet U have to be of this form. Thus the image of $t(U)$ consists precisely of irreducible closed sets of X which meet U, which is an open set of $t(X)$. It is convenient to identify this image with $t(U)$. This is the open set of $t(X)$ associated to U.

Remark 6.12. This is a very interesting phenomenon: X and $t(X)$ are two topological spaces which may not be homeomorphic but whose posets of open (closed) sets are isomorphic.

The above bijection can be better understood as follows. Define a continuous map $\alpha : X \to t(X)$ which sends an element of X to its closure. Given any open (closed) set V of $t(X)$, the corresponding open (closed) set of X is $\alpha^{-1}(V)$. For

any continuous map $f : X_1 \to X_2$, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\alpha} & t(X_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t(f)} \\ X_2 & \xrightarrow[\alpha]{} & t(X_2) \end{array}$$

commutes. This says that α is a natural transformation between the identity functor and the above functor.

This leads to a functor

$$t : \mathsf{Sheaf} \to \mathsf{Sheaf} \qquad (X, \mathcal{O}_X) \mapsto (t(X), \alpha_*(\mathcal{O}_X)). \tag{6.1}$$

What is going on is very simple. The open sets in X and $t(X)$ are in bijection. So a sheaf on X induces a sheaf on $t(X)$: the ring of sections of an open set of $t(X)$ is same as the ring of sections of the corresponding open set of X.

6.2.2. From varieties to schemes. There is a fully faithful functor from the category of finitely generated reduced commutative $\mathbb{C}$-algebras to the category of commutative $\mathbb{C}$-algebras. Equivalently, there is a functor from the category of affine varieties over $\mathbb{C}$ to the category of affine schemes over $\mathbb{C}$.

Proposition 6.13. *The following is a commutative diagram of functors.*

$$\begin{array}{ccc} \mathsf{AffVariety}(\mathbb{C}) & \longrightarrow & \mathsf{AffScheme}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathsf{Sheaf} & \xrightarrow[t]{} & \mathsf{Sheaf} \end{array}$$

The vertical functors send an affine variety (scheme) to the underlying topological space with its structure sheaf.

Proof. Suppose R is a finitely generated reduced commutative $\mathbb{C}$-algebra. Then the affine variety associated to R is $\operatorname{maxSpec}(R)$ and the affine scheme associated to R is $\operatorname{Spec}(R)$ (each with its structure sheaf). First observe that as topological spaces

$$t(\operatorname{maxSpec}(R)) = \operatorname{Spec}(R).$$

This is because the irreducible closed subsets of $\operatorname{maxSpec}(R)$ correspond to the prime ideals of R.

A maximal ideal is a closed point in $\operatorname{Spec}(R)$, so

$$\alpha : \operatorname{maxSpec}(R) \to \operatorname{Spec}(R)$$

is just the inclusion map. We now only need to check that the structure sheaf of $\operatorname{Spec}(R)$ is obtained by pushing forward the structure sheaf of $\operatorname{maxSpec}(R)$. It is enough to check this for basic open sets in $\operatorname{Spec}(R)$. Let $f \in R$. Then $\alpha^{-1}(\operatorname{Spec}(R)_f) = \operatorname{maxSpec}(R)_f$. So the basic open set associated to f in the maximal and prime spectrum correspond. The sections over this basic open set is R_f in both cases, so we are done. □

Remark 6.14. In the context of varieties, we had shown $\mathcal{O}_X(X_f) = R_f$ only when X is irreducible. But I believe it holds in general. In any case, we are using it in the proof above. We did resolve $\mathcal{O}_X(X) = R$ in general, and perhaps $\mathcal{O}_X(X_f) = R_f$ can be deduced from that.

Remark 6.15. An affine scheme $\mathrm{Spec}(R)$ may fail to be an affine variety for three reasons. The ring R may not be

- a $\mathbb{C}$-algebra. Example: $\mathbb{Z}$, $\mathbb{Z}[x]$.
- reduced. Example: $\mathbb{C}[x]/(x^2)$.
- finitely generated. Example: $\mathbb{C}[[x]]$, $\mathbb{C}[x]_{(x)}$.

The above result says that the functor (6.1) sends an affine variety to an affine scheme. We now claim further that it sends a prevariety to a prescheme (over $\mathbb{C}$)). Let X be prevariety. Let $\{U_i\}$ be a cover of X by affine open sets. Then $\{t(U_i)\}$ is an open cover of $t(X)$ (as explained above). Since U_i with the induced sheaf is affine, the above result implies that $t(U_i)$ with the pushforward sheaf is an affine scheme. Thus $t(X)$ is covered by affine schemes as required. To summarize:

Proposition 6.16. *There is a commutative diagram of functors.*

$$\begin{array}{ccc} \mathsf{Prevariety}(\mathbb{C}) & \longrightarrow & \mathsf{Prescheme}(\mathbb{C}) \\ \downarrow & & \downarrow \\ \mathsf{Sheaf} & \xrightarrow[t]{} & \mathsf{Sheaf} \end{array}$$

Definition 6.17. A prescheme X over R is of *finite type over* R if for any affine open set U, the ring of sections $\mathcal{O}_X(U)$ is a finitely generated R-algebra.

Proposition 6.18. *Let X be a prescheme over R. If there exists a finite open covering $\{U_i\}$ of X such that $\mathcal{O}_X(U_i)$ is a finitely generated R-algebra, then X is of finite type over R.*

Theorem 6.19. *The category of prevarieties is equivalent to the category of reduced (Noetherian) preschemes of finite type over $\mathbb{C}$.*

We have constructed a functor from the first category to the second. One needs to show that it is an equivalence; see Mumford [19, page 88] for a proof. The reason for writing Noetherian above is because prevarieties were defined to be Noetherian.

6.2.3. Problems.

(1) Is $t(\mathrm{maxSpec}(\mathbb{Z})) = \mathrm{Spec}(\mathbb{Z})$ as topological spaces? Identify the class of rings R for which $t(\mathrm{maxSpec}(R)) = \mathrm{Spec}(R)$ as topological spaces?
(2) What is $t(\mathrm{Spec}(R))$? What is the functor t^2 obtained by composing t with itself? It is the same sheaf. $t^2 = t$, so it is a monad?
(3) Under what conditions is the map $\alpha : X \to t(X)$ injective? surjective? bijective?

6.3. Fiber product in the category of preschemes

In this section, we discuss the fiber product in the category of preschemes. We use its existence to define the fiber of a morphism, affine spaces and projective spaces.

6.3.1. Fiber product of preschemes. We have seen that products and pullbacks exist in the category of affine schemes. Explicitly, if X, Y and Z are affine schemes with morphisms $Y \to X$ and $Z \to X$, then there is a affine scheme $Y \times_X Z$ universal wrt the diagram (5.8). This is called the fiber product of Y and Z over X.

It turns out that the fiber product exists in the category or preschemes. The standard way to show this is: Take affine open covers X, Y and Z such that the morphisms $Y \to X$ and $Z \to X$ take an affine open set inside an affine open set. Take the fiber product of these affine sets, and then patch them together using the gluing construction. The details are straightforward but tedious; see [14, Theorem 3.3].

6.3.2. Fiber of a morphism. Let X be a prescheme. Pick any $x \in X$. There is a canonical morphism

$$Z := \operatorname{Spec}(\kappa(x)) \to X$$

which sends the unique point in Z to x, and on stalks is the quotient map $\mathcal{O}_{X,x} \twoheadrightarrow \kappa(x)$. (If x is a closed point, then this map would be an isomorphisma and Z would be a closed subscheme of X.)

Definition 6.20. Let $\varphi : Y \to X$ be a morphism of preschemes. Then the fiber of φ over $x \in X$ is defined to be the fiber product $Y \times_X \operatorname{Spec}(\kappa(x))$.

Proposition 6.21. *The fiber Y_x is a prescheme over $\operatorname{Spec}(\kappa(x))$, and its underlying topological space is homeomorphic to the subset $\varphi^{-1}(y)$ of X.*

Proof. We may assume that X and Y are affine, since the general case can be deduced from this one by a gluing argument. Accordingly:

Let S be an R-algebra (with unit map $\varphi : R \to S$), and let P be a prime ideal in R. There is a ring homomorphism

$$S \to S \otimes_R \kappa(P) =: T, \quad s \mapsto s \otimes 1.$$

Since $\kappa(P)$ is a field, an ideal in T is necessarily of the form $J \otimes 1$ for some ideal J of S. Note that:

- $PS \otimes 1 = 0$ since P acts by 0 on $\kappa(P)$.
- If $\varphi^{-1}(J)$ strictly contains P, then $J \otimes 1 = T$.

One can deduce that the prime ideals in T correspond to prime ideals in S whose inverse image under φ is P. Further $\operatorname{Spec}(T)$ and the subset of $\operatorname{Spec}(S)$ consisting of these primes (with the induced topology) are homeomorphic. □

Example 6.22. Let $X = \operatorname{Spec}(R)$ be an affine scheme. Consider the canonical morphism $X \to \operatorname{Spec}(\mathbb{Z})$. Then X can be written as a disjoint union of the fibers $X_{(p)}$, one for each prime p, and the fiber $X_{(0)}$. These fibers are affine schemes corresponding to the rings

$$R \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \quad \text{and} \quad R \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This observation can be useful to figure out the prime ideals in R. For example, take $R = \mathbb{Z}[x]$. Then the rings in question are

$$\mathbb{Z}/(p)[x] = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \quad \text{and} \quad \mathbb{Q}[x] = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The prime ideals in these rings are principal ideals generated by irreducible polynomials. By taking inverse images, one obtains all the prime ideals in $\mathbb{Z}[x]$.

Example 6.23. Let $\Bbbk$ be an algebraically closed field of characteristic 0. Let $R = \Bbbk[x]$ and $S = \Bbbk[x, y]/(x - y^2) \cong \Bbbk[y]$. Let $X = \operatorname{Spec}(R)$ and let $Y = \operatorname{Spec}(S)$). Let $\varphi : Y \to X$ be the morphism induced by the algebra homomorphism $R \to S$ which sends x to $x = y^2$. (Geometrically, it sends the closed point (s^2, s) to s. Draw a picture.)

Let $t \in X$ be a closed point. So $t = (x - a)$ for some $a \in \Bbbk$. To compute the fiber over t in Y:

$$\Bbbk \otimes_R \Bbbk[y] = \Bbbk[y]/(y^2 - a).$$

($x \in R$ acts by a on $\Bbbk$ and by y^2 on S.)

- If $a \neq 0$, then the fiber product is isomorphic to $\Bbbk \times \Bbbk$. Thus the fiber Y_t consists of two points $(y - \sqrt{a})$ and $(y + \sqrt{a})$, where $\sqrt{a}$ denotes an element of $\Bbbk$ whose square is a. Both points have stalk $\Bbbk$.
- If $a = 0$, that is $t = (x)$, then the fiber Y_t is a nonreduced one-point prescheme with stalk $\Bbbk[y]/y^2$.

There is only one non-closed point in X. This is the generic point (0). Its residue field is $\Bbbk(x)$. To compute the fiber over (0) in Y:

$$\Bbbk(x) \otimes_R \Bbbk[y].$$

($x \in R$ acts by x on $\Bbbk(x)$ and by y^2 on S.) This ring has only one prime ideal, namely the zero ideal. So the spectrum consists of one point. The stalk is the ring itself. The residue field is its field of fractions. It is the splitting field of $y^2 - x$ over $\Bbbk(x)$, and in particular a degree 2 extension of $\Bbbk(x)$.

6.3.3. Base change. Suppose we are given a ring T. If T is a R-algebra, then we think of R as a base ring for T. Now let S be another R-algebra. Then we say that $S \otimes_R T$ is obtained from T by making a base extension $R \to S$. A common case is when R is a field and S is some field extension of R. For example, consider the ring $\mathbb{R}[x]$. Note that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x] = \mathbb{C}[x]$. Extending scalars from $\mathbb{R}$ to $\mathbb{C}$ changes $\mathbb{R}[x]$ to $\mathbb{C}[x]$.

The same terminology is applied to affine schemes and to preschemes in general. For example, $\mathrm{Spec}(\mathbb{R}[x])$ is an affine scheme over $\mathbb{R}$. And $\mathrm{Spec}(\mathbb{C}[x])$ is the affine scheme obtained by the base extension $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$.

6.3.4. Affine space. Let $\mathbb{A}^n_{\mathbb{Z}} := \mathrm{Spec}(\mathbb{Z}[x_1, \ldots, x_n])$. This is called the *affine space* over $\mathrm{Spec}(\mathbb{Z})$. Now let X be any prescheme. Define

$$\mathbb{A}^n_X := \mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} X.$$

(Recall that $\mathrm{Spec}(\mathbb{Z})$ is the terminal object in the category of preschemes. So any prescheme is canonically a scheme over $\mathbb{Z}$.) This is called the *affine space* over X. It is a prescheme over X.

If $X = \mathrm{Spec}(R)$ is affine, then $\mathbb{A}^n_X = \mathrm{Spec}(R[x_1, \ldots, x_n])$. In this situation, the notation $\mathbb{A}^n_R$ is also used. This is a prescheme over R. If $R = \mathbb{C}$, then we recover our earlier affine space (but keep in mind that we are now thinking of it as an affine scheme rather than an affine variety, though the two are equivalent notions in this context).

Suppose $\Bbbk'$ is a field extension of $\Bbbk$. Then base extension transforms $\mathbb{A}^n_{\Bbbk}$ to $\mathbb{A}^n_{\Bbbk'}$. There is a morphism from the latter to the former.

6.3.5. Projective space. Let $X_1 = \mathrm{Spec}(R[z])$ and $X_2 = \mathrm{Spec}(R[w])$ be two copies of the affine line $\mathbb{A}^1_R$. Consider the open sets $U_1 = \mathrm{Spec}(R[z]_{(z)})$ and $U_2 = \mathrm{Spec}(R[w]_{(w)})$. The ring isomorphism

$$R[z]_{(z)} \to R[w]_{(w)} \qquad z \mapsto 1/w$$

induces an isomorphism $U_2 \to U_1$ or preschemes. We get a prescheme by gluing X_1 and X_2 by this map. The resulting scheme is denote $\mathbb{P}^1_R$. It is called the *projective line* over R.

Starting with $n+1$ copies of affine space, and gluing them appropriately yields $\mathbb{P}^n_R$. Details which are straightforward can be found in [19, page 82]. One may check that

$$\mathbb{P}^n_R = \mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec}(R).$$

This leads a natural question. How do we define projective schemes over R? One can take an approach similar to the one for projective varieties. Start with a graded R-algebra and use homogeneous prime ideals in it. For details, see [11, Chapter 3].

6.3.6. Problems.

(1) Compute $\mathbb{Z}/(m) \otimes \mathbb{Z}/(n)$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.
(2) Show that for any ring R,
$$R[x_1, \ldots, x_n] \otimes_R R[y_1, \ldots, y_m] = R[x_1, \ldots, x_n, y_1, \ldots, y_m].$$
(3) View R and $R[y]$ as $R[x]$-algebras via the ring homomorphisms
$$R[x] \to R, \quad x \mapsto 0 \qquad R[x] \to R[y], \quad x \mapsto y^2.$$
Show that
$$R[y] \otimes_{R[x]} R = R[y]/(y^2).$$
(4) Find the images in $\mathbb{A}^2_{\mathbb{Q}}$ of the following points of $\mathbb{A}^2_{\mathbb{C}}$.
   (a) $(x - \sqrt{2}, y - \sqrt{2})$
   (b) $(x - \sqrt{2}, y - \sqrt{3})$
   (c) $(\sqrt{2}x - \sqrt{3}y)$
   (d) $(\sqrt{2}x - \sqrt{3}y - 1)$
   (e) $(x - \zeta, y - \zeta^{-1})$ where ζ is a pth root of unity, with p prime.
(5) A $\Bbbk$-scheme X is absolutely irreducible if the fiber product $X \times_{\operatorname{Spec} \Bbbk} \operatorname{Spec} \overline{\Bbbk}$ is irreducible. Classify the following closed subschemes of $\mathbb{Q}[x, y]$ as reducible, irreducible but not absolutely irreducible, or absolutely irreducible.
   (a) $x^2 - y^2$
   (b) $x^2 + y^2$
   (c) $x^2 + y^2 - 1$
   (d) $(x + y, xy - 2)$
   (e) $(x^2 - 2y^2, x^3 + 3y^3)$
(6) Let X and Y be affine schemes. Show that the underlying point set of the product $X \times Y$ is *not* the underlying point sets of X and Y in general. This can happen even for affine schemes over an algebraically closed field.
(7) Show that a prescheme is irreducible iff it has a unique generic point (that is, a dense point). Is the stalk at a generic point always a field?

6.4. Quasi-coherent sheaves

In this section, we give a brief introduction to quasi-coherent sheaves. Roughly speaking, a quasi-coherent sheaf of a prescheme X is a sheaf of abelian groups such that the sections of any open set U is a $\mathcal{O}_X(U)$-module.

6.4.1. The category of ring-modules. Let $\mathsf{RingMod}$ denote the category of ring-modules: An object is a pair (R, M), where R is a ring and M is a R-module. A morphism $(R, M) \to (S, N)$ consists of a ring homomorphism $\varphi : R \to S$ and a homomorphism $f : M \to N$ of abelian groups which is a map of R-modules (viewing N as a R-module via φ:

$$f(rm) = \varphi(r)f(m)$$

for all $r \in R$ and $m \in M$.

Remark 6.24. One can consider the category of monoid-modules in any monoidal category. Consider the monoidal category of abelian groups, with monoidal structure being the tensor product. Then a monoid-module in this category is precisely a ring-module. More precisely, $\mathsf{RingMod}$ is the category of monoid-modules in the monoidal category of abelian groups.

6.4.2. The prime spectrum of a ring-module. Consider the category of sheaves of ring-modules. We denote it by $\mathsf{Sheaf}_{\mathsf{RingMod}}$. It consists of a topological space X; further, for every open set U in X, there is specified a ring-module (R, M); for every containment $V \subseteq U$ of open sets, there is a restriction map of the corresponding ring-modules, such that the usual sheaf axioms are satisfied.

Suppose we have a sheaf of ring-modules. Then it contains a sheaf of rings, (denote it by $(X, \mathcal{O}_X)$ as usual), and a sheaf of abelian groups, (denote it by $\mathcal{F}$). And these two sheaves are compatible in the manner indicated above. It is customary to call $\mathcal{F}$ an $\mathcal{O}_X$-module.

The functor (5.5) can be extended to a functor

$$\mathsf{RingMod} \to \mathsf{Sheaf}_{\mathsf{RingMod}}$$

as follows. Let (R, M) be a ring-module. To it we need to associate a sheaf of ring-modules. Take the topological space X to be $\mathrm{Spec}(R)$. For any basic open set X_f, associate the ring-module (R_f, M_f). This defines a $\mathcal{B}$-sheaf which then uniquely extends to a sheaf. It is clear that the sheaf of rings $(X, \mathcal{O}_X)$ is just $\mathrm{Spec}(R)$, while the sheaf of abelian groups $\mathcal{F}$, is constructed by localizations of the given module M. It is customary to denote $\mathcal{F}$ by $\tilde{M}$.

Let R be any ring and let $X = \mathrm{Spec}(R)$. Then the above discussion also yields a functor

$$\mathsf{Mod}_R \to \mathsf{Mod}_{\mathcal{O}_X}, \qquad M \mapsto \tilde{M}.$$

from the category of R-modules to the category of $\mathcal{O}_X$-modules. This functor is exact. For more information, see [14, Proposition 5.2].

6.4.3. Quasi-coherent sheaves. Let $(X, \mathcal{O}_X)$ be a prescheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is *quasi-coherent* if X can be covered by affine open sets $U_i \cong \mathrm{Spec}(A_i)$, such that for each i, there is a A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

A quasi-coherent sheaf of ideals on X is a quasi-coherent sheaf of $\mathcal{O}_X$-modules such that the section over any open set U is an ideal in $\mathcal{O}_X(U)$.

Here are some standard examples.

- On any prescheme X, the structure sheaf $\mathcal{O}_X$ is quasi-coherent as a module over itself.
- Let I be any ideal of R. Then

$$0 \to I \to R \to R/I \to 0$$

is an exact sequence of R-modules. It induces an exact sequence of quasi-coherent $\mathcal{O}_X$-modules where $X = \operatorname{Spec}(R)$: $\tilde{I}$ is a quasi-coherent sheaf of ideals on X, $\tilde{R}$ is $\mathcal{O}_X$ viewed as a module over itself, $\widetilde{R/I}$ can be interpreted as follows. Let $Y = \operatorname{Spec}(R/I)$ be the closed subprescheme of X associated to I. Then $i_*(\mathcal{O}_Y)$ is a sheaf of rings on X where $i : Y \to X$, and hence an $\mathcal{O}_X$-module which is precisely $\widetilde{R/I}$.

For all practical purposes, one can manipulate quasi-coherent sheaves as if there were modules.

Theorem 6.25. *Let R be any ring and let $X = \operatorname{Spec}(R)$. Then there is an equivalence between the category of R-modules and the category of quasi-coherent $\mathcal{O}_X$-modules.*

This should extend to a (contravariant) equivalence between the category of ring-modules and the category of "affine schemes of ring-modules".

6.5. The functor of points

We apply the ideas related to the Yoneda embedding (Appendix F) to the category of preschemes. More details can be found in [11, Chapter VI].

Consider the Yoneda embedding

$$\mathsf{Prescheme} \to \mathsf{Set}^{\mathsf{Prescheme}^{\mathrm{op}}}$$

The set $h_X(Y)$ is called the set of Y-valued points of X. Thus a Y-valued point of X is simply a morphism $Y \to X$ of preschemes.

If $Y = \operatorname{Spec}(R)$, then it is customary to write $h_X(R)$ and call it the set of R-valued points of X.

One may also consider the Yoneda embedding for the category $\mathsf{Prescheme}(Z)$ of preschemes over a fixed prescheme Z. A Y-valued point of X would be a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

The underlying set of a fiber product $X_1 \times X_2$ is in general not the cartesian product of the underlying sets of X_1 and X_2. But the set of Y-valued points of $X_1 \times X_2$ is the cartesian product of the sets of Y-valued points of X_1 and X_2. This follows from Lemma F.2.

It turns out that the restricted functor

$$\mathsf{Prescheme} \to \mathsf{Set}^{\mathsf{Ring}}$$

is also fully faithful. In other words, a prescheme is determined by its R-points, as R varies over all rings. A functor $\mathsf{Ring} \to \mathsf{Set}$ is *representable* if it is of the form h_X for some prescheme X. There are criteria to determine when a functor of this kind is representable.

It is often convenient to think of a prescheme in terms of its R-points. Similar to thinking of a random variable in terms of its distribution.

Example 6.26. Suppose $\Bbbk$ is a field. A prescheme X over $\Bbbk$ is a prescheme in which the ring of sections over any open set is a $\Bbbk$-algebra. Suppose $\Bbbk'$ is any field

and X is any prescheme. A morphism $\mathrm{Spec}(\Bbbk') \to X$ is a same as a point $x \in X$ equipped with a field extension $\kappa(x) \to \Bbbk'$. Now consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\Bbbk') & \longrightarrow & X \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(\Bbbk). & \end{array}$$

This is a $\Bbbk'$-valued point of a prescheme X over $\Bbbk$ where $\Bbbk'$ is a field extension of $\Bbbk$. This amounts to specifying a point $x \in X$ whose residue field lies between $\Bbbk$ and $\Bbbk'$. If $\Bbbk = \Bbbk'$, then the residue field at x is $\Bbbk$. Such a point is called a $\Bbbk$-rational point of X. It is a $\Bbbk$-valued point of a prescheme X over $\Bbbk$.

If $\Bbbk = \mathbb{C}$ and the prescheme X comes from a prevariety, then the $\Bbbk$-rational points of X are precisely the closed points of X.

CHAPTER 7

Groups and Hopf algebras

The basic theory of monoidal categories is explained in Appendix E. In this chapter, we discuss two examples of this theory which are of immediate relevance to us.

7.1. Sets

A basic example of a symmetric monoidal category is $(\mathsf{Set}, \times)$, the category of sets under cartesian product. For the unit object, we choose the one-element set $\{\emptyset\}$. The braiding is given by switching the factors.

A monoid in $(\mathsf{Set}, \times)$ is a monoid (as defined in any elementary algebra class). What about comonoids? Observe that every set A is a comonoid in a unique way: The coproduct is the diagonal map

$$A \to A \times A, \qquad x \mapsto (x, x).$$

The counit is the unique map $A \to \{\emptyset\}$. Further note that this coproduct is cocommutative. One can check that with this canonical coproduct, every monoid is a bimonoid in a unique way. Thus a bimonoid in $(\mathsf{Set}, \times)$ is same as a monoid.

What about a Hopf monoid? Check that a Hopf monoid in $(\mathsf{Set}, \times)$ is same as a group! More precisely,

$$\mathsf{Hopf}(\mathsf{Set}, \times) \cong \mathsf{Group}.$$

Since the cartesian product is the categorical product in Set, the monoidal category $(\mathsf{Set}, \times)$ is cartesian. The above observations are instances of generalities on cartesian monoidal categories given in Section E.6: A Hopf monoid is such a category is the same as a group object, and a group object in $(\mathsf{Set}, \times)$ is a group (in the usual sense).

$$\mathsf{Hopf}(\mathsf{Set}, \times) \cong \mathsf{Group}(\mathsf{Set}, \times) \cong \mathsf{Group}.$$

7.2. Modules and vector spaces

Fix a commutative ring R. Let Mod_R denote the category of (left) R-modules. Suppose M and N are two R-modules. Then one can define their tensor product $M \otimes_R N$. This is again a R-module. This construction is explained in most books on algebra, see for instance [4, page 24]. A key observation is the following.

Proposition 7.1. *The tensor product turns* Mod_R *into a symmetric monoidal category.*

There are natural isomorphisms:

$$(M \otimes_R N) \otimes_R P \xrightarrow{\cong} M \otimes_R (N \otimes_R P).$$
$$M \otimes_R R \xrightarrow{\cong} M \xrightarrow{\cong} R \otimes_R M.$$
$$M \otimes_R N \xrightarrow{\cong} N \otimes_R M.$$

The first is the associativity constraint, the second is the unit constraint, in particualr, the unit object is R, and the third is the symmetry given by switching the tensor factors.

A (co, bi, Hopf) monoid in $(\mathsf{Mod}_R, \otimes)$ is a (co, bi, Hopf) algebra over R. This is the example most commonly treated in the literature. Unlike sets, there is no general simplification here. So to define, say a Hopf algebra over R, one has to repeat the definition of a Hopf monoid with $\otimes$ replacing $\bullet$ and R replacing I. You will find this explicitly written in numerous textbooks on Hopf algebras [1, 17, 25].

Some general contexts in which Hopf algebras arise are given below.

- Group algebra of an arbitrary group. This is cocommutative.
- Coordinate ring of an affine algebraic group. This is commutative.
- Universal enveloping algebra of a Lie algebra. This is cocommutative.

The diamond of categories given in Figure E.1 specializes as follows:

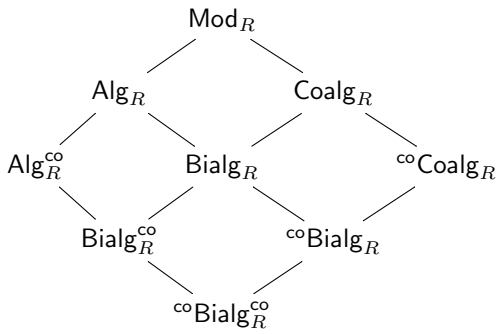


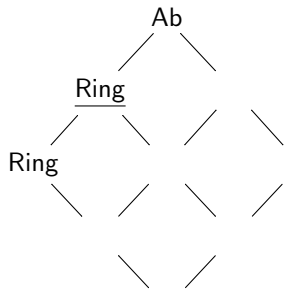
FIGURE 7.1. The monoid and comonoid constructions on modules.

Thus Alg_R is the category of R-algebras, $\mathsf{Alg}_R^{\mathsf{co}}$ is the category of commutative R-algebras, and so on. Each of them is a monoidal category with the induced tensor product. In particular, statements such as "tensor product of two R-algebras is again a R-algebra" are consequences of iterations of the monoid and comonoid constructions.

Two specializations of R are worth highlighting.

- R is a field. Let us denote it by $\Bbbk$. A R-module is the same as a vector space over $\Bbbk$. So instead of Mod_R we write $\mathsf{Vec}_\Bbbk$. It is also customary to simply write Vec with $\Bbbk$ being understood. The terminology and notations for the remaining objects are same as before. For example, a monoid in $(\mathsf{Vec}, \otimes)$ is a $\Bbbk$-algebra, and so on.
- $R = \mathbb{Z}$. Recall that a $\mathbb{Z}$-module is the same as an abelian group, a $\mathbb{Z}$-algebra is same as an arbitrary ring (not necessarily commutative), and a

commutative $\mathbb{Z}$-algebra is same as a ring. So Figure 7.1 takes the form



Here Ab denotes the category of abelian groups, Ring denotes the category of arbitrary rings, and Ring denotes the category of commutative rings. There are no standard terms for the blank entries. If we want to refer to them, then we use the term $\mathbb{Z}$-coalgebra, $\mathbb{Z}$-bialgebra, and so on.

7.3. Group-like elements of a Hopf algebra

7.3.1. The group of group-like elements. Let H be a Hopf algebra over a commutative ring $\Bbbk$. A *group-like* element of H is an invertible element for which $\Delta(x) = x \otimes x$, $\epsilon(x) = 1$ and $\mathrm{S}(x) = x^{-1}$. Observe that the product of two group-likes is a group-like:

$$\Delta(xy) = \Delta(x)\Delta(y) = (x \otimes x)(y \otimes y) = xy \otimes xy.$$

Probably: If $\Bbbk$ is a field, then a group-like element is an element $x \neq 0$ for which $\Delta(x) = x \otimes x$.

Proposition 7.2. *Under the above operation, the set of all group-like elements forms a group, the identity element is the unit of H and the inverse of x is $\mathrm{S}(x)$.*

Proof. This requires a couple of checks.

- $\Delta(1) = 1 \otimes 1$.
- If $\Delta(x) = x \otimes x$, then $\Delta(\mathrm{S}(x)) = \mathrm{S}(x) \otimes \mathrm{S}(x)$: Use that S is an anti-morphism of coalgebras [17, Theorem III.3.4] or [2, Proposition 1.22].

□

Let $\mathcal{G}(H)$ denote the group of group-like elements of H. Suppose $\varphi : H \to H'$ is a morphism of Hopf algebras. If x is a group-like in H, then $\varphi(x)$ is a group-like in H':

$$\Delta(\varphi(x)) = (\varphi \otimes \varphi)(\Delta(x)) = (\varphi \otimes \varphi)(x \otimes x) = \varphi(x) \otimes \varphi(x).$$

This induces a group homomorphism $\mathcal{G}(H) \to \mathcal{G}(H')$. This shows that there is a functor:

$$\mathcal{G} : \mathsf{HopfAlg}_{\Bbbk} \to \mathsf{Group}, \qquad H \mapsto \mathcal{G}(H). \tag{7.1}$$

Note that if the Hopf algebra H is commutative, then the group $\mathcal{G}(H)$ is abelian. Thus there is an induced functor

$$\mathcal{G} : \mathsf{HopfAlg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Ab}, \qquad H \mapsto \mathcal{G}(H), \tag{7.2}$$

where Ab is the category of abelian groups.

Lemma 7.3. *If H is a Hopf algebra over a field $\Bbbk$, then the set of group-like elements in H is linearly independent.*

PROOF. Suppose g and $\{g_i\}$ are group-like elements, and $g = \sum_i \lambda_i g_i$. We may assume that the g_i are linearly independent. Applying the counit map, we deduce that $1 = \sum_i \lambda_i$. Further,

$$\sum_i \lambda_i\, g_i \otimes g_i = g \otimes g = \sum_{i,j} \lambda_i \lambda_j\, g_i \otimes g_j.$$

Comparing coefficients yields

$$\lambda_i \lambda_j = \begin{cases} \lambda_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since each λ_i is idempotent and $\Bbbk$ is a field, either $\lambda_i = 0$ or $\lambda_i = 1$. Further, since their sum is 1, exactly one of them equals 1 and the rest are 0. So g equals one of the g_i. □

7.3.2. The linearization functor. Let $\Bbbk$ be a commutative ring. Consider the *linearization functor*

$$\Bbbk(-)\colon (\mathsf{Set}, \times, \{*\}) \longrightarrow (\mathsf{Mod}_\Bbbk, \otimes, \Bbbk),$$

which sends a set to the free $\Bbbk$-module with basis the given set. This functor is monoidal:

$$\Bbbk(A \times B) \cong \Bbbk(A) \otimes \Bbbk(B) \quad \text{and} \quad \Bbbk(\{\emptyset\}) \cong \Bbbk.$$

Below we discuss implications of this statement given by Proposition E.13.

- Every set A carries a unique comonoid structure in $(\mathsf{Set}, \times, \{*\})$. The coproduct $\Delta : A \to A \times A$ is $\Delta(x) = (x, x)$ and the counit $\epsilon : A \to \{*\}$ is $\epsilon(x) = *$.

  It follows that $\Bbbk A$ is a coalgebra in which all elements of A are group-like, that is, $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$ for $x \in A$. This is known as the *coalgebra of a set*.
- If the set A is a monoid (in the usual sense), then it is canonically a bimonoid in $(\mathsf{Set}, \times, \{*\})$, and hence $\Bbbk A$ is a bialgebra.
- A monoid A is a Hopf monoid iff A is a group. (The antipode sends an element to its inverse.) Hence for any group A, the group algebra $\Bbbk A$ is a Hopf algebra. Its antipode is the linearization of the map of $x \mapsto x^{-1}$.

Example 7.4. Let $H = \Bbbk[x, x^{-1}]$ denote the $\Bbbk$-algebra of Laurent polynomials in the variable x. Then the coproduct

$$\Delta : H \to H \otimes H, \quad x^n \mapsto x^n \otimes x^n,$$

and the counit

$$\epsilon : H \to \Bbbk, \quad x^n \mapsto 1,$$

turn H into a Hopf algebra. The antipode is given by

$$\mathrm{S} : H \to H, \quad x^n \mapsto x^{-n}.$$

Let $A = (\mathbb{Z}, +)$ be the group of integers under addition. Then observe that

$$H \xrightarrow{\cong} \Bbbk(\mathbb{Z}), \qquad x \mapsto 1,\ x^{-1} \mapsto -1.$$

Example 7.5. Let $H = \Bbbk[x]/(x^n - 1)$ be the $\Bbbk$-algebra of polynomials in the variable x modulo the equation $x^n - 1$. Thus H is an n-dimensional algebra with basis $1, x, \ldots, x^{n-1}$. The coproduct

$$\Delta : H \to H \otimes H, \quad x^k \mapsto x^k \otimes x^k,$$

and the counit

$$\epsilon : H \to \Bbbk, \quad x^k \mapsto 1,$$

turn H into a Hopf algebra. The antipode is given by

$$\mathrm{S} : H \to H, \quad x^k \mapsto x^{n-k}.$$

Let $A = (\mathbb{Z}_n, +)$ be the group of integers modulo n under addition. Then observe that

$$H \xrightarrow{\cong} \Bbbk(\mathbb{Z}_n), \qquad x \mapsto 1.$$

7.3.3. An adjunction between groups and Hopf algebras. For any set S, let $\Bbbk S$ denote the free module with basis S. We know that if S is a group, then $\Bbbk S$ carries the structure of a Hopf algebra in which every element of S is group-like. Thus, there is a functor

$$\mathsf{Group} \to \mathsf{HopfAlg}_{\Bbbk}, \qquad S \mapsto \Bbbk S. \tag{7.3}$$

This functor goes into the full subcategory of cocommutative Hopf algebras.

If the group S is abelian, then the Hopf algebra $\Bbbk S$ is commutative. Thus, there is an induced functor

$$\mathsf{Ab} \to \mathsf{HopfAlg}_{\Bbbk}^{\mathsf{co}}, \qquad S \mapsto \Bbbk S \tag{7.4}$$

where Ab denotes the category of abelian groups.

Proposition 7.6. *The functors (7.3) and (7.1) are adjoints of each other. Explicitly, for any group S and Hopf algebra H, there is a natural isomorphism*

$$\mathsf{Group}(S, \mathcal{G}(H)) \xrightarrow{\cong} \mathsf{HopfAlg}_{\Bbbk}(\Bbbk S, H). \tag{7.5}$$

Proof. The essential check is: Any group homomorphism $\varphi : S \to \mathcal{G}(H)$ extends uniquely to a morphism $\tilde{\varphi} : \Bbbk S \to H$ of Hopf algebras.

There is no choice but to extend φ linearly to $\tilde{\varphi}$. Since φ is a monoid homomorphism, $\tilde{\varphi}$ is an algebra homomorphism. To check that it is a coalgebra homomorphism: Let $g_i \in S$.

$$\Delta(\tilde{\varphi}(\sum \lambda_i g_i)) = \Delta(\sum \lambda_i \varphi(g_i)) = \sum \lambda_i \Delta(\varphi(g_i)) = \sum \lambda_i \varphi(g_i) \otimes \varphi(g_i).$$

The last step uses that $\varphi(g_i)$ is group-like in H.

$$(\tilde{\varphi} \otimes \tilde{\varphi})\Delta(\sum \lambda_i g_i) = (\tilde{\varphi} \otimes \tilde{\varphi})(\sum \lambda_i g_i \otimes g_i) = \sum \lambda_i \varphi(g_i) \otimes \varphi(g_i).$$

□

Corollary 7.7. *For any Hopf algebra H,*

$$\mathsf{HopfAlg}_{\Bbbk}(\Bbbk\mathbb{Z}, H) \cong \mathcal{G}(H) \tag{7.6}$$

Proof. Put $S := \mathbb{Z}$ in (7.5). Since $\mathbb{Z}$ is the infinite cyclic group, a group homomorphism from $\mathbb{Z}$ to say a group S' corresponds to an element of S' (namely, the image of the generator $1 \in \mathbb{Z}$). □

Proposition 7.8. *Suppose $\Bbbk$ is a field. Then:*

- *For any group S, we have $\mathcal{G}(\Bbbk S) = S$.*
- *The functor (7.3) is fully faithful.*

PROOF. By definition, all elements of S are group-like in the Hopf algebra $\Bbbk S$. Since these form a basis of $\Bbbk S$, there cannot be any more group-like elements by Lemma 7.3. Hence $\mathcal{G}(\Bbbk S) = S$.

The second part is a formal consequence of the first part: [18, Theorem IV.3.1] says that for any adjunction $(\mathcal{F}, \mathcal{G})$, if $\mathcal{G}\mathcal{F} = \mathrm{id}$, then $\mathcal{G}$ is fully faithful. Let us explicitly see how this works. We need to show that for any groups S and S', the map

$$\mathsf{Group}(S, S') \to \mathsf{HopfAlg}_{\Bbbk}(\Bbbk S, \Bbbk S')$$

is a bijection. This follows by putting $H := \Bbbk S'$ in (7.5), and using the first part. Unwinding this argument, the inverse map is constructed as follows. Suppose $\varphi : \Bbbk S \to \Bbbk S'$ is a morphism of Hopf algebras. So it preserves group-likes. By the first part, this induces a map $S \to S'$. Since φ is a morphism of algebras, this map is a group homomorphism. This is the required inverse map. □

Proposition 7.9. *The functors (7.4) and (7.2) are adjoints of each other. Explicitly, for any abelian group S and commutative Hopf algebra H, there is a natural isomorphism*

$$\mathsf{Ab}(S, \mathcal{G}(H)) \xrightarrow{\cong} \mathsf{HopfAlg}^{\mathsf{co}}_{\Bbbk}(\Bbbk S, H).$$

PROOF. Since Ab is a full subcategory of Group, and $\mathsf{HopfAlg}^{\mathsf{co}}_{\Bbbk}$ is a full subcategory of $\mathsf{HopfAlg}_{\Bbbk}$, this adjunction is a consequence of Proposition 7.6. □

Proposition 7.10. *Suppose $\Bbbk$ is a field. Then:*

- *For any abelian group S, we have $\mathcal{G}(\Bbbk S) = S$.*
- *The functor (7.4) is fully faithful.*

PROOF. This follows from Proposition 7.8. □

Lemma 7.11. *Let S be an abelian group. Then: S is finitely generated as an abelian group iff $\Bbbk S$ is finitely generated as a commutative algebra.*

Also true in the nonabelian setting.

PROOF. Forward implication. Suppose U is a finite generating set for S. Then U along with all inverses is a finite generating set for $\Bbbk S$. For instance, as an abelian group $\mathbb{Z}$ is generated by 1, while as a commutative algebra $\Bbbk\mathbb{Z}$ is generated by 1 and -1.

Backward implication. Take a finite set of generators of $\Bbbk S$ and write them in the canonical basis S. Let U be the finite subset of S consisting of elements which appear in these expressions. Let S' be the subgroup of S generated by U. Then $\Bbbk S'$ is a subalgebra of $\Bbbk S$, and since it contains a generating set, $\Bbbk S' = \Bbbk S$. This implies $S' = S$, and hence S is finitely generated. □

7.4. Problems

(1) Do the operations of union and intersection define monoidal structures on Set?
(2) An ideal I of an algebra A is characterized by the property that the quotient A/I is an algebra. The dual notion is that of a coideal of a coalgebra. Make this explicit.

(3) Let $\Bbbk$ be a ring with nontrivial idempotents. Show that group-like elements in a Hopf algebra over $\Bbbk$ need not be linearly independent.

(4) A *primitive* element of H is an element x for which $\Delta(x) = 1 \otimes x + x \otimes 1$. The set of all primitive elements of H is a subspace of H.

Show that a morphism of Hopf algebras preserves primitive elements.

What is the left adjoint to the functor of primitive elements? What would be the analogue of (7.6)?

CHAPTER 8

Affine algebraic groups

In this chapter, we introduce the basic language of affine algebraic groups. Fix a field $\Bbbk$. We do not assume that $\Bbbk$ is algebraically closed or of characteristic 0.

Review standard notions from group theory such as subgroups, normal subgroups, direct and semidirect products, center of a group, nilpotent groups, solvable groups, etc.

8.1. Affine varieties

Let $\Bbbk$ be any field. An affine variety over $\Bbbk$ is a pair $(X, \Bbbk^n)$ where X is the set of solutions of a system of equations in n variables. A morphism $(X, \Bbbk^n) \to (Y, \Bbbk^m)$ is a map $X \to Y$ obtained by restricting a polynomial map $\Bbbk^n \to \Bbbk^m$. This defines the category of affine varieties over $\Bbbk$ which we denote by $\mathsf{AffVariety}(\Bbbk)$.

8.1.1. Products in affine varieties. The category $\mathsf{AffVariety}(\Bbbk)$ is a cartesian monoidal category (meaning that it has finite products). Let us recall this construction.

- Any affine variety of the form $(\{x\}, \Bbbk^n)$ is a terminal object. Fix one of them and call it J. This is the product over the empty set.
- The product of $(X, \Bbbk^n)$ and $(Y, \Bbbk^m)$ is given by $(X \times Y, \Bbbk^{n+m})$. First of all, this is an affine variety: If the polynomials $\{f_i\}$ define X and $\{g_j\}$ define Y, then $\{f_i, g_j\}$ define $X \times Y$. The canonical maps $X \times Y \to X$ and $X \times Y \to Y$ are the projections on the two coordinates.

Existence of the product over the empty set and over the two-element set is same as existence of finite products.

8.1.2. The coordinate ring of an affine variety. The Galois connection between $\mathbb{V}$ and $\mathbb{I}$ in Section 2.2.1 works over any field $\Bbbk$. Affine varieties over $\Bbbk$ are characterized by the property $X = \mathbb{V}(\mathbb{I}(X))$. In general, it seems that there is no good characterization of the ideals with the property $I = \mathbb{I}(\mathbb{V}(I))$. If $\Bbbk$ is algebraically closed, then these are precisely the radical ideals by the Nullstellensatz. The categorical implications of these facts are discussed next.

Recall that to every affine variety $X \subseteq \Bbbk^n$, one can associate its coordinate ring

$$\Bbbk[X] := \Bbbk[x_1, \dots, x_n]/\mathbb{I}(X).$$

If $X \to Y$ is a morphism of varieties, then there is an induced $\Bbbk$-algebra homomorphism $\Bbbk[Y] \to \Bbbk[X]$. This yields a functor

$$\Bbbk[-] : \mathsf{AffVariety} \to (\mathsf{Alg}^{\mathsf{co}})^{\mathrm{op}}. \tag{8.1}$$

The rhs is the opposite of the category of commutative $\Bbbk$-algebras.

Proposition 8.1. *There is a correspondence between X and the set of all $\Bbbk$-algebra homomorphisms $\Bbbk[X] \to \Bbbk$.*

PROOF. Given $x \in X$, the homomorphism $\Bbbk[X] \to \Bbbk$ is obtained by evaluating at x. Conversely, suppose $\varphi : \Bbbk[X] \to \Bbbk$ is a homomorphism. Then φ is evaluation at a point which is in $\mathbb{V}(\mathbb{I}(X))$. But $X = \mathbb{V}(\mathbb{I}(X))$. □

Proposition 8.2. *The functor (8.1) is fully faithful, that is, for any affine varieties X and Y, the map*

$$\mathsf{AffVariety}(X, Y) \xrightarrow{\cong} \mathsf{Alg}^{\mathsf{co}}(\Bbbk[Y], \Bbbk[X])$$

is a bijection.

The inverse map can be understood as follows. Suppose $f : \Bbbk[Y] \to \Bbbk[X]$ is a $\Bbbk$-algebra homomorphism. Then it induces a map

$$\mathsf{Alg}(\Bbbk[X], \Bbbk) \to \mathsf{Alg}(\Bbbk[Y], \Bbbk), \qquad \alpha \mapsto alpha f,$$

which by the correspondence of Proposition 8.1 yields a map $X \to Y$.

Recall that a fully faithful functor gives rise to an equivalence. It remains to describe the image of (8.1).

Theorem 8.3. *Suppose $\Bbbk$ is an algebraically closed field. A commutative $\Bbbk$-algebra A is of the form $\Bbbk[X]$ iff A is finitely generated and reduced.*

PROOF. This is a restatement of Theorem 2.29 (which is valid for any algebraically closed field). □

Thus if $\Bbbk$ is algebraically closed, there is a satisfactory answer. But in the general case, it seems that there is no good description of the image. The best one can say is the following.

Proposition 8.4. *A commutative $\Bbbk$-algebra A is of the form $\Bbbk[X]$ iff A is finitely generated and no nonzero element of A goes to zero under all $\Bbbk$-algebra homomorphisms $A \to \Bbbk$ (and so in particular A is reduced).*

PROOF. Forward implication is clear. Backward implication: Since A is finitely generated, $A \cong \Bbbk[x_1, \ldots, x_n]/I$ for some n and some ideal I. Let $X \subseteq \Bbbk^n$ be the zero set of I. Note that any homomorphism $A \to \Bbbk$ is evaluation at a point of X. So by the second hypothesis, the vanishing ideal of X must be I. Thus $\Bbbk[X] \cong A$. □

Proposition 8.5. *Suppose $A \neq 0$ is a finitely generated commutative $\Bbbk$-algebra, and there is no $\Bbbk$-algebra homomorphism $A \to \Bbbk$. Then A is not of the form $\Bbbk[X]$.*

PROOF. This is a special case of Proposition 8.4, every nonzero element of A satisfies the homomorphism condition vacuously. Let us spell this out.

Suppose A is of the form $\Bbbk[X]$. Then elements of X correspond to homomorphisms $A \to \Bbbk$. So by hypothesis, $X = \emptyset$, which implies $A \cong 0$. This is a contradiction. □

8.1.3. The $\Bbbk$-points of a commutative $\Bbbk$-algebra. Recall that the set of $\Bbbk$-points of a commutative $\Bbbk$-algebra A is the set of all $\Bbbk$-algebra homomorphisms $A \to \Bbbk$. We look at situations in which A is and is not determined by its $\Bbbk$-points.

Definition 8.6. Let A be a finitely generated $\Bbbk$-algebra. We say that A is determined by its $\Bbbk$-points if A is of the form $\Bbbk[X]$ for some affine variety X. Equivalently, A is determined by its $\Bbbk$-points if for $A = \Bbbk[x_1, \ldots, x_n]/I$, the ideal I has the property $I = \mathbb{I}(\mathbb{V}(I))$.

Some simple negative examples are given below.

- Consider the $\mathbb{R}$-algebra $A = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$. Then there are no $\mathbb{R}$-algebra homomorphisms $A \to \mathbb{R}$. So A cannot be the coordinate ring of any affine variety over $\mathbb{R}$.
- Consider the $\mathbb{C}$-algebra $A = \mathbb{C}[x]/(x^2)$. Then there is exactly one $\mathbb{C}$-algebra homomorphisms $A \to \mathbb{C}$, namely $1 \mapsto 1$ and $x \mapsto 0$. But $x \neq 0$ in A. So A cannot be the coordinate ring of any affine variety over $\mathbb{C}$. Also note that A is not reduced and x is a nonzero nilpotent element.
- Let $A = \Bbbk[x]/(x^n - 1)$. Suppose $\Bbbk = \mathbb{R}$. Then $X := \mathbb{V}(x^n - 1)$ is either $\{1\}$ or $\{1, -1\}$ depending on whether n is odd or even. In the former case, $\mathbb{R}[X] = \mathbb{R}$, while in the latter case $\mathbb{R}[X] = \mathbb{R}[x]/(x^2 - 1)$. Thus we see that A is not determined by its $\mathbb{R}$-points.

  In contrast, if $\Bbbk := \mathbb{C}$, then A is determined by its $\mathbb{C}$-points.

Now we give some positive examples.

Lemma 8.7. *Suppose $\Bbbk$ is an infinite field. Then a nontrivial polynomial in $\Bbbk[x_1, \ldots, x_n]$ cannot vanish on all points of $\Bbbk^n$.*

Proof. Let f be a nontrivial polynomial. We proceed by induction on n. Suppose $n = 1$. Then f can have only finitely many zeroes, since each zero corresponds to a linear factor of f. But $\Bbbk$ is assumed to be infinite. So f cannot vanish at all points of $\Bbbk$. This is the induction base.

For $n > 1$, write f as a polynomial in x_n, with coefficients in $\Bbbk[x_1, \ldots, x_{n-1}]$. By the induction hypothesis, there is a point $(a_1, \ldots, a_{n-1}) \in \Bbbk^{n-1}$ such that $f(a_1, \ldots, a_{n-1}, x_n)$ is a nontrivial polynomial in the variable x_n. Now by the $n = 1$ case, there is a a_n such that $f(a_1, \ldots, a_{n-1}, a_n) \neq 0$. So f cannot vanish on all points of $\Bbbk^n$. This completes the induction step. □

Lemma 8.8. *Suppose $\Bbbk$ is an infinite field. Let h be a nontrivial polynomial in $x_1, \ldots, x_n$. Then no nontrivial polynomial in these variables can vanish on the basic open set $X_h := \{x \in \Bbbk^n \mid h(x) \neq 0\}$.*

Proof. Suppose f is a nontrivial polynomial in $x_1, \ldots, x_n$ which vanishes on X_h. Then fh is a nontrivial polynomial which vanishes on $\Bbbk^n$. This is a contradiction to Proposition 8.7. □

Theorem 8.9. *Suppose $\Bbbk$ is an infinite field. The following $\Bbbk$-algebras are determined by their $\Bbbk$-points.*

(1) $\Bbbk[x]$.
(2) $\Bbbk[x_1, \ldots, x_n]$.
(3) $\Bbbk[x, y]/(xy - 1)$.
(4) $\Bbbk[(x_{ij}), y]/(y \det(x_{ij}) - 1)$, *with* $1 \leq i, j \leq n$.

PROOF. (2) reduces to Lemma 8.7, and (1) is its $n = 1$ case.

For (3), put $A := \Bbbk[x, y]/(xy - 1)$. Note that $\Bbbk$-algebra homomorphisms $A \to \Bbbk$ correspond to nonzero elements of $\Bbbk$. Now let f be a nonzero element of A. Write f as a polynomial in x multiplied by a power of y. Then by the $n = 1$ case of Lemma 8.8, this polynomial in x cannot vanish on all $x \neq 0$, which amounts to saying that f cannot vanish on all $A \to \Bbbk$.

For (4), we argue as in (3). □

8.1.4. Problems.

(1) Let $\Bbbk$ be a finite field. Show that every subset of $\Bbbk^n$ is an affine variety, and any set map between such subsets is a morphism of affine varieties.
(2) Suppose $\Bbbk$ is the finite field with q elements, and $\mathbb{A}$ is the affine line over $\Bbbk$. Show that $\mathbb{I}(\mathbb{A}) = (x^q - x) \subseteq \Bbbk[x]$, and hence $\Bbbk[\mathbb{A}] = \Bbbk[x]/(x^q - x)$.
(3) Show that Proposition 2.10 works over any field $\Bbbk$.
(4) Show that $A := \Bbbk[(x_{ij})]/(\det(x_{ij}) - 1)$, with $1 \leq i, j \leq n$ is determined by its $\Bbbk$-points.
(5) Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$ whose zero set $\mathbb{V}(f)$ in $\mathbb{R}^2$ is nonempty and not irreducible as an affine variety over $\mathbb{R}$.
(6) Show that the affine varieties $\mathbb{V}(x^2 + y^2 - 1)$ and $\mathbb{V}(xy - 1)$ are isomorphic over $\mathbb{C}$ but not isomorphic over $\mathbb{R}$.
(7) Let $\Bbbk$ be any field. Consider the $\Bbbk$-algebra

$$R := \Bbbk[x, y]/(x^2 + 2xy - 4y^2).$$

Give an example of $\Bbbk$ for which
   (a) R is not determined by its $\Bbbk$-points.
   (b) R is determined by its $\Bbbk$-points.

8.2. Affine algebraic groups

Recall that to any cartesian monoidal category, one can associate the category of its group objects (Definition E.14). Let us specialize this to affine varieties.

Definition 8.10. An *affine algebraic group*, aag for short, is a group object in the category of affine varieties. A morphism between affine algebraic groups is a morphism between group objects. We denote AAG denote the category of affine algebraic groups.

Observe that the forgetful functor $\mathsf{AffVariety} \to \mathsf{Set}$ preserves products, and hence by Proposition E.16 induces a functor on the category of group objects.

$$\mathsf{AAG} = \mathsf{Group}(\mathsf{AffVariety}) \to \mathsf{Group} = \mathsf{Group}(\mathsf{Set}).$$

In particular, the underlying set of an aag is a group, and the underlying map of a morphism of aags is a group homomorphism. This leads to the following explicit description of AAG.

Proposition 8.11. *An aag is an affine variety G which is also a group such that*

$$G \times G \to G \quad (g, h) \mapsto gh, \qquad \text{and} \qquad G \to G \quad g \mapsto g^{-1}$$

are morphisms of affine varieties.

A morphism between aags is a morphism $G \to H$ of affine varieties which is also a group homomorphism.

(Presumably, if the multiplication is a morphism, then the inverse is automatically a morphism.)

PROOF. Suppose G is a group object in $\mathsf{AffVariety}$. So G is an affine variety, and by the forgetful functor, G is also a group (in the usual sense). The group operations must clearly be morphisms of affine varieties. The required commutative diagrams for G are now automatically satisfied since two morphism in $\mathsf{AffVariety}$ are equal iff their underlying maps in Set are equal. This proves the first part.

A morphism between aags is a morphism $G \to H$ of group objects. So this is a morphism of affine varieties, and by the forgetful functor, $G \to H$ is a group homomorphism. The following diagrams must commute

$$\begin{array}{ccc} G \times G & \longrightarrow & H \times H \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array} \qquad\qquad \begin{array}{ccc} G & & \\ & \searrow & \\ & & H \\ & \nearrow & \\ J & & \end{array}$$

in $\mathsf{AffVariety}$. Using the same comment as above, this is automatic since the diagrams commute in Set. □

Remark 8.12. In general, the topology on $G \times G$ is not the product topology, and G is *not* a topological group. (Any topological group is automatically Hausdorff.)

Let us now consider the coordinate ring of an aag. An important property of the functor (8.1) is that it preserves products: For any affine varieties X and Y, there is a natural isomorphism

$$\Bbbk[X \times Y] \cong \Bbbk[X] \otimes \Bbbk[Y]. \tag{8.2}$$

(Recall that the tensor product is the categorical coproduct in the category of commutative algebras, and hence the categorical product in the opposite category.)

Recall from (E.15) that a cogroup object in the category of commutative algebras is same as a commutative Hopf algebra. More precisely,

$$\mathsf{Cogroup}(\mathsf{Alg}^{\mathsf{co}}) \cong \mathsf{HopfAlg}^{\mathsf{co}}$$

where the latter is the category of commutative Hopf algebras over $\Bbbk$. Passing to the opposite categories,

$$\mathsf{Group}((\mathsf{Alg}^{\mathsf{co}})^{\mathrm{op}}) \cong (\mathsf{HopfAlg}^{\mathsf{co}})^{\mathrm{op}}.$$

Since the functor (8.1) preserves products, by Proposition E.16 it induces a functor on the category of group objects:

$$\Bbbk[-] : \mathsf{AAG} \to (\mathsf{HopfAlg}^{\mathsf{co}})^{\mathrm{op}}. \tag{8.3}$$

Thus, for any affine algebraic group G, its coordinate ring $\Bbbk[G]$ is a commutative Hopf algebra, and a morphism of $G \to H$ of aags induces a morphism $\Bbbk[H] \to \Bbbk[G]$ of commutative Hopf algebras.

Explicitly, the coproduct of $\Bbbk[G]$ is given by the composite map

$$\Bbbk[G] \to \Bbbk[G \times G] \xrightarrow{\cong} \Bbbk[G] \otimes \Bbbk[G].$$

The first map is obtained by applying (8.1) to the multiplication $G \times G \to G$, while the second map is obtained from (8.2). The antipode of $\Bbbk[G]$ is obtained by applying (8.1) to the inverse map $G \to G$.

8.3. Examples

In this section, we discuss many examples of aags. In most of then, we first specify an ideal $I \subseteq \Bbbk[x_1, \ldots, x_n]$ and then define the aag as $\mathbb{V}(I)$. We then say that the coordinate ring of the aag is $A := \Bbbk[x_1, \ldots, x_n]/I$. This passage requires knowing that A is determined by its $\Bbbk$-points. In our examples, this is made possible by Theorem 8.9. For that purpose, throughout this section, we assume that $\Bbbk$ is an infinite field.

Example 8.13. Any singleton $\{*\}$ with its unique group structure is an aag. Its coordinate ring is $\Bbbk$. It is a commutative and cocommutative Hopf algebra. The product and coproduct are given by the canonical identifications:

$$\Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk, \quad \text{and} \quad \Bbbk \xrightarrow{\cong} \Bbbk \otimes \Bbbk.$$

The unit, counit and antipode are all the identity map $\Bbbk \to \Bbbk$.

This is an instance of a general principle: The unit object in any monoidal category is a Hopf monoid and in particular, a (co, bi)monoid. Further, if the monoidal category is (co)cartesian, then the unit object is the (initial) terminal object, and it is a (co)group object.

Example 8.14. Consider the affine variety $G_a := \Bbbk$. It is an aag with the (abelian) group structure being addition:

$$G_a \times G_a \to G_a, \quad (b, b') \mapsto b + b'.$$

Its coordinate ring is $\Bbbk[x]$. (This simple fact requires $\Bbbk$ to be infinite.) It is a commutative and cocommutative Hopf algebra. The coproduct is given by

$$\Delta : \Bbbk[x] \to \Bbbk[x] \otimes \Bbbk[x], \quad x^n \mapsto \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-1}.$$

To check that we dualized correctly:

$$\langle \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-1}, (b, b') \rangle = \sum_{i=0}^{n} \binom{n}{i} b^i (b')^{n-i} = (b + b')^n = \langle x^n, b + b' \rangle.$$

(This is the binomial theorem!) In particular, $\Delta(x) = 1 \otimes x + x \otimes 1$. Thus x is a primitive element of $\Bbbk[x]$.

The counit is given by

$$\epsilon : \Bbbk[x] \to \Bbbk, \quad 1 \mapsto 1, \ x^n \mapsto 0 \text{ for } n \geq 1.$$

The antipode is given by

$$\mathrm{S} : \Bbbk[x] \to \Bbbk[x], \quad x^n \mapsto (-1)^n x^n.$$

It is instructive to check explicitly that $\Bbbk[x]$ with product, coproduct and antipode as defined satisfies all the axioms of a Hopf algebra. For instance, the emergence of the binomial coefficients in the coproduct can be seen from

$$\Delta(x^n) = \Delta(x) \ldots \Delta(x) = (1 \otimes x + x \otimes 1)^n.$$

Example 8.15. Consider the affine variety $\Bbbk^n$. It is an aag with the (abelian) group structure being addition:

$$\Bbbk^n \times \Bbbk^n \to \Bbbk^n, \quad ((b_i), (b'_i)) \mapsto (b_i + b'_i).$$

The cases $n = 0, 1$ specialize to the previous two examples. The coordinate ring $\Bbbk[x_1, \dots, x_n]$ is a commutative and cocommutative Hopf algebra. The coproduct is given by

$$\Delta : \Bbbk[x_1, \dots, x_n] \to \Bbbk[x_1, \dots, x_n] \otimes \Bbbk[x_1, \dots, x_n], \quad x_i \mapsto 1 \otimes x_i + x_i \otimes 1.$$

In other words, all the generators are primitive. Since $\Bbbk[x_1, \dots, x_n]$ is a free commutative algebra, and Δ is a morphism of algebras, knowing Δ on the generators determines it uniquely. For instance,

$$\Delta(x_1 x_2) = 1 \otimes x_1 x_2 + x_1 \otimes x_2 + x_2 \otimes x_1 + x_1 x_2 \otimes 1.$$

The counit is given by

$$\epsilon : \Bbbk[x_1, \dots, x_n] \to \Bbbk, \quad 1 \mapsto 1, \ x_i \mapsto 0.$$

The antipode is given by

$$\mathrm{s} : \Bbbk[x_1, \dots, x_n] \to \Bbbk[x_1, \dots, x_n], \quad x_i \mapsto -x_i.$$

You should write down the formula on any monomial.

Example 8.16. Consider the affine variety $G_m := \mathbb{V}(xy - 1)$. It is an aag under multiplication:

$$G_m \times G_m \to G_m, \quad ((a, b), (a', b')) \mapsto (aa', bb').$$

Since this map is defined by polynomials, it is a morphism of affine varieties. This turns G_m into an abelian group: the identity element is $(1, 1)$, and the inverse map is

$$G_m \to G_m, \quad (a, b) \mapsto (b, a).$$

This is also a morphism of affine varieties. So G_m is an aag.

The coordinate ring $\Bbbk[G_m] = \Bbbk[x, y]/(xy - 1)$ is a commutative Hopf algebra. The (cocommutative) coproduct is given by

$$\Delta : \Bbbk[G_m] \to \Bbbk[G_m] \otimes \Bbbk[G_m], \quad x \mapsto x \otimes x, \ y \mapsto y \otimes y.$$

To see that we dualized correctly:

$$\langle x \otimes x, ((a, b), (a', b')) \rangle = aa' = \langle x, (aa', bb') \rangle.$$

Thus x and y are group-like elements of $\Bbbk[G_m]$. The counit is given by

$$\epsilon : \Bbbk[G_m] \to \Bbbk, \quad x \mapsto 1, \ y \mapsto 1.$$

The antipode is given by

$$\mathrm{s} : \Bbbk[G_m] \to \Bbbk[G_m], \quad x \mapsto y, \ y \mapsto x.$$

It is instructive to directly check that $\Bbbk[G_m]$ satisfies all the axioms of a Hopf algebra. This Hopf algebra is often denoted $\Bbbk[x, x^{-1}]$, see Example 7.4. It is obtained by linearizing the group of integers under addition.

Example 8.17. Consider the affine space $\Bbbk^{n^2+1}$ in which the first n^2 coordinates are indexed by $\{x_{ij}\}$, for $1 \le i, j \le n$, and the last coordinate is indexed by y. Define

$$GL_n := \mathbb{V}(y \det(x_{ij}) - 1).$$

Note that elements (points) of GL_n correspond to a matrix (a_{ij}) whose determinant is not 0 (or equivalently, to invertible linear endomorphisms of $\Bbbk^n$). (The last coordinate b is the inverse of the determinant.)

Note that $GL_1 = G_m$. The present discussion can be viewed as a generalization of that example. Also it makes sense to put $GL_0 = \{*\}$, since there is the identity linear endomorphisms of the 0 space. This will yield the first example.

GL_n is an aag under matrix multiplication:

$$GL_n \times GL_n \to GL_n, \quad ((a_{ij}, b), (a'_{ij}, b')) \mapsto (\sum_k a_{ik}a'_{kj}, bb').$$

Since this map is defined by polynomials, it is a morphism of affine varieties. The point $(I, 1)$, where I the identity matrix of size n, is the identity element for this multiplication. Recall that for an invertible matrix A, the entries of A^{-1} are expressible as polynomials in the a_{ij}'s and the inverse of $\det(A)$. This is known as Cramer's rule. Further, $\det(A^{-1}) = \det(A)^{-1}$. So the inverse $GL_n \to GL_n$ is also a morphism of affine varieties. Thus GL_n is an aag. This is known as the *general linear group*.

The coordinate ring $\Bbbk[GL_n] = \Bbbk[(x_{ij}), y]/(y\det(x_{ij}) - 1)$ is a commutative Hopf algebra. The coproduct is given by

$$\Delta : \Bbbk[GL_n] \to \Bbbk[GL_n] \otimes \Bbbk[GL_n], \quad x_{ij} \mapsto \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \ y \mapsto y \otimes y.$$

(You should check that this is the correct dualization.) This is not cocommutative reflecting the fact that matrix multiplication is not commutative.

The counit is given by

$$\epsilon : \Bbbk[GL_n] \to \Bbbk, \quad y \mapsto 1, \ x_{ii} \mapsto 1, \ x_{ij} \mapsto 0 \text{ for } i \neq j.$$

The antipode is given by

$$\mathrm{s} : \Bbbk[GL_n] \to \Bbbk[GL_n], \quad x_{ij} \mapsto (-1)^{i+j} y \det(-), \ y \mapsto \det(x_{ij}),$$

where $-$ stands for the matrix (x_{ij}) with the ith column and the jth row deleted.

Observe that

$$GL_n \to G_m, \qquad ((a_{ij}), b) \mapsto (\det(a_{ij}), b)$$

is a morphism of aags.

Example 8.18. Consider the affine space $\Bbbk^{n^2}$ in which the coordinates are indexed by $\{x_{ij}\}$, for $1 \leq i, j \leq n$. Define

$$SL_n := \mathbb{V}(\det(x_{ij}) - 1).$$

This is known as the *special linear group*. It is an aag under matrix multiplication. This can be checked directly or can also be deduced by noting that SL_n is isomorphic to a closed subgroup of GL_n under the map

$$SL_n \to GL_n, \qquad (a_{ij}) \mapsto ((a_{ij}), 1).$$

Thus we have a diagram of morphisms of aags

$$SL_n \hookrightarrow GL_n \twoheadrightarrow G_m,$$

with SL_n being the kernel of the second map.

Example 8.19. Observe that a closed subgroup of an aag is again an aag. This gives us further examples. For instance, the following are all closed subgroups of GL_n.

- the group D_n of all invertible diagonal matrices of size n.

- the group T_n of all invertible upper triangular matrices of size n.
- the group U_n of all invertible upper triangular matrices of size n with all diagonal entries 1.

The coordinate ring $\Bbbk[D_n]$ is a commutative Hopf algebra with a basis of group-like elements indexed by n-tuples of integers. More precisely, $\Bbbk[D_n] \cong \Bbbk\mathbb{Z}^n$.

8.4. Subgroups

We begin with a basic observation.

Lemma 8.20. *Suppose G is an aag. For any $g \in G$, the following are isomorphisms of affine varieties.*

$$\lambda_g : G \to G \quad x \mapsto gx, \qquad \rho_g : G \to G \quad x \mapsto xg, \qquad \kappa_g : G \to G \quad x \mapsto gxg^{-1}.$$

In particular, they are homeomorphisms of the underlying topology of G.

Proof. They can all be expressed as composites of morphisms involving the product and inverse of G. So they are morphisms. Furthermore, note that $\lambda_{g^{-1}}$ is the inverse to λ_g, etc; so they are isomorphisms. □

Recall:

- For any continuous map $\varphi : X \to Y$ between topological spaces and any subset U of X, we have $\varphi(\overline{U}) \subseteq \overline{\varphi(U)}$.
- For any homeomorphism $\varphi : X \to X$ and any subset U of X, we have $\overline{\varphi(U)} = \varphi(\overline{U})$.

We will now apply this to the homeomorphisms of Lemma 8.20.

Lemma 8.21. *Suppose G is an aag, and H is a subgroup of G. Then*

1. *The (Zariski) closure $\overline{H}$ is a subgroup of G. In particular: If H is (Zariski) closed, then H is an aag.*
2. *If $K \subseteq H$ is a subgroup with K normal in H, then $\overline{K}$ is normal in $\overline{H}$.*
3. *If A, B and C are subsets with $[A, B] \subseteq C$, then $[\overline{A}, \overline{B}] \subseteq \overline{C}$. Further, if C is a subgroup, then $(\overline{A}, \overline{B}) \subseteq \overline{C}$.*
4. *If H is abelian, nilpotent or solvable, then so is $\overline{H}$.*
5. *If there exists a $U \leq H$ such that U is open and dense in $\overline{H}$, then $\overline{H} = U \cdot U = H$, where $U \cdot U = \{u_1 u_2 \mid u_1, u_2 \in U\}$.*

Here $[A, B] = \{aba^{-1}b^{-1} \mid a \in A, b \in B\}$ and (A, B) is the subgroup generated by $[A, B]$.

This result is also given in [26, Section 4.3].

Proof. (1). For any $x \in H$, $x\overline{H} = \overline{xH} = \overline{H}$. Thus for any $y \in \overline{H}$, $Hy \subseteq \overline{H}$. Hence $\overline{H}y = \overline{Hy} \subseteq \overline{H}$. Therefore $\overline{H}\,\overline{H} = \overline{H}$. Also $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$.

(2). For any $x \in H$, $xKx^{-1} \subseteq K \subseteq \overline{K}$, so $x\overline{K}x^{-1} = \overline{xKx^{-1}} \subseteq \overline{K}$. Now for a fixed $y \in \overline{K}$, the map

$$G \to G, \quad x \mapsto xyx^{-1}$$

takes H into $\overline{K}$, and hence takes $\overline{H}$ into $\overline{K}$.

(3). Similar to (2).

(4). Suppose H is abelian. Then $[H,H] = \{e\}$ whose closure is itself. So $[\overline{H},\overline{H}] = \{e\}$ which says that $\overline{H}$ is abelian. Now suppose H is nilpotent. Then the lower central series

$$H \supseteq (H,H) \supseteq (H,(H,H)) \supseteq (H,(H,(H,H))) \supseteq \dots$$

terminates at $\{e\}$. Then the lower central series of $\overline{H}$ also terminates at $\{e\}$: a typical term is contained in the closure of the corresponding term for H. So $\overline{H}$ is nilpotent. The argument for solvable is similar.

(5). For any $x \in \overline{H}$, the open set xU^{-1} must meet the dense open set U, so $x \in U \cdot U$, as required. □

Lemma 8.22. *Let G be an aag. Then the irreducible and connected components of G coincide, and there are finitely many of them.*

Let G^0 denote the connected component containing the unit e. Then G^0 is a closed, normal subgroup of finite index in G, and the cosets of G^0 in G are the connected components of G.

Proof. Any aag is a Noetherian space. So by Proposition 1.22, we can decompose G uniquely into finitely many irreducible components: $G = \bigcup_{j=1}^{m} X_m$. Pick $x \in X_1$ such that x is not in any of the remaining X_i's. Let z be any element of G. The homeomorphism $y \mapsto zx^{-1}y$ takes x to z. So z must belong to exactly one irreducible component. This shows that the irreducible components are disjoint and they equal the connected components.

Since G^0 is a connected component, it is closed. For any $x \in G^0$, by Lemma 8.20, xG^0 and $x^{-1}G^0$ are connected components and they both meet G^0, hence $xG^0 = x^{-1}G^0 = G^0$. This shows G^0 is closed under taking products and inverses, and hence it is a subgroup of G. Further, for any $y \in G$, by Lemma 8.20, yG^0y^{-1} is a connected component and it meets G^0, hence $yG^0y^{-1} = G^0$. This shows that G^0 is a normal subgroup. Again by Lemma 8.20, each coset yG^0 is a connected component, and since the cosets exhaust G, every connected component has this form. Since there are finitely many of them, we deduce that G^0 has finite index in G. □

In general, irreducible implies connected but not vice-versa. But for an aag, the two notions coincide. An aag G is called *connected* if $G = G^0$.

8.5. Modules over an affine algebraic group

Recall that a (left) module of a group object A in a cartesian category C is a (left) module of the underlying monoid. This defines the category $\mathsf{Mod}_A(\mathsf{C})$.

An aag is a group object in $\mathsf{AffVariety}$. So for any aag G, we have the category $\mathsf{Mod}_G(\mathsf{AffVariety})$. An explicit description G-modules and morphisms between G-modules is given below.

Proposition 8.23. *Suppose G is an aag. Then a (left) module over G is an affine variety X such that the group G acts (on the left) on the underlying set of X and the map*

$$G \times X \to X$$

is a morphism of affine varieties.

A morphism between (left) modules X and Y over G is a morphism $X \to Y$ of affine varieties which is also a map of (left) G-sets.

PROOF. Similar to the proof of Proposition 8.11. □

In this situation, we say that the aag G acts (on the left) on the affine variety X.

Example 8.24. Any aag G acts on itself (on the left) by left translation:

$$G \times G \to G, \qquad (g,h) \mapsto gh.$$

Similarly, G acts on itself (on the left) by right translation: $(g,h) \mapsto hg^{-1}$, and by conjugation: $(g,h) \mapsto ghg^{-1}$.

Let G be an aag and $\Bbbk[G]$ be the commutative Hopf algebra of functions on G. Recall from (8.2) that the functor $\Bbbk[-]$ is monoidal. Hence there is an induced contravariant functor

$$\mathsf{Mod}_G(\mathsf{AffVariety}) \to \mathsf{Comod}^{\Bbbk[G]}(\mathsf{Alg}^{\mathsf{co}}).$$

The latter is the category of (left) comodules of the Hopf algebra $\Bbbk[G]$. In particular, if G acts (on the left) on X, then $\Bbbk[G]$ coacts (on the left) on $\Bbbk[X]$:

$$\Bbbk[X] \to \Bbbk[G] \otimes \Bbbk[X].$$

The preceding discussion is valid if left is replaced by right. When dealing with group objects, the categories of left modules and right modules are equivalent, hence whether one wants to work with left or right is a matter of convention.

Example 8.25. Put $V := \Bbbk^n$. There is a canonical (right) action of GL_n on V:

$$V \times GL_n \to V, \qquad ((v_i),(a_{ij})) \mapsto (\sum_i v_i a_{ij}). \tag{8.4}$$

This induces a (right) coaction of $\Bbbk[GL_n]$ on $\Bbbk[V]$.

$$\Bbbk[V] \to \Bbbk[V] \otimes \Bbbk[GL_n], \qquad x_i \mapsto \sum_j x_j \otimes x_{ji}. \tag{8.5}$$

This is a morphism of $\Bbbk$-algebras, hence specifying it on the generators x_i suffices.

More generally, any closed subgroup G of GL_n acts on V, and the corresponding Hopf algebra $\Bbbk[G]$ coacts on $\Bbbk[V]$.

Theorem 8.26. *Suppose $\Bbbk$ is an infinite field. Let G be a subgroup of GL_n (over $\Bbbk$). Then the elements of G can be simultaneously diagonalized iff the Hopf algebra $\Bbbk[\overline{G}]$ has a basis of group-like elements.*

PROOF. Forward implication. By conjugating (which is an isomorphism), we may assume that G is a subgroup of D_n. Since D_n is a closed subgroup of GL_n, it follows that $\overline{G}$ is also a subgroup of D_n. So $\Bbbk[\overline{G}]$ is a quotient Hopf algebra of $\Bbbk[D_n]$. Since the latter is spanned by group-likes, so is any quotient of it.

Backward implication. Let $\{b_i\}$ be the basis of group-likes of $\Bbbk[\overline{G}]$. Since G is a subgroup of GL_n, there is an induced action of G on V, and an induced coaction of $\Bbbk[G]$ on $\Bbbk[V]$. Observe from (8.5) that the coaction preserves the linear part of $\Bbbk[V]$ spanned by the x_i, so in fact, there is a map

$$\rho : V^* \to V^* \otimes \Bbbk[GL_n] \to V^* \otimes \Bbbk[G].$$

For $f \in V^*$, write $\rho(f) = \sum_i f_i \otimes b_i$. We deduce from here that $\rho(f_i) = f_i \otimes b_i$ and $f = \sum_i f_i$. By repeating this procedure, we can construct a basis of V^*, say

$h_1, \ldots, h_n$ such that for each h_i, the coaction has the form $\rho(h_i) = h_i \otimes b_{k_i}$ with b_{k_i} being some group-like element of $\Bbbk[G]$.

Now let $v_1, \ldots, v_n$ be the basis of V dual to $h_1, \ldots, h_n$. Then for any $g \in G$, $g \cdot v_i = b_{k_i}(g)v_i$. Thus, in this basis, all elements of G are diagonal matrices. □

8.6. Linearity of affine algebraic groups

Definition 8.27. An affine algebraic group or aag is called a *linear algebraic group*, lag for short, if it is isomorphic to a closed subgroup of $GL_n(\Bbbk)$ for some n.

We want to show that any aag is isomorphic to a lag. Let us try to understand what this amounts to.

For any $g \in G$, consider the isomorphism $\lambda_g : G \to G$ of affine varieties. This induces an isomorphism $\lambda_g^* : \Bbbk[G] \to \Bbbk[G]$. This yields a group homomorphism

$$G \to GL_{\Bbbk[G]}.$$

The problem is that $\Bbbk[G]$ is infinite-dimensional. So the task is to find a finite-dimensional G-invariant subspace of $\Bbbk[G]$.

Recall that for any variety X, elements (points) of X correspond to $\Bbbk$-algebra homomorphisms $f : \Bbbk[G] \to \Bbbk$. Let G be an aag. The identity element of G corresponds to the counit $\epsilon : \Bbbk[G] \to \Bbbk$. The product in G corresponds to the convolution product on $\mathsf{Alg}^{\mathsf{co}}(\Bbbk[G], \Bbbk)$.

Now view $g : \Bbbk[G] \to \Bbbk$. Then λ_g^* coincides with the composite:

$$\Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G] \xrightarrow{\mathrm{id} \otimes g} \Bbbk[G] \otimes \Bbbk \xrightarrow{\cong} \Bbbk[G].$$

This is because composing the above composite map with $f : \Bbbk[G] \to \Bbbk$ is precisely the convolution product $f * g$. So equivalently, the task is to find a finite-dimensional subcomodule of $\Bbbk[G]$ (viewed as a comodule over itself).

Lemma 8.28. *Let $\Bbbk$ be a field. Suppose V is a comodule of a coalgebra A over $\Bbbk$. Let S be a finite subset of V. Then there exists a finite-dimensional subcomodule of V containing S.*

Proof. Let $\rho : V \to V \otimes A$ denote the comodule map. Since a sum of subcomodules is again a subcomodule, we may assume that $S = \{v\}$ is a singleton set. Pick a basis $\{a_i\}$ of A and write $\rho(v) = \sum_i v_i \otimes a_i$, where all but finitely many v_i are zero. Write $\Delta(a_i) = \sum_{j,k} r^i_{jk} a_j \otimes a_k$. Then, applying the coassociativity diagram of ρ to v, we get

$$\sum_i \rho(v_i) \otimes a_i = \sum_{j,k} v_i \otimes r^i_{jk} a_j \otimes a_k.$$

Comparing the coefficients of a_k, we obtain $\rho(v_k) = \sum_{i,j} v_i \otimes r^i_{jk} a_j$. Hence the finite-dimensional subspace spanned by v and the v_i is a subcomodule of V. □

Theorem 8.29. *Suppose H is a commutative Hopf algebra and V is a vector space of dimension $n \geq 0$. Then the following are equivalent.*

1. *A morphism $\Bbbk[x_{11}, \ldots, x_{nn}, 1/\det] \to H$ of Hopf algebras.*
2. *A right H-comodule structure on V.*

Proof. If $n = 0$, then there is a unique morphism $\Bbbk \to H$ of Hopf algebras, and there is a unique H-comodule structure on $V = 0$ (the comodule map must be the zero map).

The general case is as follows. Let $\{v_1, \ldots, v_n\}$ be a basis for V. The Hopf algebra morphism

$$\Bbbk[x_{11}, \ldots, x_{nn}, 1/\det] \twoheadrightarrow H, \qquad x_{ij} \mapsto a_{ij} \tag{8.6}$$

corresponds to the comodule structure

$$\rho(v_j) = \sum_i v_i \otimes a_{ij}. \tag{8.7}$$

It needs to be checked that this correspondence works. This can be done directly, or deduced from Proposition G.1 and Theorem G.2 (where these ideas are developed in a more abstract setting). □

Theorem 8.30. *Let H be a finitely generated Hopf algebra over a field $\Bbbk$. Then there exists a surjective morphism of Hopf algebras of the form*

$$\Bbbk[x_{11}, \ldots, x_{nn}, 1/\det] \twoheadrightarrow H$$

for some n.

Proof. View H as a comodule over itself. By Lemma 8.28, there exists a finite-dimensional subcomodule V of H which contains the algebra generators of H. Let $\{v_1, \ldots, v_n\}$ be a basis for V and write $\rho(v_j) = \sum_i v_i \otimes a_{ij}$ where ρ is the restriction of Δ to V. Then by the preceding discussion,

$$\Bbbk[x_{11}, \ldots, x_{nn}, 1/\det] \twoheadrightarrow H, \qquad x_{ij} \mapsto a_{ij}$$

is a morphism of Hopf algebras. It remains to show that this map is surjective. The counitality condition on Δ implies $v_j = \sum_i \epsilon(v_i) a_{ij}$. So the image of the above map contains v_j, and hence is all of A. □

Corollary 8.31. *Any aag is isomorphic to a lag (the base field $\Bbbk$ is assumed to be infinite).*

Proof. Let G be an aag. Then $\Bbbk[G]$ is a finitely generated commutative Hopf algebra. By Theorem 8.30, there is a surjective morphism of Hopf algebras $\Bbbk[GL_n] \twoheadrightarrow \Bbbk[G]$ for some n, or equivalently, G is isomorphic to a closed subgroup of GL_n. □

The proof given above is constructive. The algorithm to construct a faithful representation of an aag G is summarized below.

- Let $H = \Bbbk[G]$. Pick a finite set S of algebra generators for H. Also pick a basis $\{a_i\}$ of H.
- For each $s \in S$, write $\Delta(s) = \sum_i s_i \otimes a_i$ for unique s_i. Let V be the finite-dimensional subspace of H spanned by all the s and nonzero s_i. We are guaranteed that V is a right subcomodule of H.
- Pick a basis $\{v_1, \ldots, v_n\}$ of V and define the matrix (a_{ij}) using (8.7).
- The map
$$G \to GL_n, \qquad \alpha \mapsto (a_{ij}(\alpha))_{1 \le i,j \le n}.$$
is the required injective morphism of aags.

Example 8.32. Take $G := G_a$. Then $H = \Bbbk[x]$. Choose $S = \{x\}$ and the basis of H to be $1, x, x^2, \ldots$. Write $\Delta(x) = 1 \otimes x + x \otimes 1$. So V is the subspace of H

spanned by $\{1, x\}$. This is also a basis, so take $v_1 = 1$ and $v_2 = x$. Writing the coproduct in this basis,

$$\rho(1) = 1 \otimes 1 + x \otimes 0,$$
$$\rho(x) = 1 \otimes x + x \otimes 1.$$

So the matrix a_{ij} is 2×2 with entries $1, 0, x, 1$. This yields the closed embedding

$$G \to GL_2, \qquad \alpha \mapsto \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}.$$

Example 8.33. Take $G := G_m$. Then $H = \Bbbk[x, x^{-1}]$. Choose $S = \{x, x^{-1}\}$ and the basis of H to be $\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots$. Write $\Delta(x) = x \otimes x$ and $\Delta(x^{-1}) = x^{-1} \otimes x^{-1}$. So V is the subspace of H spanned by $\{x, x^{-1}\}$. This is also a basis, so take $v_1 = x$ and $v_2 = x^{-1}$. Writing the coproduct in this basis,

$$\rho(x) = x \otimes x + x^{-1} \otimes 0,$$
$$\rho(x^{-1}) = x \otimes 0 + x^{-1} \otimes x^{-1}.$$

So the matrix a_{ij} is 2×2 with entries $x, 0, 0, x^{-1}$. This yields the closed embedding

$$G \to GL_2, \qquad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

8.7. Problems

Doublecheck these claims.

(1) Describe the initial object, terminal object, product and coproduct in AAG.

Same for $\mathsf{HopfAlg}^{\mathsf{co}}$.

(2) Write down an explicit formula for the coproduct of $\Bbbk[x_1, \ldots, x_n]$.

(3) Show that $U_2 \cong G_a$. Show that there is an aag structure on the affine variety $\Bbbk^3$ such that $U_3 \cong \Bbbk^3$ as aags.

(4) Show that the automorphisms of $G_a = \Bbbk$ are the multiplications by nonzero elements of $\Bbbk$.

What are the automorphisms of G_m?

(5) Prove that G_a and G_m are not isomorphic aags.

(6) Consider the affine variety $X = \Bbbk$. It is not an aag under multiplication (because 0 is not invertible). But check that it is a monoid in AffVariety. What structure does $\Bbbk[X]$ carry? Describe it explicitly.

(7) Let $\Bbbk := \mathbb{C}$. Consider the affine variety $\mu_n := \mathbb{V}(x^n - 1)$. Check that this is an aag under multiplication induced from $\Bbbk$. Describe the Hopf algebra structure on $\Bbbk[\mu_n]$. Show that it is of the form $\Bbbk G$ for some group G. Identify G.

What happens if $\Bbbk = \mathbb{R}$?

(8) Suppose $\Bbbk$ is algebraically closed. Prove that T_n, D_n and U_n are all connected. Verify that T_n is the semidirect product of D_n and U_n.

(9) A monomial matrix of size n is a matrix of size n which has precisely one nonzero entry in each row and each column. Let N_n denote the set of monomial matrices of size n. Show that N_n is a closed subgroup of GL_n (hence an aag) with $N_n^0 = D_n$. In particular, N_n is not connected.

(10) Let G be a connected aag. Show that any finite normal subgroup H lies in the center of G.

(11) Show by example that the image of a morphism $G \to H$ of aags may not be a closed subgroup of H.
(12) Show by example that the subgroup of an aag generated by two non-irreducible closed subsets need not be closed.
(13) Let A be a finite dimensional $\Bbbk$-algebra. Prove that the group of automorphisms of A is a closed subgroup of GL_A.
(14) Show that the group of upper triangular matrices is solvable.
(15) Let $\Bbbk := \mathbb{R}$. Consider the affine variety $X := \mathbb{V}(x^2 + y^2 - 1)$. Check that the product

$$(a, b)(a', b') := (aa' - bb', ab' + ba')$$

turns X into an aag. Describe the coproduct, counit and antipode of the Hopf algebra $\Bbbk[X]$. Is it cocommutative?

CHAPTER 9

Affine group schemes

In this chapter, $\Bbbk$ stands for a commutative ring.

9.1. Affine group schemes

Recall that the category of affine schemes $\mathsf{AffScheme}$ is cartesian. (The product in this category is described indirectly via the coproduct in the contravariantly equivalent category of rings.) This allows us to make the following definition.

Definition 9.1. An *affine group scheme* is a group object in the category of affine schemes $\mathsf{AffScheme}$. A morphism between affine group schemes is a morphism between group objects.

We let AGS denote the category of affine group schemes.

Since we could not explicitly describe the product for affine schemes, making the definition of an affine group scheme any more explicit is out of question. Contrast this with the situation for affine algebraic groups.

More generally, let $\mathsf{AGS}(\Bbbk)$ denote the category of affine group schemes over $\Bbbk$. This is the category of group objects in $\mathsf{AffScheme}(\Bbbk)$. The product in this category is the fiber product over $\Bbbk$. We obtain the previous case by setting $\Bbbk := \mathbb{Z}$.

Theorem 9.2. *The category of affine group schemes over $\Bbbk$ is contravariantly equivalent to the category of commutative Hopf algebras over $\Bbbk$.*

$$\mathsf{AGS}(\Bbbk) \cong (\mathsf{HopfAlg}_{\Bbbk}^{\mathsf{co}})^{\mathrm{op}}.$$

Proof. Recall from Theorem 5.44 that the category of affine schemes over $\Bbbk$ is contravariantly equivalent to the category of commutative $\Bbbk$-algebras. This implies:

$$\mathsf{Group}(\mathsf{AffScheme}(\Bbbk)) \cong \mathsf{Cogroup}(\mathsf{Alg}_{\Bbbk}^{\mathsf{co}})^{\mathrm{op}}.$$

Recall that the category of cogroup objects in $\mathsf{Alg}_{\Bbbk}^{\mathsf{co}}$ is precisely the category of commutative Hopf algebras over $\Bbbk$. The result follows. □

9.2. The functor of points

Recall that the functor (5.9) is an instance of the Yoneda embedding. So it preserves products and induces a functor

$$\mathsf{AGS} \to \mathsf{Group}^{\mathsf{AGS}^{\mathrm{op}}} \quad \text{or equivalently,} \quad \mathsf{AGS} \to \mathsf{Group}^{\mathsf{Ring}}. \tag{9.1}$$

Further, by Proposition F.6, this functor is also fully faithful.

More generally, there is a fully faithful functor

$$\mathsf{AGS}(\Bbbk) \to \mathsf{Group}^{\mathsf{Alg}_{\Bbbk}^{\mathsf{co}}}, \qquad X \mapsto h_X, \tag{9.2}$$

or equivalently, in view of Theorem 9.2,

$$\mathsf{HopfAlg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}^{\mathsf{Alg}_{\Bbbk}^{\mathsf{co}}}, \qquad H \mapsto h^H. \tag{9.3}$$

Explicitly, suppose H is a commutative Hopf algebras over $\Bbbk$, and $X = \mathrm{Spec}(H)$ is the corresponding affine group scheme. Then

$$h_X = h^H : \mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}, \qquad h_X(A) = h^H(A) = \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, A).$$

In other words, for each commutative $\Bbbk$-algebra A, we are looking at the set of all $\Bbbk$-algebra homomorphisms from H to A. This is a group under convolution, see Corollary E.20. This is called the group of A-valued points of X or of H. Explicitly, if $f, g : H \to A$, then their convolution product is

$$f * g : H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.$$

Thus X being an affine group scheme as opposed to simply an affine scheme has the effect that the set of A-valued points of X is a group as opposed to simply a set.

Definition 9.3. A functor $\mathsf{Ring} \to \mathsf{Group}$ is *representable* if it is of the form h_X for some affine group scheme X.

More generally, a functor $\mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}$ is representable if it is of the form h_X for some affine group scheme X over $\Bbbk$, or equivalently, if it is of the form h^H for some commutative Hopf algebra H over $\Bbbk$.

Let $\mathsf{AGS}'(\Bbbk)$ denote the category of representable functors $\mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}$. Since (9.2) is fully faithful,

$$\mathsf{AGS}(\Bbbk) \to \mathsf{AGS}'(\Bbbk), \qquad X \mapsto h_X \tag{9.4}$$

is an equivalence of categories.

In view of the equivalence (9.4), instead of saying "representable functors", we will reuse the term "affine group schemes" for the objects of $\mathsf{AGS}'(\Bbbk)$ as well. This can be potentially confusing, but the context should make it clear as to which of the two categories $\mathsf{AGS}(\Bbbk)$ or $\mathsf{AGS}'(\Bbbk)$ we are talking about.

Proposition 9.4. *Suppose $\mathcal{F} : \mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}$ is a functor such that the functor $\mathcal{G} : \mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Set}$ obtained from $\mathcal{F}$ by composing with the forgetful functor is an affine scheme. Then $\mathcal{F}$ is an affine group scheme.*

More precisely, suppose there is a commutative $\Bbbk$-algebra H such that $\mathcal{G}(A) = \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, A)$. Then H carries the structure of a commutative Hopf algebra such that the group structure on $\mathcal{F}(A) = \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, A)$ is convolution.

Proof. This is an instance of Proposition F.5. □

Explicitly, the coproduct, the counit and the antipode are constructed as follows.

$$\mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H \otimes H, A) \xrightarrow{\cong} \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, A) \times \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, A) \to \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, A). \tag{9.5}$$

The first map is the categorical coproduct property of $H \otimes H$. The second map is the group product. Now put $A := H \otimes H$ and let the coproduct $\Delta : H \to H \otimes H$ be the image of the identity under the above composite map.

The counit $H \to \Bbbk$ is the unit element of the group $\mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, \Bbbk)$. The antipode $H \to H$ is the inverse of the identity $H \to H$ in the group $\mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(H, H)$.

The above result in conjunction with Theorem 5.46 yields:

Theorem 9.5. *Suppose $\mathcal{F} : \mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}$ is a functor such that the elements of $\mathcal{F}(A)$ are the solutions over A of some family of equations (defined over $\Bbbk$). Then $\mathcal{F}$ is an affine group scheme, that is, there is a commutative Hopf algebra S over $\Bbbk$ such that $\mathcal{F}$ is isomorphic to h^S, where $h^S(T) = \mathsf{Alg}_{\Bbbk}^{\mathsf{co}}(S, T)$.*

Definition 9.6. An affine (group) scheme is called *algebraic* if the corresponding (Hopf) algebra is finitely generated as an algebra.

This happens for example if the affine group scheme is defined by equations involving finitely many, say n, variables.

Proposition 9.7. *Suppose $\mathcal{F}$ is as in Theorem 9.5 and the equations involve n variables. Then $\mathcal{F}(A)$ is a subset of A^n, and the product and inverse maps*

$$\mathcal{F}(A) \times \mathcal{F}(A) \to \mathcal{F}(A) \quad \text{and} \quad \mathcal{F}(A) \to \mathcal{F}(A)$$

are induced from polynomial maps $A^n \times A^n \to A^n$ and $A^n \to A^n$.

Proof. Apply Theorem 9.5. Since the equations involve finitely many variables, the Hopf algebra S is finitely generated. Express the coproduct and antipode of S using polynomials in these generators. (Because of the relations, these expressions are not unique in general.) The result follows. □

Definition 9.8. Suppose G and G' are affine group schemes. A morphism $G \to G'$ is called a *closed embedding* if the correpsonding map on (Hopf) algebras is surjective.

9.3. Examples

We now give many examples of affine group schemes. In each one of them, the functor

$$\mathsf{Alg}_{\Bbbk}^{\mathsf{co}} \to \mathsf{Group}$$

will be defined by equations. So by Theorem 9.5, it will be an affine group scheme.

In the discussion below, A denotes a commutative $\Bbbk$-algebra.

- $G_a(A) = A$, where the group structure on A is addition. This is a functor because any $\Bbbk$-algebra homomorphism is linear and preserves addition.
- $G_m(A)$ is the group of invertible elements of A. Here we ignore the additive structure of A. The product of A turns the underlying set of A into a monoid. $G_m(A)$ is the group of units of this monoid. This is a functor because any $\Bbbk$-algebra homomorphism is a morphism of the underlying monoids and hence preserves units.
- $GL_n(A)$ is the group of all invertible $n \times n$ matrices with entries in A. Check that an algebra homomorphism $A \to B$ induces a group homomophism $GL_n(A) \to \mathrm{GL}_n(B)$. Hence GL_n is a functor.
- $SL_n(A)$ is the group of all $n \times n$ matrices with entries in A with determinant 1.
- $\mu_n(A) = \{x \in A \mid x^n = 1\}$ is the group of nth roots of unity.

As remarked above, all these functors are represented by a Hopf algebra. The coproduct, counit and antipode can all be explicitly determined. For instance, for GL_2, the representating algebra is

$$H := \Bbbk[x_{11}, x_{12}, x_{21}, x_{22}, 1/\det].$$

The coproduct is constructed from (9.5). Explicitly,

$$\begin{pmatrix} \Delta(x_{11}) & \Delta(x_{12}) \\ \Delta(x_{21}) & \Delta(x_{22}) \end{pmatrix} = \begin{pmatrix} x_{11}\otimes 1 & x_{12}\otimes 1 \\ x_{21}\otimes 1 & x_{22}\otimes 1 \end{pmatrix} \begin{pmatrix} 1\otimes x_{11} & 1\otimes x_{12} \\ 1\otimes x_{21} & 1\otimes x_{22} \end{pmatrix}.$$

The rhs is matrix multiplication in the algebra $H \otimes H$. Thus

$$\Delta(x_{ij}) = \sum_{k=1}^{2} x_{ik} \otimes x_{kj}.$$

We leave it to you to check that in all the above examples except the last, the Hopf algebras are as defined in Section 8.3. There we were assuming $\Bbbk$ to be an infinite field, but the definitions make sense for any $\Bbbk$. In the present context, $\Bbbk$ is any commutative ring. The point about assuming $\Bbbk$ to be an infinite field is that the Hopf algebra over $\Bbbk$ is then determined by its $\Bbbk$ points. For the last example, the relevant Hopf algebra is discussed in Example 7.5.

The following are examples of morphisms.

- $\det : Gl_n \to G_m$.
- $SL_n \to \mathrm{GL}_n$.

Theorem 9.9. *Let $\Bbbk$ be any field. Every algebraic affine group scheme over $\Bbbk$ is isomorphic to a closed subgroup of some GL_n.*

Proof. This is equivalent to Theorem 8.30. □

9.4. Character group

Let G be an affine group scheme. A *character* of G is a morphism $G \to G_m$. The set of characters of G, denoted $\mathsf{AffScheme}'(G, G_m)$, is a group under pointwise multiplication.

Proposition 9.10. *Let G be an affine group scheme, and H the corresponding Hopf algebra. Then there is a group isomorphism*

$$\mathsf{AffScheme}'(G, G_m) \to \mathcal{G}(H),$$

where $\mathcal{G}(H)$ denotes the set of group-like elements of H.

Proof. Recall that the Hopf algebra of G_m is $\Bbbk[x, x^{-1}] = \Bbbk\mathbb{Z}$. So, there are bijections

$$\mathsf{AffScheme}'(G, G_m) \longleftrightarrow \mathsf{HopfAlg}^{\mathsf{co}}(\Bbbk\mathbb{Z}, H) \longleftrightarrow \mathcal{G}(H).$$

The second bijection uses (7.6). The fact that the product in $\mathcal{G}(H)$ corresponds to pointwise multiplication is a straightforward check. □

9.5. Diagonalizable affine group schemes

An affine group scheme is called *diagonalizable* if the corresponding Hopf algebra is of the form $\Bbbk S$ for some group S. Let $\mathsf{DiagAGS}'(\Bbbk)$ denote the full subcategory of $\mathsf{AGS}'(\Bbbk)$ whose objects are diagonalizable affine group schemes.

Theorem 9.11. *Let $\Bbbk$ be a field. Then there is a contravariant equivalence between the category of diagonalizable affine group schemes $\mathsf{DiagAGS}'(\Bbbk)$ and the category of abelian groups Ab.*

Under this equivalence, the subcategory of algebraic diagonalizable affine group schemes is equivalence to the subcategory of finitely generated abelian groups.

PROOF. The first part follows from Proposition 7.10. The second part follows from Lemma 7.11. □

9.6. Problems

(1) Let G be an affine group scheme, and H the corresponding Hopf algebra. Show that morphisms of the form $G \to G_a$ correspond to primitive elements of H, and the group structure on $\mathsf{AffScheme}'(G, G_a)$ is pointwise addition.
(2) Consider the affine group schemes G_a and G_m over any commutative ring $\Bbbk$.
   (a) Show that there are no nontrivial morphisms from G_m to G_a.
   (b) If $\Bbbk$ is reduced, then show that there are no nontrivial morphisms from G_a to G_m.
   (c) For each $b \neq 0$ in $\Bbbk$ with $b^2 = 0$, find a nontrivial morphism from G_a to G_m.

APPENDIX A

Category theory

Review basic notions from category theory:

- category, functor, natural transformation,
- opposite category, contravariant functor,
- notion of isomorphism between objects in a category, any functor preserves isomorphisms.

Wikipedia has a good discussion. For a more comprehensive treatment, see the book by Mac Lane [18].

For objects V and W of a category C, we let $\mathsf{C}(V, W)$ denote the class of morphisms from V to W. A functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ induces a map

$$\mathsf{C}(V, W) \to \mathsf{D}(\mathcal{F}(V), \mathcal{F}(W)).$$

We say that $\mathcal{F}$ is

- *full* if this map is surjective,
- *faithful* if this map is injective, and
- *fully faithful* if this map is bijective,

for all V and W.

A *full subcategory* of C is a subcategory of C for which the inclusion functor is full. In other words, whenever objects A and B of C belong to the subcategory, all morphisms in C between A and B also belong to the subcategory.

Let C^{op} denote the *opposite category* of C; these categories have the same objects, and

$$\mathsf{C}^{\mathrm{op}}(A, B) := \mathsf{C}(B, A).$$

For categories C and C', let $\mathsf{C} \times \mathsf{C}'$ denote their cartesian product: objects are pairs (A, A') where A is an object in C and A' is an object in C', and morphisms are pairs $(f, f') : (A, A') \to (B, B')$, where $f : A \to B$ is a morphism in C and $f' : A' \to B'$ is a morphism in C'.

A category is *discrete* if the only morphisms in the category are identities. In particular, there are no morphisms between distinct objects.

A category is *indiscrete* if there is a unique morphism between any two objects. It follows that every morphism in an indiscrete category is an isomorphism.

Remark A.1. One may think of a category as a graph in which arrows can be composed whenever the target of the first arrow is the source of the second arrow. Initial object is like a source, terminal object is like a sink. A discrete category is a graph whose only arrows are identity loops. To pass from C to C^{op}, one reverses all the arrows.

Standard examples of categories to keep in mind.

- Set of sets

- Vec of vector spaces
- Group of groups
- Top of topological spaces
- Ring of (commutative) rings (with identity).
- $\mathsf{Alg}^{\mathsf{co}}$ of commutative algebras over $\mathbb{C}$.
- $\overline{\mathsf{Alg}}^{\mathsf{co}}$ of finitely generated reduced commutative algebras over $\mathbb{C}$.
- The *one-arrow category* has one object and one morphism.
- A *one-object category* has one object. Its structure is completely specified by the composition of loops on that object. Thus, a one-object category is same as a monoid.

We will briefly review some of the notions listed below. The material is taken from [2, Appendix A].

- equivalence of categories,
- limit: terminal object, product, pullback, inverse limit,
- colimit: initial object, coproduct, pushout, direct limit,
- adjoint functors.

A.1. Products and coproducts

The notions reviewed here can also be found in [18, Sections III.3–5 and VIII.2].

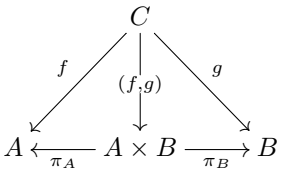
Definition A.2. Let A and B be objects of a category C. An object $A \times B$ along with arrows

$$A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$$

is said to be a *product* of A and B, if it satisfies the following property: given arrows

$$A \xleftarrow{f} C \xrightarrow{g} B,$$

there exists a unique arrow $(f, g) : C \to A \times B$ for which



commutes. If the product exists, it is unique up to isomorphism.

The product of an arbitrary family of objects is defined similarly. The product of an empty family is a *terminal object*: an object J with a unique arrow from any other object in the category.

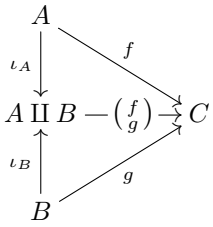
Definition A.3. Dually, a *coproduct* of two objects A and B is an object $A \amalg B$ with arrows

$$A \xrightarrow{\iota_A} A \amalg B \xleftarrow{\iota_B} B$$

such that given any arrows

$$A \xrightarrow{f} C \xleftarrow{g} B,$$

there is a unique arrow $\begin{pmatrix} f \\ g \end{pmatrix} : A \amalg B \to C$ for which



commutes.

An *initial* object I has a unique arrow to any other object in the category.

A.2. Adjunctions

Various equivalent formulations of the notion of adjunction are discussed in Mac Lane's book [18, Section IV.1, Theorem 2]. We briefly discuss two of these.

Definition A.4. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ be a pair of functors. We say that $\mathcal{F}$ is a *left adjoint* to $\mathcal{G}$ or that $\mathcal{G}$ is a *right adjoint* to $\mathcal{F}$ if there exists a natural isomorphism

$$\mathsf{D}(\mathcal{F}(-), -) \xrightarrow{\cong} \mathsf{C}\big(-, \mathcal{G}(-)\big)$$

of functors $\mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Set}$.

In other words, the functors are adjoint if there exists a bijection

$$\mathsf{D}(\mathcal{F}(A), X) \xrightarrow{\cong} \mathsf{C}\big(A, \mathcal{G}(X)\big) \tag{A.1}$$

which is natural in A and X.

Proposition A.5. *Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ be functors. Then $\mathcal{F}$ is left adjoint to $\mathcal{G}$ (or $\mathcal{G}$ is a right adjoint to $\mathcal{F}$) iff there exist natural transformations $\eta\colon \mathrm{id} \Rightarrow \mathcal{G}\mathcal{F}$ and $\xi\colon \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}$ such that the following diagrams commute.*

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(\eta_A)} & \mathcal{F}\mathcal{G}\mathcal{F}(A) \\ & {\scriptstyle \mathrm{id}} \searrow & \downarrow {\scriptstyle \xi_{\mathcal{F}(A)}} \\ & & \mathcal{F}(A) \end{array} \qquad \begin{array}{ccc} \mathcal{G}\mathcal{F}\mathcal{G}(X) & \xleftarrow{\eta_{\mathcal{G}(X)}} & \mathcal{G}(X) \\ {\scriptstyle \mathcal{G}(\xi_X)} \downarrow & \swarrow {\scriptstyle \mathrm{id}} & \\ \mathcal{G}(X) & & \end{array} \tag{A.2}$$

The tuple $(\mathcal{F}, \mathcal{G}, \eta, \xi)$ is called an *adjunction*. The transformations η and ξ are the *unit* and *counit* of the adjunction respectively.

Proof. Starting with (A.1), one sets $X = \mathcal{F}(A)$ and defines η_A as the morphism corresponding to the identity. Setting $A = \mathcal{G}(X)$ one obtains ξ_X. Conversely, given η and ξ, one defines (A.1) by sending $f\colon \mathcal{F}(A) \to X$ to the composite

$$A \xrightarrow{\eta_A} \mathcal{G}\mathcal{F}(A) \xrightarrow{\mathcal{G}(f)} \mathcal{G}(X). \tag{A.3}$$

The inverse correspondence sends $g\colon A \to \mathcal{G}(X)$ to the composite

$$\mathcal{F}(A) \xrightarrow{\mathcal{F}(g)} \mathcal{F}\mathcal{G}(X) \xrightarrow{\xi_X} X. \tag{A.4}$$

□

Example A.6. Consider the linearization functor $\Bbbk(-) : \mathsf{Set} \to \mathsf{Vec}$: For any set A, $\Bbbk(A)$ is the vector space with basis A. This functor satisfies the following universal property. Any map from a set A to a vector space V extends uniquely to a linear map $\Bbbk(A) \to V$. This says that $\Bbbk(-)$ is the left adjoint to the forgetful functor $\mathsf{Vec} \to \mathsf{Set}$:

Consider the functor

$$\mathcal{S} : \mathsf{Set} \to \mathsf{Alg}^{\mathsf{co}}$$

which assigns to a set $\{x_1, \ldots, x_n\}$, the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$. This functor satisfies the following universal property. Any map from a set A to an algebra V extends uniquely to an algebra homomorphism from $\mathcal{S}(A)$ to V. This says that the functor $\mathcal{S}$ is the left adjoint to the forgetful functor

$$\mathsf{Alg}^{\mathsf{co}} \to \mathsf{Set}.$$

A.3. Equivalences

Definition A.7. An *equivalence* of categories C and D is a tuple $(\mathcal{F}, \mathcal{G}, \eta, \xi)$, where

$$\mathcal{F} \colon \mathsf{C} \to \mathsf{D} \quad \text{and} \quad \mathcal{G} \colon \mathsf{D} \to \mathsf{C}$$

are functors, and

$$\eta \colon \mathrm{id} \Rightarrow \mathcal{G}\mathcal{F} \quad \text{and} \quad \xi \colon \mathcal{F}\mathcal{G} \Rightarrow \mathrm{id}$$

and natural isomorphisms.

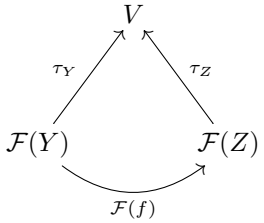
An *adjoint equivalence* is an adjunction that is also an equivalence.

If $(\mathcal{F}, \mathcal{G}, \eta, \xi)$ is an adjoint equivalence, then so is $(\mathcal{G}, \mathcal{F}, \xi^{-1}, \eta^{-1})$. Not every equivalence is an adjoint equivalence. However, given an equivalence $(\mathcal{F}, \mathcal{G}, \eta, \xi)$, there is always an adjoint equivalence of the form $(\mathcal{F}, \mathcal{G}, \eta', \xi')$. In fact, one may always choose $\eta' = \eta$ or $\xi' = \xi$ (but not both at the same time); see the comments following the proof of [5, Proposition 3.4.3].

A.4. Colimits of functors

We review the concept of colimit. More information can be found in the books by Borceux [5, Chapter 2] or Mac Lane [18, Chapters III and V].

Definition A.8. Let $\mathcal{F} \colon \mathsf{D} \to \mathsf{C}$ be a functor. Consider an object V in C equipped with morphisms $\tau_Y \colon \mathcal{F}(Y) \to V$, one for each object Y in D, and such that for each morphism $f \colon Y \to Z$ in D the following diagram commutes.



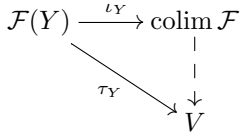
Such a structure is called a *cone* from the base $\mathcal{F}$ to the vertex V.

Definition A.9. The colimit of a functor $\mathcal{F} \colon \mathsf{D} \to \mathsf{C}$ is an object of C, denoted

$$\operatorname{colim} \mathcal{F} \quad \text{or} \quad \operatorname*{colim}_X \mathcal{F}(X),$$

together with morphisms $\iota_Y \colon \mathcal{F}(Y) \to \operatorname{colim} \mathcal{F}$ for each object Y in D, satisfying the following properties.

- The maps ι_Y form a cone from the base $\mathcal{F}$ to the vertex $\operatorname{colim}\mathcal{F}$. In other words,
$$\iota_Z \mathcal{F}(f) = \iota_Y$$
for each morphism $f\colon Y \to Z$ in D.
- For any cone from $\mathcal{F}$ to a vertex V in C, there is a unique morphism $\operatorname{colim}\mathcal{F} \to V$, such that for each object Y in D the following diagram commutes, where τ_Y is the structure map of the cone to V.



The colimit may not exist, but if it does, then it is unique up to isomorphism.

The cone with vertex $\operatorname{colim}\mathcal{F}$ is called the *limiting cone* or *universal cone* (from $\mathcal{F}$).

Let $\mathcal{G}\colon \mathsf{E} \to \mathsf{D}$ be another functor and $\mathcal{F}\mathcal{G}\colon \mathsf{E} \to \mathsf{C}$ the composite. Suppose the colimit of $\mathcal{G}$ exists and let $\iota_Y\colon \mathcal{G}(Y) \to \operatorname{colim}\mathcal{G}$ be a universal cone. Then
$$\mathcal{F}\mathcal{G}(\iota_Y)\colon \mathcal{F}\mathcal{G}(Y) \to \mathcal{F}(\operatorname{colim}\mathcal{G})$$
is a cone from $\mathcal{F}\mathcal{G}$ to $\mathcal{F}(\operatorname{colim}\mathcal{G})$. Therefore, there is a canonical map
$$\operatorname{colim}\mathcal{F}\mathcal{G} \to \mathcal{F}(\operatorname{colim}\mathcal{G}).$$
When this map is an isomorphism, or equivalently when the above cone is universal, we say that $\mathcal{F}$ *preserves* the colimit of $\mathcal{G}$, and we have
$$\operatorname{colim}\mathcal{F}\mathcal{G} \cong \mathcal{F}(\operatorname{colim}\mathcal{G}).$$

Proposition A.10. *A functor which is part of an equivalence, preserves colimits.*

Proof. See [18, Theorem V.3.1,Section IX.2]. □

Theorem A.11. *A right adjoint preserves all existing limits, and dually a left adjoint preserves all existing colimits.*

In particular, a right adjoint preserves terminal objects and products, and dually a left adjoint preserves initial objects and coproducts.

Proof. See [18, Theorems V.5.1]. □

A.5. Problems

(1) Describe the initial object, terminal object, product and coproduct in the categories Set, Vec, Group, Top, Ring, $\mathsf{Alg}^{\mathsf{co}}$ and $\overline{\mathsf{Alg}}^{\mathsf{co}}$.
(2) Check explicitly that in Example A.6, the forgetful functors preserve products, while $\Bbbk(-)$ and $\mathcal{S}$ preserves coproducts. (We know this should be the case from Theorem A.11.)
(3) Consider the functor
$$\mathcal{S} : \mathsf{Vec} \to \mathsf{Alg}^{\mathsf{co}}$$
which assigns to a vector space V its symmetric algebra $\mathcal{S}(V)$. The functor $\mathcal{S}$ is the left adjoint to the forgetful functor
$$\mathsf{Alg}^{\mathsf{co}} \to \mathsf{Vec}.$$

Check that the forgetful functor preserves products, while $\mathcal{S}$ preserves coproducts. How does this adjunction relate to those in Example A.6?

(4) Check that the forgetful functor

$$\mathsf{Group} \to \mathsf{Set}$$

preserves products. Describe the left adjoint to this functor and check that it preserves coproducts.

APPENDIX B

Closure operators

Closure operators on posets arise in many different fields of mathematics, and in particular, in ring theory and algebraic geometry. In this section, we review their definition and basic properties to the extent required. Apparently introduced by Birkhoff in the first edition of this book on Lattice theory, so says Gratzer. A detailed treatment of closure operators can be found in [8, Chapter 7]. Other good sources are [12, 21].

B.1. Closure operators

Let 2^I denote the poset of subsets of a finite set I.

Definition B.1. Let P be any poset (not necessarily finite). A *closure operator* on P is a map

$$c\colon P \to P$$

such that for every $x, y \in P$, we have:

- $x \leq c(x)$;
- if $x \leq c(y)$, then $c(x) \leq c(y)$.

An element $z \in P$ is *closed* if $c(z) = z$, or equivalently if $z = c(x)$ for some $x \in P$. Let P_c denote the set of closed sets in P.

Example B.2. Let G be any group. Consider the operator

$$c\colon 2^G \to 2^G$$

defined as follows: $c(A)$ is the subgroup of G generated by the subset $A \subseteq G$. In other words, $c(A)$ is the smallest subgroup of G which contains A.

This is a closure operator. The closed sets are precisely those subsets of G which are subgroups of G.

Example B.3. Let R be any ring (commutative with identity). Consider the operator

$$c\colon 2^R \to 2^R$$

defined as follows: $c(A)$ is the ideal of R generated by the subset $A \subseteq R$. In other words, $c(A)$ is the smallest ideal of R which contains A.

This is a closure operator. The closed sets are precisely those subsets of R which are ideals of R.

Example B.4. Let $V := \mathbb{R}^n$. Consider the operator

$$c\colon 2^V \to 2^V$$

defined as follows: $c(A)$ is the convex hull of A.

This is a closure operator. The closed sets are precisely the convex subsets of V.

B.2. Topological closure operators

A closure operator $t\colon 2^I \to 2^I$ is *topological* if

- $t(\emptyset) = \emptyset$;
- $t(A \cup B) = t(A) \cup t(B)$.

Example B.5. Let T be any topological space. Consider the operator

$$c\colon 2^T \to 2^T$$

defined as follows: $c(A) = \overline{A}$ is the closure of the set A in T. In other words, $c(A)$ is the smallest closed set of T which contains A.

This is a topological closure operator. The closed sets of this operator are precisely the closed sets of T.

Remark B.6. Matroids, antimatroids, convex geometries are closure operators with more structure (just like topological closure operators). In particular, any matroid, antimatroid or convex geometry is an example of a closure operator.

B.3. Joins and meets

Let P be a poset. For any subset S of P, we write $\bigvee S$ for the join (least upper bound) of elements in S, whenever it exists. Similarly, we write $\bigwedge S$ for the meet (greatest lower bound) of elements in S, whenever it exists.

Definition B.7. A *lattice* is a poset in which $\bigvee S$ and $\bigwedge S$ exist for all finite subsets S. A *complete lattice* is a poset in which $\bigvee S$ and $\bigwedge S$ exist for all subsets S.

If the poset is finite, then there is no distinction between the two notions. The Boolean poset 2^I for *any* set I is an example of a complete lattice.

Proposition B.8. *Let c be a closure operator on a poset P. Assume further that P is a complete lattice. Then P_c is a complete lattice, under the order inherited from P, such that for every subset S of P_c,*

$$\bigwedge_{P_c} S = \bigwedge_{P} S \qquad \text{and} \qquad \bigvee_{P_c} S = c\Big(\bigvee_{P} S\Big).$$

This is a part of [8, Proposition 7.2]. It shows that an arbitrary meet in P of closed sets is a closed set, but an arbitrary join in P of closed sets may not be closed set; so to get the join in P_c, one has to take the closure of the join in P.

Proof. Let $x \in S$. Since x is a closed set, $c\big(\bigwedge_P S\big) \le x$. Thus $c\big(\bigwedge_P S\big) \le \bigwedge_P S$, from which it follows that $\bigwedge_P S$ is a closed set. This shows that arbitrary meets exist in P_c and agree with the meets in P.

If some closed set is greater than all elements of S, then it is greater than $\bigvee_P S$ and hence also greater than $c\big(\bigvee_P S\big)$. This shows that the latter is the join of S in P_c. □

B.4. Galois connection

Let P and Q be posets and let $f : P \to Q$ and $g : Q \to P$ be order-reversing maps. One says that f and g form a *Galois connection* if

$$x \le g(y) \iff y \le f(x) \tag{B.1}$$

for all $x \in P$ and $y \in Q$.

Proposition B.9. *Suppose f and g form a Galois connection. Then*

(1) *gf and fg are closure operators on P and Q respectively,*
(2) *f and g restrict to inverse bijections between the closed sets of gf and the closed sets of fg,*
(3) *$f = fgf$ and $g = gfg$,*
(4) *the image of f equals the image of fg which is same as the set of closed sets of fg, and similarly, the image of g equals the image of gf which is same as the set of closed sets of gf.*

PROOF. (1). We show gf is a closure operator on P: Putting $y = f(x)$ in (B.1) gives $x \leq gf(x)$. This is the first property of a closure operator. To verify the second property, suppose $x \leq gf(z)$. Putting $y = f(z)$ in (B.1) gives $f(z) \leq f(x)$. Applying g and noting it is order-reversing yields $gf(x) \leq gf(z)$.

(2). Follows from (1).

(3). We show $f(x) = fgf(x)$. Since gf is a closure operator, $x \leq gf(x)$. Applying f and noting it is order-reversing yields $fgf(x) \leq f(x)$. Since fg is a closure operator, $f(x) \leq fg(f(x))$. Combining the two conclusions proves the result.

(4). Follows from (3). □

Remark B.10. Instead of (B.1), one may also consider the condition:

$$g(y) \leq x \iff f(x) \leq y.$$

This does not lead to anything new since one may pass from this situation to the previous one by reversing the partial orders on P and Q.

If one reverses the partial orders on only one of P or Q, then both f and g become order-preserving and condition (B.1) changes to

$$x \leq g(y) \iff f(x) \leq y,$$

or

$$g(y) \leq x \iff y \leq f(x).$$

The categorical significance of these identities will be made clear in the next section.

B.5. Problems

(1) For a closure operator c, show that
   - $c\big(c(x)\big) = c(x)$;
   - if $x \leq y$, then $c(x) \leq c(y)$.
(2) Let V be a vector space. Construct a closure operator on 2^V following the strategy of Example B.2. You may think of similar examples for other algebraic structures.
(3) Show that arbitrary intersection of closed sets is again a closed set for any closure operator on 2^I. This is part of the claim made in Proposition B.8. It agrees with our familiar experience that intersection of ideals is an ideal, intersection of subgroups is a subgroup, and so on.
(4) Show that topological closures with ground set I and *topologies* on I are equivalent notions: given a topological closure t, its closed sets are the closed sets of a topology on the set I; conversely, any topology on I arises in this manner from a unique topological closure operator.

APPENDIX C

Posets

C.1. Posets as categories

A poset P can be viewed as a category which we denote by P: Objects of P are the elements of P. Morphisms are as follows. If $x \leq y$ in P, then there is a unique morphism from x to y in P, else there is no morphism from x to y.

The reflexivity in P yields the identity morphisms in P, transitivity in P allows morphisms to be composed in P.

In a poset P,

- the smallest element, if any, is an initial object in P,
- the largest element, if any, a terminal object in P,
- the meet of x and y, if it exists, is the product of x and y in P,
- the join of x and y, if it exists, is the coproduct of x and y in P.

The assertions about the meet and join work for any collection of elements of P (not just two or finite).

Let P^{op} denote the opposite poset to P: elements are the same as those of P but the order is reversed. Then the category associated to P^{op} is precisely the opposite category P^{op}.

C.2. Order-preserving maps as functors

An order-preserving map $F : P \to Q$ between posets is same as a functor $\mathcal{F} : \mathsf{P} \to \mathsf{Q}$ between the corresponding categories. Note that an order-reversing map yields a contravariant functor.

Proposition C.1. *Let P and Q be posets and let $F : P \to Q$ and $G : Q \to P$ be order-preserving maps. Then the functor $\mathcal{F} : \mathsf{P} \to \mathsf{Q}$ is the left adjoint of $\mathcal{G} : \mathsf{Q} \to \mathsf{P}$ (or equivalently, G is the right adjoint of F) iff*

$$F(x) \leq y \iff x \leq G(y) \tag{C.1}$$

for all $x \in P$ and $y \in Q$.

Definition C.2. Let P and Q be posets and let $F : P \to Q$ and $G : Q \to P$ be order-preserving maps. We say that (F, G) is a adjunction if (C.1) holds. In this situation, we let $P_c = \mathrm{Image}(G)$ and $Q_c = \mathrm{Image}(F)$. We view them as subposets of P and Q respectively.

We say that an order-preserving map $F : P \to Q$ *preserves existing joins* if whenever the join $\bigvee S$ exists in P, then the join $\bigvee F(S)$ exists in Q and $f(\bigvee S) = \bigvee F(S)$. Preservation of existing meets is defined dually.

Theorem C.3. *Suppose (F, G) is an adjunction. Then F preserves all existing joins and G preserves all existing meets.*

PROOF. This follows from Theorem A.11: In the example at hand, joins are coproducts and meets are products. □

Theorem C.4. *Suppose (F, G) is an adjunction, where in addition, P and Q are complete lattices. Then*

(1) *F preserves arbitrary joins and G preserves arbitrary meets.*
(2) *P_c and Q_c are complete lattices. Further, they are isomorphic as complete lattices (with the inverse isomorphisms provided by the restrictions of F and G).*

PROOF. (1) follows from Theorem C.3, the only additional point being that since P and Q are complete lattices, all joins and meets will exist.

The first part of (2) follows from Proposition B.8. For the second part of (2): First note that (F, G) restricts to an adjunction between P_c and Q_c. In this case, F and G are inverse poset isomorphisms as well. Now apply (1). □

C.3. Problems

(1) Prove Theorem C.3 directly. In doing this, you will in principle see all the ingredients that go into the proof of Theorem A.11.
(2) Let $G : (c, d) \to (a, b)$ be an increasing right continuous function such that

$$\lim_{y \to c+} G(y) = a \qquad \text{and} \qquad \lim_{y \to d-} G(y) = b.$$

Then there is a unique increasing function $F : (a, b) \to (c, d)$ such that

$$F(x) \leq y \iff x \leq G(y).$$

Further, F is left continuous and

$$\lim_{x \to a+} F(x) = c \qquad \text{and} \qquad \lim_{x \to b-} F(x) = d.$$

(This result is useful in measure theory and probability theory.)

APPENDIX D

Sheaves

In this section, we introduce the basic notation for sheaves. For more details, see [14, Section II.1] or [11, Section I.1.3].

D.1. Sheaves

Let X be a topological space. The category of open sets in X, denoted by Open_X, is defined as follows: Objects are open sets in X and there is a unique morphism $U \to V$ whenever $U \subseteq V$. (Alternatively, open sets in X form a poset under inclusion, and Open_X is precisely the category associated to this poset.)

Definition D.1. A *presheaf* on X is a functor

$$\mathcal{F} : \mathsf{Open}_X^{\mathrm{op}} \to \mathsf{Set}.$$

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves is a natural transformation between the functors. Let $\mathsf{Presheaf}(X)$ denote the category of presheaves on X.

We make this definition more explicit. A presheaf on X consists of the following data:

- for each open set U, there is a set $\mathcal{F}(U)$, and
- for each $V \subseteq U$, there is a map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$,

such that

- $\rho_{U,U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map, and
- if $W \subseteq V \subseteq U$, then $\rho_{U,W} = \rho_{V,W}\rho_{U,V}$.

A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves consists of a map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U, such that whenever $V \subseteq U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho_{U,V}} \\ \mathcal{F}(V) & \xrightarrow[\varphi(V)]{} & \mathcal{G}(V) \end{array}$$

commutes.

The set $\mathcal{F}(U)$ is called the *sections* of $\mathcal{F}$ over the set U, the set $\mathcal{F}(X)$ is called the *global sections* of $\mathcal{F}$, and the map $\rho_{U,V}$ is called the *restriction* of U to V.

Definition D.2. A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following gluing condition:

- If $\{V_j\}$ is an open cover of an open set $U \subseteq X$, and if there are elements $s_j \in \mathcal{F}(V_j)$ for each j satisfying

$$\rho_{V_i,V_i\cap V_j}(s_i) = \rho_{V_j,V_i\cap V_j}(s_j)$$

for each i, j, then there is a unique element $s \in \mathcal{F}(U)$ such that

$$\rho_{U,V_j}(s) = s_j$$

for all j.

A morphism of sheaves is a morphism of the underlying presheaves. Let $\mathsf{Sheaf}(X)$ denote the category of sheaves on X.

Example D.3. The following are some familiar examples of sheaves.

- Continuous real-valued functions on a topological space.
- Smooth real-valued functions on a smooth manifold.
- Holomorphic functions on a Riemann surface.

Let $\mathcal{F}$ be the sheaf of differentiable functions on $\mathbb{R}$, and let $\mathcal{G}$ be the sheaf of continuous functions on $\mathbb{R}$. Since every differentiable function is also continuous, there is a morphism of sheaves $\mathcal{F} \to \mathcal{G}$.

The sheaf axiom implies that $\mathcal{F}(\emptyset)$ is a singleton (see EH).

Remark D.4. The category Set in the definition of a (pre)sheaf can be replaced by any category C. In this manner, one defines a (pre)sheaf of rings, a (pre)sheaf of abelian groups, a (pre)sheaf of $\mathbb{C}$-algebras, etc.

D.2. Stalks

For $x \in X$, let $\mathsf{Open}_{X,x}$ denote the subcategory of Open_X consisting of open sets containing x. Let $\mathcal{F}$ be a presheaf (sheaf) on X. Consider the restriction of $\mathcal{F}$:

$$\mathsf{Open}_{X,x}^{\mathrm{op}} \to \mathsf{Set}.$$

This functor has a colimit. It is called the *stalk* of $\mathcal{F}$ at the point x, and is denoted by $\mathcal{F}_x$. An explicit construction is as follows. An element of $\mathcal{F}_x$ is a pair (U, s), where U is a neighbourhood of x and $s \in \mathcal{F}(U)$. Two such pairs (U, s) and (V, t) define the same element of $\mathcal{F}_x$ iff there is a neighbourhood W of x with $W \subseteq U \cap V$, such that

$$\rho_{U,W}(s) = \rho_{V,W}(t).$$

One says that elements of $\mathcal{F}_x$ are *germs* of sections of $\mathcal{F}$ at the point x.

By general principles, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves (sheaves) on X induces a morphism $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ on the stalks, for any point $x \in X$.

D.3. The generic stalk

Let X be an *irreducible* topological space, that is, any two nonempty open sets in X intersect. Let $\mathsf{Open}_{X,gen}$ denote the subcategory of Open_X consisting of all nonempty open sets. Let $\mathcal{F}$ be a presheaf (sheaf) on X. Consider the restriction of $\mathcal{F}$:

$$\mathsf{Open}_{X,gen}^{\mathrm{op}} \to \mathsf{Set}.$$

This functor has a colimit. It is called the *generic stalk* of $\mathcal{F}$, and is denoted by $\mathcal{F}_{gen}$. An explicit construction is as follows. An element of $\mathcal{F}_{gen}$ is a pair (U, s), where U is a nonempty open set and $s \in \mathcal{F}(U)$. Two such pairs (U, s) and (V, t) define the same element of $\mathcal{F}_{gen}$ iff there is a nonempty open set W contained in $U \cap V$, such that

$$\rho_{U,W}(s) = \rho_{V,W}(t).$$

(Check that this is an equivalence relation. This uses the fact that X is irreducible.) By general principles, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves (sheaves) on X induces a morphism $\varphi_{gen} : \mathcal{F}_{gen} \to \mathcal{G}_{gen}$ on the generic stalks. Draw picture

Remark D.5. I believe that the colimit exists even if the space is not irreducible. For its construction, we will have to take transitive closure of the relation defined above.

D.4. The category of all sheaves

So far, we have talked about sheaves on a single topological space. Now we construct a category which involves sheaves where the topological space is allowed to vary.

Definition D.6. Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on X, we define the *direct image sheaf* $f_*\mathcal{F}$ on Y by

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

for any open set V in Y. This defines a functor

$$f_* : \mathsf{Sheaf}(X) \to \mathsf{Sheaf}(Y).$$

We can now define the category Sheaf: An object is a pair $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ is a sheaf on X. A morphism $(X, \mathcal{F}) \to (Y, \mathcal{G})$ is a pair $(f, f^\#)$ of a continuous map $f : X \to Y$ and a map of sheaves $f^\# : \mathcal{G} \to f_*\mathcal{F}$ on Y.

Example D.7. Let $\mathsf{Manifold}$ be the category of smooth manifolds: objects are smooth manifolds and morphisms are smooth maps. There is a functor

$$\mathsf{Manifold} \to \mathsf{Sheaf}$$

constructed as follows. It sends a manifold M to the pair $(M, \mathcal{O}_M)$: the first component is the underlying topological space of M and the second component is the sheaf of smooth functions of M. Further, given a smooth map $f : M \to N$ of manifolds, there is an induced map

$$f^\# : \mathcal{O}_N \to f_*\mathcal{O}_M \qquad g \mapsto g \circ f.$$

Add material on $\mathcal{B}$-sheaves.

Proposition D.8. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves (of rings). Then φ is an isomorphism iff the induced map on the stalk $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for all $P \in X$.*

Proof. See [14, Proposition 1.1]. □

D.5. Problems

(1) Let X be the two-element set $\{0, 1\}$, and make X into a topological space by taking each of the four subsets to be open. A presheaf on X is thus a collection of four sets with certain maps between them satisfying some conditions. Describe these conditions. What additional conditions would you need to make this into a sheaf?

Do the same in the case when the topology on X has as open sets $\emptyset$, $\{0\}$ and $\{0, 1\}$.

(2) Show that there is a presheaf which is not a sheaf because the existence part of the gluing condition fails. Similarly, show that there is a presheaf which is not a sheaf because the uniqueness part of the gluing condition fails.
(3) Show that the colimit of any presheaf exists.
(4) Show that the forgetful functor $\mathsf{Sheaf}(X) \to \mathsf{Presheaf}(X)$ has a left adjoint. This says that there is a canonical way to sheafify a presheaf.
(5) Show that the functor $f_* : \mathsf{Presheaf}(X) \to \mathsf{Presheaf}(Y)$ has a left adjoint. It is usually denoted by f^{-1}. Explicitly, given a presheaf $\mathcal{G}$ on Y, the presheaf $f^{-1}\mathcal{G}$ is defined as follows. Its value on an open set U in X is the colimit of $\mathcal{G}$ over all open sets in Y containing $f(U)$.

    An important point is that $f^{-1}\mathcal{G}$ may not be a sheaf even if $\mathcal{G}$ is. So to obtain the left adjoint of $f_* : \mathsf{Sheaf}(X) \to \mathsf{Sheaf}(Y)$, one combines this result with the previous exercise.

APPENDIX E

Monoidal categories

This material is mostly taken from [2, Chapter 1] (available on my homepage) where you can find more details as well as plenty of historical references.

E.1. Braided monoidal categories

E.1.1. Monoidal categories.

Definition E.1. A *monoidal category* $(\mathsf{C}, \bullet)$ is a category C with a functor

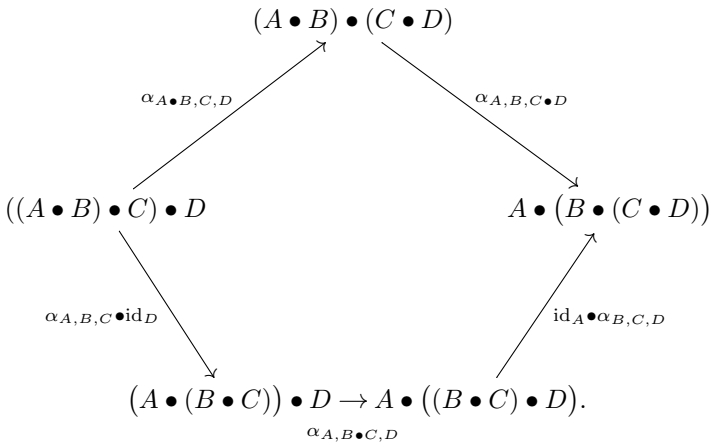
$$\bullet\colon \mathsf{C}\times\mathsf{C}\to\mathsf{C},$$

together with a natural isomorphism

$$\alpha_{A,B,C}\colon (A\bullet B)\bullet C \xrightarrow{\cong} A\bullet(B\bullet C)$$

which satisfies the pentagon axiom:

(E.1)

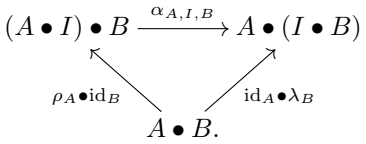


Further, C has a distinguished object I with natural isomorphisms

$$\lambda_A\colon A\to I\bullet A, \quad \text{and} \quad \rho_A\colon A\to A\bullet I,$$

which satisfy the triangle axiom:

(E.2)

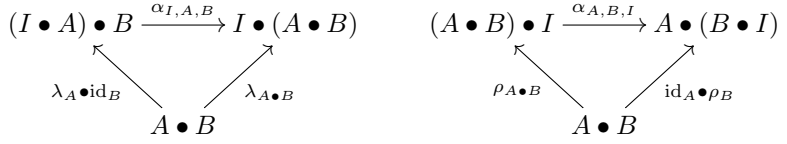


It follows that

(E.3)
$$\lambda_I = \rho_I,$$

and the following diagrams commute.

(E.4)



We refer to $A \bullet B$ as the *tensor product* of A and B and to I as the *unit object* of C. The natural isomorphism α above is called the *associativity constraint* and λ and ρ are called the *unit constraints*.

We often omit α from the notation and identify the objects $(A \bullet B) \bullet C$ and $A \bullet (B \bullet C)$. The identification is denoted $A \bullet B \bullet C$ and referred to as the unbracketed tensor product of A, B and C. We often omit the subindexes A, B, C from α, λ and ρ if they can be told from the context.

If $(\mathsf{C}, \bullet)$ is monoidal, then so is $(\mathsf{C}^{\mathrm{op}}, \bullet)$. The *transpose* of $(\mathsf{C}, \bullet)$ is the monoidal category $(\mathsf{C}, \tilde{\bullet})$ where

$$A \,\tilde{\bullet}\, B := B \bullet A.$$

If C and C' are monoidal categories, then so is $\mathsf{C} \times \mathsf{C}'$ with tensor product

$$(A, A') \bullet (B, B') := (A \bullet B, A' \bullet B').$$

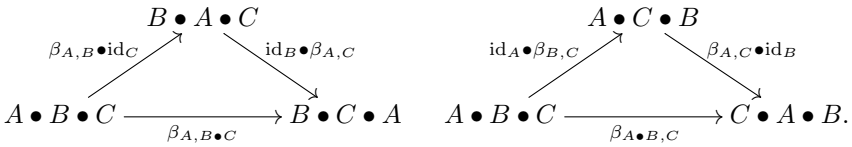
E.1.2. Braided monoidal categories.

Definition E.2. A *braided monoidal category* is a monoidal category $(\mathsf{C}, \bullet)$ together with a natural isomorphism

$$\beta_{A,B} \colon A \bullet B \to B \bullet A,$$

which satisfies the axioms

(E.5)

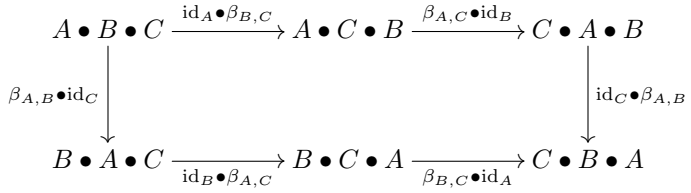


The natural isomorphism β is called a *braiding*. We often omit the subindexes A and B from β if they can be told from the context.

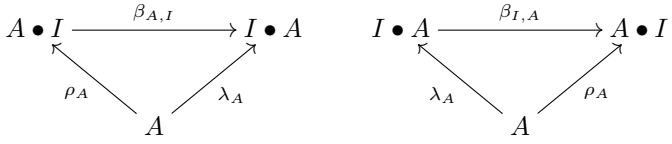
A monoidal category is *symmetric* if it is equipped with a braiding β which satisfies $\beta^2 = \mathrm{id}$. In this case β is called a *symmetry*.

It follows from the axioms that the following diagrams commute.

(E.6)



(E.7)



It follows from (E.3) and (E.7) that

$$\beta_{I,I} = \mathrm{id}_{I \bullet I}. \tag{E.8}$$

If β is a braiding for $(\mathsf{C}, \bullet)$, then so is its inverse β^{-1} defined by

$$(\beta^{-1})_{A,B} := (\beta_{B,A})^{-1} \colon A \bullet B \to B \bullet A.$$

The monoidal category $(\mathsf{C}^{\mathrm{op}}, \bullet)$ is braided, with the *opposite braiding* defined by

$$(\beta^{\mathrm{op}})_{A,B} := \beta_{B,A},$$

as is the category $(\mathsf{C}, \tilde{\bullet})$, with the *transpose braiding*

$$(\beta^{t})_{A,B} := \beta_{B,A}.$$

If C and C' are braided monoidal categories with braidings β and β', then so is $\mathsf{C} \times \mathsf{C}'$ with braiding

$$\begin{array}{ccc} (A, A') \bullet (B, B') & \dashrightarrow & (B, B') \bullet (A, A') \\ \| & & \| \\ (A \bullet B, A' \bullet B') & \xrightarrow[(\beta_{A,B}, \beta'_{A',B'})]{} & (B \bullet A, B' \bullet A'). \end{array}$$

E.2. Hopf monoids

A monoidal category allows one to define monoids and comonoids; if the category is braided, then one can also define bimonoids and Hopf monoids as well as different types of monoids. These objects along with their notations are summarized in Table E.1.

TABLE E.1. Categories of "monoids" in monoidal categories.

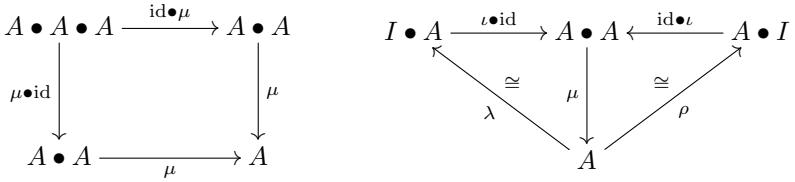
Category	Description	Category	Description
$\mathsf{Mon}(\mathsf{C})$	Monoids	$\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$	Com. bimonoids
$\mathsf{Comon}(\mathsf{C})$	Comonoids	${}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C})$	Cocom. bimonoids
$\mathsf{Bimon}(\mathsf{C})$	Bimonoids	${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$	Com. & cocom. bimonoids
$\mathsf{Hopf}(\mathsf{C})$	Hopf monoids	$\mathsf{Hopf}^{\mathsf{co}}(\mathsf{C})$	Com. Hopf monoids
$\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$	Com. monoids	${}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{C})$	Cocom. Hopf monoids
${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$	Cocom. comonoids	${}^{\mathsf{co}}\mathsf{Hopf}^{\mathsf{co}}(\mathsf{C})$	Com. & cocom. Hopf monoids

E.2.1. Monoids and comonoids.

Definition E.3. A *monoid* in a monoidal category $(\mathsf{C}, \bullet)$ is a triple (A, μ, ι) where

$$\mu \colon A \bullet A \to A \quad \text{and} \quad \iota \colon I \to A$$

(the product and the unit) satisfy the associativity and unit axioms, which state that the following diagrams commute.



A morphism $(A, \mu, \iota) \to (A', \mu', \iota')$ of monoids is a map $A \to A'$ which commutes with μ and μ', and ι and ι'.

Similarly a *comonoid* in a monoidal category $(\mathsf{C}, \bullet)$ is a triple (C, Δ, ϵ) where

$$\Delta\colon C \to C \bullet C \quad \text{and} \quad \epsilon\colon C \to I$$

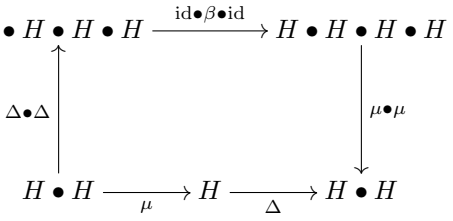
(the coproduct and the counit) satisfy the coassociativity and counit axioms. These are obtained from the monoid axioms by replacing μ by Δ and ι by ϵ, and reversing the arrows with those labels. A morphism $(C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ of comonoids is a map $C \to C'$ which commutes with Δ and Δ', and ϵ and ϵ'.

We denote the categories of monoids and of comonoids in $(\mathsf{C}, \bullet)$ by $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ respectively. The notions of monoid and comonoid are dual in the sense that $\mathsf{Mon}(\mathsf{C})$ is equivalent to $\mathsf{Comon}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.

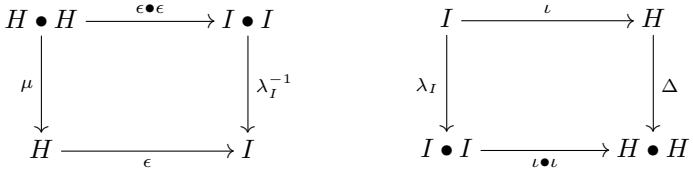
E.2.2. Bimonoids.

Definition E.4. A *bimonoid* in a braided monoidal category $(\mathsf{C}, \bullet, \beta)$ is a quintuple $(H, \mu, \iota, \Delta, \epsilon)$ where (H, μ, ι) is a monoid, (H, Δ, ϵ) is a comonoid, and the two structures are compatible in the sense that the following four diagrams commute.

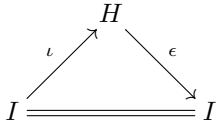
(E.9)



(E.10)



(E.11)



A morphism of bimonoids is a morphism of the underlying monoids and comonoids.

In a braided monoidal category $(\mathsf{C}, \bullet, \beta)$, if (A_1, μ_1) and (A_2, μ_2) are monoids, then so is $A_1 \bullet A_2$, with structure maps

$$\begin{aligned} A_1 \bullet A_2 \bullet A_1 \bullet A_2 &\xrightarrow{\mathrm{id}\bullet\beta\bullet\mathrm{id}} A_1 \bullet A_1 \bullet A_2 \bullet A_2 \xrightarrow{\mu_1\bullet\mu_2} A_1 \bullet A_2 \\ I &\xrightarrow{\lambda_I = \rho_I} I \bullet I \xrightarrow{\iota_1\bullet\iota_2} A_1 \bullet A_2. \end{aligned} \tag{E.12}$$

Dually, if (C_1, Δ_1) and (C_2, Δ_2) are comonoids, then so is $C_1 \bullet C_2$.

In this manner, $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ are monoidal categories. One can then give the following alternative description for bimonoids.

Proposition E.5. *A bimonoid is an object H in a braided monoidal category with maps*

$$\mu\colon H \bullet H \to H \qquad \Delta\colon H \to H \bullet H$$
$$\iota\colon I \to H \qquad \epsilon\colon H \to I$$

such that (H, μ, ι) is a monoid, (H, Δ, ϵ) is a comonoid, and μ and ι are morphisms of comonoids, or equivalently, Δ and ϵ are morphisms of monoids.

We denote the category of bimonoids in $(\mathsf{C}, \bullet, \beta)$ by $\mathsf{Bimon}(\mathsf{C})$.

E.2.3. Convolution monoids. Let $(\mathsf{C}, \bullet)$ be a monoidal category.

Definition E.6. For C a comonoid and A a monoid in C, define the *convolution monoid* as the set $\mathsf{C}(C, A)$ of all maps in C from C to A with the following product.

For $f, g \in \mathsf{C}(C, A)$, we let the product $f * g$ be the composite morphism

$$C \xrightarrow{\Delta} C \bullet C \xrightarrow{f\bullet g} A \bullet A \xrightarrow{\mu} A.$$

This is an associative product called *convolution*. The map $\iota\epsilon\colon C \to A$ serves as the unit for this product and is called the *convolution unit*. The set $\mathsf{C}(C, A)$ is thus an ordinary monoid, that is, a monoid in the monoidal category $(\mathsf{Set}, \times)$ of sets with cartesian product.

If C is a linear monoidal category, then $\mathsf{C}(C, A)$ has the structure of an (associative) algebra. This is called the *convolution algebra*.

Proposition E.7. *Let C and C' be comonoids and A and A' be monoids in C. Let*

$$j\colon C' \to C \quad \text{and} \quad k\colon A \to A'$$

be a morphism of comonoids and a morphism of monoids, respectively. Then the maps

$$\mathsf{C}(C, A) \to \mathsf{C}(C', A), \qquad f \mapsto fj$$

and

$$\mathsf{C}(C, A) \to \mathsf{C}(C, A'), \qquad f \mapsto kf$$

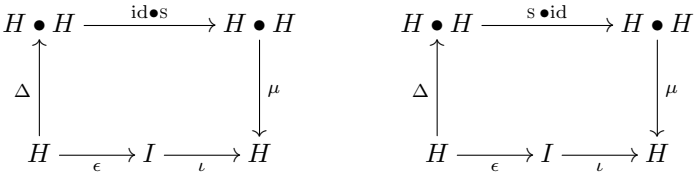
are morphisms of convolution monoids.

The proof is straightforward.

E.2.4. Hopf monoids. Let $(\mathsf{C}, \bullet, \beta)$ be a braided monoidal category. For a bimonoid H in C, consider the convolution monoid $\mathrm{End}(H) := \mathsf{C}(H, H)$.

Definition E.8. A *Hopf monoid* in C is a bimonoid H for which the identity map $\mathrm{id}\colon H \to H$ is invertible in the convolution monoid $\mathrm{End}(H)$. Explicitly, there must exist a map $\mathrm{s}\colon H \to H$ such that

(E.13)



commute. The map s is the *antipode* of H.

The antipode of a bimonoid H may exist or not, but if it does, then it is unique (and H is a Hopf monoid).

Proposition E.9. *Let H and H' be Hopf monoids. A morphism of bimonoids $H \to H'$ necessarily commutes with the antipodes and the convolution units.*

Proof. We prove the first claim. Let $k\colon H \to H'$ be a morphism of bimonoids. According to Proposition E.7, we have morphisms of convolution monoids

$$\mathsf{C}(H', H') \to \mathsf{C}(H, H'), \qquad f \mapsto fk$$

and

$$\mathsf{C}(H, H) \to \mathsf{C}(H, H'), \qquad f \mapsto kf.$$

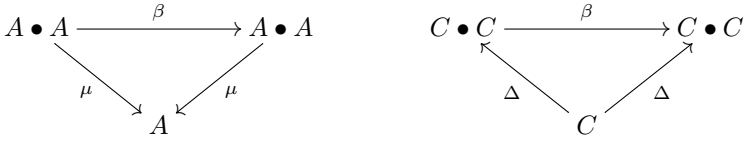
It follows that both $\mathrm{s}'k$ and $k\,\mathrm{s}$ are the inverse of k in $\mathsf{C}(H, H')$, so they must coincide. □

A morphism of Hopf monoids $H \to H'$ is defined to be a morphism of the underlying bimonoids. In view of Proposition E.9, such morphisms preserve the extra structure present in a Hopf monoid.

We denote the category of Hopf monoids in $(\mathsf{C}, \bullet, \beta)$ by $\mathsf{Hopf}(\mathsf{C})$.

E.2.5. Commutative monoids. In addition to playing a role in the definition of bimonoids, the braiding in a braided monoidal category is related to another aspect: the possibility of defining different types of monoids. Presently, we discuss the well-known example of commutative monoids.

Definition E.10. A *commutative monoid* (resp. *cocommutative comonoid*) in a braided monoidal category $(\mathsf{C}, \bullet, \beta)$ is a monoid A (resp. comonoid C) such that the left-hand (resp. right-hand) diagram below commutes.



A morphism of commutative monoids (resp. cocommutative comonoids) is a morphism of the underlying monoids (resp. comonoids).

We denote the category of commutative monoids and cocommutative comonoids in $(\mathsf{C}, \bullet, \beta)$ by $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$ and ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$ respectively. The definition implies that they are full subcategories of $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ respectively.

We say that a bimonoid or Hopf monoid is *(co)commutative* if its underlying (co)monoid is (co)commutative. Following the above notation, this defines categories $\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$, ${}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C})$ and ${}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})$ and three more with bimonoids replaced by Hopf monoids as shown in Table E.1.

E.3. Trivial examples

- The smallest symmetric monoidal category is the *one-arrow category* $(\mathsf{I}, \bullet)$. It has one object and one morphism with the tensor product and braiding defined in the obvious manner.
- In any monoidal category, the unit object I is a monoid and a comonoid, with the product and coproduct being

$$I \xrightarrow{\cong} I \bullet I \quad \text{and} \quad I \bullet I \xrightarrow{\cong} I$$

and the unit and counit being the identity morphism $I \to I$. Further, if the monoidal category is braided, then I is a bimonoid and a Hopf monoid (with the antipode being the identity morphism).

E.4. The diamond of categories

In this section, we study the iterations of the monoid and comonoid constructions. The result is summarized in the diamond shown in Figure E.1. From any vertex, there are two paths going down, the one to the left is the monoid construction, while the one to the right is the comonoid construction.

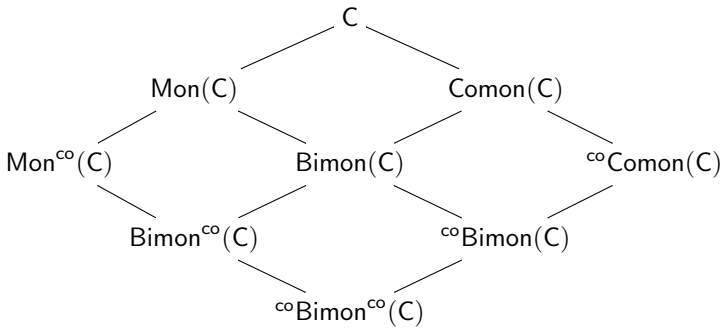


FIGURE E.1. Iterations of the monoid and comonoid constructions.

The apex of the diamond is any category C. If C is monoidal, then we can construct the first layer of the diamond. If C is braided monoidal, then we can construct the middle layer of the diamond. If C is symmetric monoidal, then we can construct the remaining two layers as well.

Stage 1. Suppose $(\mathsf{C}, \bullet)$ is a monoidal category. Then we can construct two categories from it, namely, $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$.

Stage 2. Now suppose in addition that $(\mathsf{C}, \bullet)$ is braided. Then, as mentioned in Section E.2.2, the categories $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ are themselves monoidal. So we can apply the monoid and comonoid constructions on either of them. The result of this iteration is given below.

Proposition E.11. *Suppose* C *is a braided monoidal category. Then*

$$\mathsf{Mon}(\mathsf{Comon}(\mathsf{C})) \cong \mathsf{Bimon}(\mathsf{C}) \cong \mathsf{Comon}(\mathsf{Mon}(\mathsf{C})),$$
$$\mathsf{Mon}(\mathsf{Mon}(\mathsf{C})) \cong \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}),$$
$$\mathsf{Comon}(\mathsf{Comon}(\mathsf{C})) \cong {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}).$$

The category of monoids in $\mathsf{Comon}(\mathsf{C})$ is equivalent to the category of comonoids in $\mathsf{Mon}(\mathsf{C})$, and both these categories are equivalent to the category of bimonoids in C. This follows from Proposition E.5.

But what is the category of monoids in $\mathsf{Mon}(\mathsf{C})$? An object in this category is an object in C equipped with two "compatible" monoidal structures. One can show by the *Eckmann–Hilton argument* that the compatibility forces the two monoidal structures to coincide and further they are commutative. From this one can deduce that the category of monoids in $\mathsf{Mon}(\mathsf{C})$ is equivalent to the category of commutative monoids in C.

Similarly, the category of comonoids in $\mathsf{Comon}(\mathsf{C})$ is equivalent to the category of cocommutative comonoids in C.

The classical Eckmann–Hilton argument is given below.

Eckmann–Hilton. Consider a set with two binary operations $+$ and $\times$ and two elements 0 and 1. Consider the axioms

$$(x + y) \times (z + t) = (x \times z) + (y \times t),$$
$$x + 0 = x = 0 + x,$$
$$x \times 1 = x = 1 \times x.$$

By setting $x = t = 1$ and $y = z = 0$, we deduce $1 = 0$. Then, by setting $y = z = 0$, we get that the operations $*$ and $+$ coincide. Next, by setting $x = t = 0$, we get that $+$ is commutative. Further manipulations give that $+$ is associative.

However, the categories $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$, $\mathsf{Bimon}(\mathsf{C})$ and ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$ may fail to be monoidal. (Related: The monoidal categories $\mathsf{Mon}(\mathsf{C})$ and $\mathsf{Comon}(\mathsf{C})$ may fail to be braided.)

Stage 3. Now suppose in addition that the braiding β is a symmetry. In this case, if A and B are monoids, then $\beta_{A,B} \colon A \bullet B \to B \bullet A$ is a morphism of monoids with respect to the monoid structure (E.12). It follows that $\mathsf{Mon}(\mathsf{C})$ is a symmetric monoidal category. Dually, $\mathsf{Comon}(\mathsf{C})$ is a symmetric monoidal category. Iterating these results and applying Proposition E.11, we deduce that $\mathsf{Bimon}(\mathsf{C})$, $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$, and ${}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})$ are symmetric monoidal categories as well. In particular, the tensor product of two bimonoids is again a bimonoid, and the tensor product of two (co)commutative (co)monoids is again (co)commutative.

So the monoid and comonoid constructions can be iterated indefinitely.

Proposition E.12. *Suppose* C *is a symmetric monoidal category. Then*

$$\mathsf{Mon}(\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})) \cong \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}),$$
$$\mathsf{Comon}({}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})) \cong {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}),$$
$$\mathsf{Mon}(\mathsf{Bimon}(\mathsf{C})) \cong \mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}) \cong \mathsf{Comon}(\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})),$$
$$\mathsf{Mon}({}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})) \cong {}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C}) \cong \mathsf{Comon}(\mathsf{Bimon}(\mathsf{C})),$$
$$\mathsf{Mon}({}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C})) \cong {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}) \cong \mathsf{Comon}(\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C})).$$

It is clear that no new categories are obtained beyond those of (co)commutative (co, bi)monoids.

We now extend the above considerations to Hopf monoids. Let $(\mathsf{C}, \bullet, \beta)$ be a braided monoidal category. In general, $\mathsf{Hopf}(\mathsf{C})$ fails to be a monoidal category. However, if β is a symmetry, then the tensor product $A \bullet B$ of two Hopf monoids A and B is another Hopf monoid. The bimonoid structure is as in Section E.2.2 and the antipode is $\mathrm{s}_A \bullet \mathrm{s}_B$. It follows that if C is a symmetric monoidal category, then so is $\mathsf{Hopf}(\mathsf{C})$.

E.5. Monoidal functors

Suppose $(\mathsf{C}, \bullet)$ and $(\mathsf{D}, \bullet)$ are braided monoidal category. A functor $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ is *monoidal* if for any objects A and B in C, there are natural isomorphisms

$$\mathcal{F}(A) \bullet \mathcal{F}(B) \xrightarrow{\cong} \mathcal{F}(A \bullet B) \quad \text{and} \quad I \xrightarrow{\cong} \mathcal{F}(I).$$

Proposition E.13. *Suppose* $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ *is a monoidal functor between braided monoidal categories. Then* $\mathcal{F}$ *induces functors*

$$\mathsf{Mon}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D}), \quad \mathsf{Comon}(\mathsf{C}) \to \mathsf{Comon}(\mathsf{D}),$$
$$\mathsf{Bimon}(\mathsf{C}) \to \mathsf{Bimon}(\mathsf{D}), \quad \mathsf{Hopf}(\mathsf{C}) \to \mathsf{Hopf}(\mathsf{D}).$$

Proof. Suppose A is a monoid in C. Then define the multiplication and unit of $\mathcal{F}(A)$ by

$$\mathcal{F}(A) \bullet \mathcal{F}(A) \xrightarrow{\cong} \mathcal{F}(A \bullet A) \to \mathcal{F}(A) \quad \text{and} \quad J \xrightarrow{\cong} \mathcal{F}(J) \to \mathcal{F}(A).$$

Check that: this turns $\mathcal{F}(A)$ into a monoid; if $A \to B$ is a morphism of monoids, then the induced map $\mathcal{F}(A) \to \mathcal{F}(B)$ is also a morphism of monoids. This induces the functor $\mathsf{Mon}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D})$.

By the same argument, one gets an induced functor on comonoids. For the induced functor on bimonoids and Hopf monoids, one needs to check further: If A is a bimonoid, then so is $\mathcal{F}(A)$. In addition, if A has an antipode s, then $\mathcal{F}(A)$ also has an antipode and it is given by $\mathcal{F}(\mathrm{s})$. □

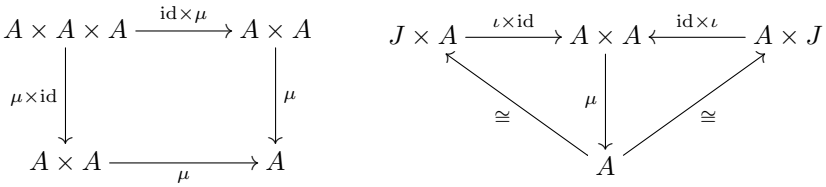
E.6. Cartesian monoidal categories

E.6.1. (Co)cartesian monoidal categories. Let C be a category with finite products. Then $(\mathsf{C}, \times, J)$ is a symmetric monoidal category: $A \times B$ is a chosen product of A and B and J is a chosen terminal object in C. The associativity, unit constraints, and the braiding are defined via the universal property for products. We say in this case that the monoidal category C is *cartesian*.

Explicitly, a monoid in $(\mathsf{C}, \times, J)$ is a triple (A, μ, ι) where

$$\mu \colon A \times A \to A \qquad \text{and} \qquad \iota \colon J \to A$$

(the product and the unit) satisfy the associativity and unit axioms, which state that the following diagrams commute.



What about comonoids? It is easy to see that every object C of C has a unique comonoid structure with respect to $\times$. Indeed, the counit $\epsilon_C \colon C \to J$ is the unique map to the terminal object J, and the coproduct $\Delta_C \colon C \to C \times C$ is the *diagonal*:

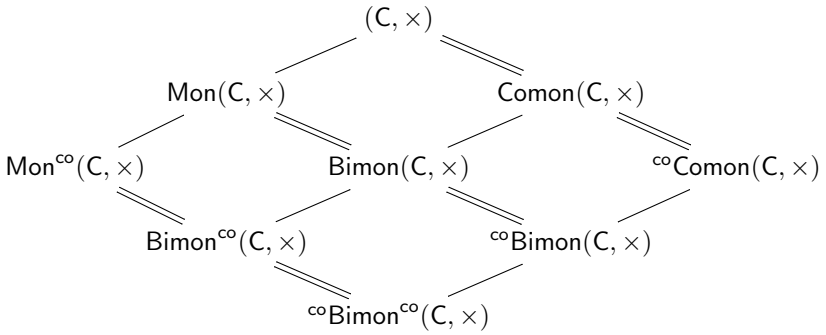
$$\Delta_C = (\mathrm{id}_C, \mathrm{id}_C).$$

Moreover, (C, Δ, ϵ) is cocommutative.

This yields equivalences of categories

$$\begin{aligned} {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}, \times) &\cong \mathsf{Comon}(\mathsf{C}, \times) \cong \mathsf{C}, \\ {}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C}, \times) &\cong \mathsf{Bimon}(\mathsf{C}, \times) \cong \mathsf{Mon}(\mathsf{C}, \times), \\ {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}, \times) &\cong \mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}, \times) \cong \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}, \times). \end{aligned}$$

Thus, in this situation, the diamond of Figure E.1 collapses on to one of its sides, so there are only three categories to consider instead of nine as shown below.



Dually, any category with finite coproducts is symmetric monoidal, with the monoidal structure given by a choice of coproduct $A \amalg B$ and the unit given by a chosen initial object I. Such monoidal categories $(\mathsf{C}, \amalg, I)$ are called *cocartesian*. Every object in a cocartesian monoidal category is a monoid in a unique way. So this does not give anything new. The useful notion here is that of comonoids.

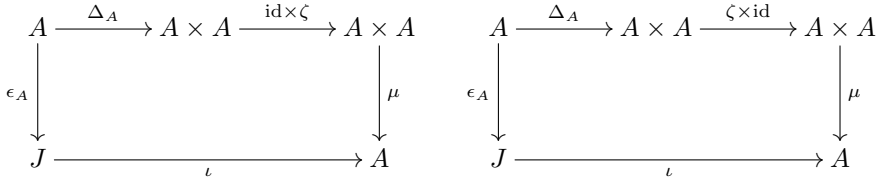
There are equivalences of categories

$$\begin{aligned} \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}, \amalg) &\cong \mathsf{Mon}(\mathsf{C}, \amalg) \cong \mathsf{C}, \\ \mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}, \amalg) &\cong \mathsf{Bimon}(\mathsf{C}, \amalg) \cong \mathsf{Comon}(\mathsf{C}, \amalg), \\ {}^{\mathsf{co}}\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}, \amalg) &\cong {}^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C}, \amalg) \cong {}^{\mathsf{co}}\mathsf{Comon}(\mathsf{C}, \amalg). \end{aligned}$$

The diamond of Figure E.1 collapses on to one of its sides.

E.6.2. (Co)group objects and Hopf monoids.

Definition E.14. A *group object* in a cartesian monoidal category $(\mathsf{C}, \times, J)$ is a monoid (A, μ, ι) together with a morphism $\zeta : A \to A$ such that the following diagrams commute.



We refer to ζ as the inverse morphism.

A morphism between group objects $A \to B$ is a morphism between their underlying monoids.

For any category C with finite products, we use the notation $\mathsf{Group}(\mathsf{C})$ to denote the category of group objects in C.

Dually, a *cogroup* in a cocartesian monoidal category is a comonoid C equipped with a morphism $\zeta : C \to C$ such that diagrams dual to the ones above commute (reverse all arrows). We use the notation $\mathsf{Cogroup}(\mathsf{C})$ to denote the category of cogroup objects in C.

Observe that

$$\begin{aligned} \mathsf{Hopf}(\mathsf{C}, \times) = {}^{\mathsf{co}}\mathsf{Hopf}(\mathsf{C}, \times) = \mathsf{Group}(\mathsf{C}, \times), \\ \mathsf{Hopf}(\mathsf{C}, \amalg) = \mathsf{Hopf}^{\mathsf{co}}(\mathsf{C}, \amalg) = \mathsf{Cogroup}(\mathsf{C}, \amalg). \end{aligned} \tag{E.14}$$

Note that $\mathsf{Group}(\mathsf{C})$ is a full subcategory of $\mathsf{Mon}(\mathsf{C})$. This is an instance of the fact that $\mathsf{Hopf}(\mathsf{C})$ is a full subcategory of $\mathsf{Bimon}(\mathsf{C})$:

Given a monoid A in C, the inverse morphism ζ, if it exists, is unique. This is because ζ is the inverse of id in the convolution monoid $\mathsf{C}(A, A)$. So in the passage from $\mathsf{Mon}(\mathsf{C})$ to $\mathsf{Group}(\mathsf{C})$, we are picking those monoids which are group objects.

Example E.15. Here are some examples of cartesian monoidal categories and the standard names that are employed for the corresponding group objects.

Cartesian monoidal category	Category of group objects
Sets	groups
Topological spaces	topological groups
Manifolds	Lie groups
Affine varieties	algebraic groups
Schemes	group schemes
Affine schemes	affine group schemes

As is evident, the motivation for the terminology "groups objects" comes from the example of sets: The category of group objects in Set is equivalent to the category of groups, that is,

$$\mathsf{Group}(\mathsf{Set}) = \mathsf{Group}.$$

Proposition E.16. *Suppose* C *and* D *are categories with finite products, and* $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ *is a functor which preserves products, then* $\mathcal{F}$ *induces functors*

$$\mathsf{Mon}(\mathsf{C}) \to \mathsf{Mon}(\mathsf{D}) \quad \textit{and} \quad \mathsf{Group}(\mathsf{C}) \to \mathsf{Group}(\mathsf{D}).$$

PROOF. If $\mathcal{F}$ preserves products, then $\mathcal{F}$ is a monoidal functor wrt these monoidal structures. So the result follows by applying Proposition E.13. Alternatively, one can proceed more directly as below.

Suppose A is a monoid in C. Then define the multiplication and unit of $\mathcal{F}(A)$ by

$$\mathcal{F}(A) \times \mathcal{F}(A) \xrightarrow{\cong} \mathcal{F}(A \times A) \to \mathcal{F}(A) \quad \text{and} \quad J \xrightarrow{\cong} \mathcal{F}(J) \to \mathcal{F}(A).$$

Check that: this turns $\mathcal{F}(A)$ into a monoid; if $A \to B$ is a morphism of monoids, then the induced map $\mathcal{F}(A) \to \mathcal{F}(B)$ is also a morphism of monoids; and if A is a group object, then so is $\mathcal{F}(A)$. □

For example, the forgetful functor $\mathsf{Top} \to \mathsf{Set}$ preserves products. This implies in particular that the underlying set of a topological group is a group.

E.6.3. Cartesian category arising from a symmetric monoidal category. Let $(\mathsf{C}, \bullet, I)$ be a symmetric monoidal category. Consider the category of commutative monoids $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$. If A and B are commutative monoids, then so is $A \bullet B$. We claim that this is the categorical coproduct of $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$.

Proposition E.17. *For any symmetric monoidal category $(\mathsf{C}, \bullet)$, the categorical coproduct of $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$ is $\bullet$.*

The canonical maps are

$$A \xrightarrow{\cong} A \bullet I \xrightarrow{\mathrm{id} \bullet \mu} A \bullet B \quad \text{and} \quad B \xrightarrow{\cong} I \bullet B \xrightarrow{\mu \bullet \mathrm{id}} A \bullet B.$$

If C is a commutative monoid and $f : A \to C$ and $g : B \to C$ are morphisms of commutative monoids, then there is a unique morphism $A \bullet B \to C$ given by the composite

$$A \bullet B \xrightarrow{f \bullet g} C \bullet C \xrightarrow{\mu} C$$

which commutes with the canonical maps.

Corollary E.18. *Recall that $\mathsf{Alg}_R^{\mathsf{co}}$ denotes the category of commutative R-algebras, where R is a commutative ring. The categorical coproduct in $\mathsf{Alg}_R^{\mathsf{co}}$ is the tensor product.*

In particular, the categorical product in Ring *is the tensor product.*

PROOF. Apply the result to $(\mathsf{Mod}_R, \otimes)$. □

Now we can consider comonoids and cogroup objects in this cocartesian monoidal category. These are familiar notions:

$$\mathsf{Bimon}^{\mathsf{co}}(\mathsf{C}) \cong \mathsf{Comon}(\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})) \quad \text{and} \quad \mathsf{Hopf}^{\mathsf{co}}(\mathsf{C}) \cong \mathsf{Cogroup}(\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})). \tag{E.15}$$

We are familiar with the first fact. The second fact says that a commutative Hopf monoid is same as a cogroup in the category of commutative monoids. The extra ingredient in one case is the antipode and in the other case is the inverse morphism. It can be deduced by applying (E.14) to $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$.

Dually,

$$^{\mathsf{co}}\mathsf{Bimon}(\mathsf{C}) \cong \mathsf{Mon}(^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})) \quad \text{and} \quad ^{\mathsf{co}}\mathsf{Hopf}(\mathsf{C}) \cong \mathsf{Group}(^{\mathsf{co}}\mathsf{Comon}(\mathsf{C})). \tag{E.16}$$

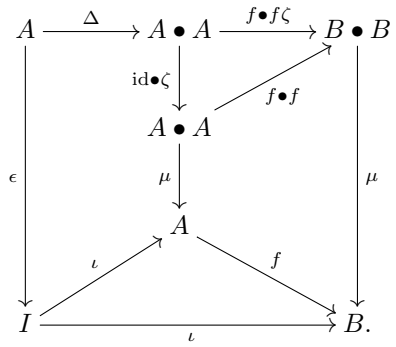
E.6.4. Example of a convolution monoid which is a group.

Proposition E.19. *Suppose* $(\mathsf{C}, \bullet)$ *is a cocartesian monoidal category. Let* A *be a cogroup object and* B *be any object in* C*. Then the convolution monoid* $\mathsf{C}(A, B)$ *is a group.*

Dually: Suppose C *is a cartesian monoidal category. Let* A *be any object and* B *be a group object in* C*. Then the convolution monoid* $\mathsf{C}(A, B)$ *is a group.*

Proof. It suffices to prove the first part. The second part can then be deduced by passing to the opposite category.

Let $\zeta : A \to A$ denote the inverse morphism of A. Suppose $f : A \to B$ is any morphism in C. Then $f\zeta$ is the inverse of f in the convolution monoid. One half of this claim is checked below; the other half can be deduced by symmetry.



Since C is a cocartesian monoidal category, the unit object I is same as the initial object in C, each object in C is a monoid in a unique manner. We have written μ for the product and ι for the unit map. Thus the square and the lower triangle commute. The pentagon commutes because A is a cogroup object. The upper triangle commutes because of functoriality of $\bullet$. □

Corollary E.20. *Let* H *be a commutative Hopf monoid and let* B *be a commutative monoid. Then the set of monoid morphisms* $H \to B$ *under convolution is a group.*

Proof. Apply Proposition E.19 to the category of commutative monoids and use (E.15). □

E.7. Modules and comodules

Definition E.21. Let (A, μ, ι) be a monoid in $(\mathsf{C}, \bullet)$. A *left* A*-module* is a pair (M, χ) where

$$\chi \colon A \bullet M \to M$$

satisfies the usual associativity and unit axioms. A *right* A*-module* is defined similarly in terms of a structure map $M \bullet A \to M$.

A morphism $(M, \chi) \to (M', \chi')$ of left A-modules is a map $M \to M'$ which commutes with χ and χ'. We denote the category of left A-modules in $(\mathsf{C}, \bullet)$ by $\mathsf{Mod}_A(\mathsf{C})$.

Assume now that $(\mathsf{C}, \bullet, \beta)$ is a braided monoidal category. Let A_i be a monoid in $(\mathsf{C}, \bullet)$, $i = 1, 2$, and consider the monoid $A_1 \bullet A_2$ defined in (E.12). If (M_i, χ_i) is a left A_i-module, then $M_1 \bullet M_2$ is a left $A_1 \bullet A_2$-module with structure map

$$A_1 \bullet A_2 \bullet M_1 \bullet M_2 \xrightarrow{\mathrm{id}\bullet\beta\bullet\mathrm{id}} A_1 \bullet M_1 \bullet A_2 \bullet M_2 \xrightarrow{\chi_1\bullet\chi_2} M_1 \bullet M_2.$$

It follows that if H is a bimonoid, and (M_1, χ_1) and (M_2, χ_2) are left H-modules, then so is $M_1 \bullet M_2$, with structure map

$$H \bullet M_1 \bullet M_2 \xrightarrow{\Delta \bullet \mathrm{id} \bullet \mathrm{id}} H \bullet H \bullet M_1 \bullet M_2 \xrightarrow{\mathrm{id} \bullet \beta \bullet \mathrm{id}} H \bullet M_1 \bullet H \bullet M_2 \xrightarrow{\chi_1 \bullet \chi_2} M_1 \bullet M_2.$$

In addition, the unit object I is a left H-module with structure map

$$H \bullet I \xrightarrow{\rho_H^{-1}} H \xrightarrow{\epsilon} I.$$

In this manner, the category $\mathsf{Mod}_H(\mathsf{C})$ is monoidal. A monoid in $\mathsf{Mod}_H(\mathsf{C})$ is called an *H-module-monoid*. A comonoid in $\mathsf{Mod}_H(\mathsf{C})$ is called an *H-module-comonoid*.

Let C be a comonoid. Dualizing the above definitions, we obtain the notions of left C-comodule, right C-comodule, and C-D-bicomodule, where D is another comonoid. Let $\mathsf{Comod}^C(\mathsf{C})$ be the category of left C-comodules. comodule-monoidcomodule-comonoidIf H is a bimonoid, $\mathsf{Comod}^H(\mathsf{C})$ is monoidal, and one has the notions of *H-comodule-monoid* and *H-comodule-comonoid*.

APPENDIX F

The Yoneda embedding

Any category with small hom-sets can be canonically embedded in a functor category. This is known as the Yoneda embedding. Further, it preserves products.

F.1. Functor categories

For any categories C and D, let D^{C} denote the category of functors from C to D: Objects are functors from C to D and morphisms are natural transformations. Such a category is called a *functor category*.

- If C is the one-arrow category, then $\mathsf{D}^{\mathsf{C}} = \mathsf{D}$.
- If C is the discrete category indexed by the natural numbers (starting at 0), then $\mathsf{Vec}^{\mathsf{C}}$ is the category of graded vector spaces.

If the categorical product exists in D, then it also exists in D^{C}, and can be computed pointwise. Suppose $\mathcal{F}$ and $\mathcal{G}$ are functors from C to D, then define the functor $\mathcal{F} \times \mathcal{G}$ by

$$(\mathcal{F} \times \mathcal{G})(A) := \mathcal{F}(A) \times \mathcal{G}(A)$$

where on the right we take the categorical product in D. It is straightforward to check that $\mathcal{F} \times \mathcal{G}$ is the categorical product of $\mathcal{F}$ and $\mathcal{G}$.

More generally if some kind of limit exists in D, then the same kind of limit also exists in D^{C}. This is also true for colimits.

Proposition F.1. *Suppose* D *is a cartesian monoidal category. Then: A functor* $\mathcal{F} : \mathsf{C} \to \mathsf{D}$ *is a monoid (group object) in* $(\mathsf{D}^{\mathsf{C}}, \times)$ *iff for each object* A *in* C, $\mathcal{F}(A)$ *is a monoid (group object) in* $(\mathsf{D}, \times)$ *and for each morphism* f *in* C, $\mathcal{F}(f)$ *is a morphism of monoids (group objects) in* $(\mathsf{D}, \times)$. *Thus, there are equivalences of categories*

$$\mathsf{Mon}(\mathsf{D}^{\mathsf{C}}, \times) \cong \mathsf{Mon}(\mathsf{D}, \times)^{\mathsf{C}} \quad \textit{and} \quad \mathsf{Group}(\mathsf{D}^{\mathsf{C}}, \times) \cong \mathsf{Group}(\mathsf{D}, \times)^{\mathsf{C}}.$$

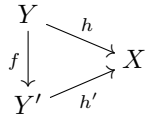
Proof. Straightforward. □

F.2. Slice categories

Let D be a category and X an object of D. The *slice category over* X, denoted $\mathsf{D} \downarrow X$, is defined as follows. The objects are the morphisms

$$h : Y \to X$$

of D with target X. A morphism from $h : Y \to X$ to $h' : Y' \to X$ is a morphism $f : Y \to Y'$ in D such that



commutes. Composition and identities are inherited from D.

The slice category of objects *under* X denoted $X \downarrow \mathsf{D}$ is defined similarly by using morphisms with source X.

Note that the product in $\mathsf{D} \downarrow X$ amounts to a pullback from X. Dually, the coproduct in $X \downarrow \mathsf{D}$ amounts to a pushout from X.

F.3. The Yoneda lemma

Recall that for any categories C and D, D^{C} denotes the category of functors from C to D.

Let C be any category (with small hom-sets). Define a functor

$$\mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}} \qquad X \mapsto h_X \tag{F.1}$$

where $h_X(Y) = \mathsf{C}(Y, X)$, the set of morphisms from Y to X.

- For any morphism $Y \to Y'$ in C, there is a map $\mathsf{C}(Y', X) \to \mathsf{C}(Y, X)$ which sends $Y' \to X$ to the composite $Y \to Y' \to X$. So h_X is a contravariant functor from C to Set.
- For any morphism $X \to X'$ in C, there is a natural transformation from h_X to $h_{X'}$: There is a map $h_X(Y) \to h_{X'}(Y)$ which sends $Y \to X$ to the composite $Y \to X \to X'$. This map is natual is Y. This means that for any morphism $Y \to Y'$, the following diagram commutes.

$$\begin{array}{ccc} h_X(Y') & \longrightarrow & h_{X'}(Y') \\ \downarrow & & \downarrow \\ h_X(Y) & \longrightarrow & h_{X'}(Y) \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{C}(Y', X) & \longrightarrow & \mathsf{C}(Y', X') \\ \downarrow & & \downarrow \\ \mathsf{C}(Y, X) & \longrightarrow & \mathsf{C}(Y, X') \end{array}$$

  Both composites send $Y' \to X$ to $Y \to Y' \to X \to X'$.

It seems that this construction is equivalent to the functor

$$\mathsf{C} \times \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}, \quad (X, Y) \mapsto \mathsf{C}(Y, X).$$

Equivalently, one may also consider the functor

$$\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}^{\mathsf{C}} \qquad X \mapsto h^X$$

where $h^X(Y) = \mathsf{C}(X, Y)$, the set of morphisms from X to Y.

Lemma F.2. *The functor $\mathsf{C} \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is fully faithful. Further if the categorical product exists in C, then this functor preserves products.*

The categorical product exists in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ since it is a functor category and the product exists in Set.

Proof. To show that the functor is fully faithful, we need to show that for any objects X and X', the map

$$\mathsf{C}(X, X') \to \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}(h_X, h_{X'}) \tag{F.2}$$

is a bijection. We do this by constructing the inverse map. Suppose we are given a natural transformation from h_X to $h_{X'}$. Then consider the map $h_X(X) \to h_{X'}(X)$. The image of the identity $X \to X$ yields a morphism $X \to X'$. Naturality shows that the natural transformation associated to this morphism is precisely the one that we started with.

Further for any objects X and X', there is a natural isomorphism

$$h_X \times h_{X'} \cong h_{X \times X'}.$$

This can be seen from the following calculation.

$$\begin{aligned}(h_X \times h_{X'})(Y) \cong h_X(Y) \times h_{X'}(Y) \cong \mathsf{C}(Y, X) \times \mathsf{C}(Y, X') \\ \cong \mathsf{C}(Y, X \times X') \cong h_{X \times X'}(Y).\end{aligned}$$

All isomorphisms are natural in X, X' and Y. □

This is known as the Yoneda lemma. It embeds a category into a larger functor category. Often it is easier to work with the functor category from which results about the original category can be deduced.

Remark F.3. The categorical coproduct always exists in $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$. (Define it pointwise using the coproduct in Set.) The coproduct may also exist in C, but the Yoneda embedding will not preserve coproducts in general.

Proposition F.4. *A functor $\mathcal{F} : \mathsf{D} \to \mathsf{Set}$ is a (monoid) group in $\mathsf{Set}^{\mathsf{D}}$ iff for each object A in D, $\mathcal{F}(A)$ is a (monoid) group and for each morphism f in D, $\mathcal{F}(f)$ is a (monoid) group homomorphism. Thus*

$$\mathsf{Mon}(\mathsf{Set}^{\mathsf{D}}) \cong \mathsf{Monoid}^{\mathsf{D}} \quad \mathsf{Group}(\mathsf{Set}^{\mathsf{D}}) \cong \mathsf{Group}^{\mathsf{D}}.$$

Proof. Apply Proposition F.1. □

Proposition F.5. *If C is a cartesian monoidal category, then an object A is a group (monoid) in C iff h_A is a group (monoid) in the functor category $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$.*

Proof. Suppose A is a monoid in C. Then Proposition E.16 applied to the Yoneda embedding implies that h_A is a monoid in the functor category. Explicitly, for each object Y, $\mathsf{C}(Y, A)$ is a monoid: The product is the composite map

$$\mathsf{C}(Y, A) \times \mathsf{C}(Y, A) \xrightarrow{\cong} \mathsf{C}(Y, A \times A) \to \mathsf{C}(Y, A).$$

The first map is the universal property of the product, the second map is the monoid structure of A. The unit element is the image of the map

$$\mathsf{C}(Y, J) \to \mathsf{C}(Y, A).$$

Recall that J is the terminal object, so $\mathsf{C}(Y, J)$ is a singleton.

Conversely, suppose h_A is a monoid in the functor category. Then there are maps

$$\mathsf{C}(Y, A \times A) \xrightarrow{\cong} \mathsf{C}(Y, A) \times \mathsf{C}(Y, A) \to \mathsf{C}(Y, A) \quad \text{and} \quad \mathsf{C}(Y, J) \to \mathsf{C}(Y, A).$$

natural in Y. Set $Y = A \times A$ in the first and $Y = J$ in the second. The images of the identity map yield morphisms $A \times A \to A$ and $J \to A$. Check that these turn A into a monoid.

Further, check that the two constructions described above are inverse to each other. □

Proposition F.6. *Suppose C is a cartesian monoidal category. Then the Yoneda embedding (F.1) induces fully faithful functors:*

$$\mathsf{Mon}(\mathsf{C}) \to \mathsf{Monoid}^{\mathsf{C}^{\mathrm{op}}} \quad \textit{and} \quad \mathsf{Group}(\mathsf{C}) \to \mathsf{Group}^{\mathsf{C}^{\mathrm{op}}}. \tag{F.3}$$

PROOF. Suppose X and X' are monoids in C. Then h_X are $h_{X'}$ are monoids in the functor category. We need to check that the bijection (F.2) restricts to a bijection between the subsets of monoid homomorphisms. This is straightforward.

Same for group objects. □

In particular, for a group object B in a cartesian monoidal category, the set of morphisms $A \to B$ has the structure of a group. Dually, for a cogroup object A in a cocartesian monoidal category, the set of morphisms $A \to B$ has the structure of a group. The group multiplication is both cases is convolution, see Proposition E.19.

APPENDIX G

Functors from commutative monoids to groups

Let $(\mathsf{C}, \bullet)$ be a symmetric monoidal category. In this section, we discuss two ways to construct a functor from the category of commutative monoids $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C}, \bullet)$ to the category of groups Group.

Suppose A is a commutative monoid A. The two ways to associate a group to A are briefly described below.

- The group of A-points of a commutative Hopf monoid H in $(\mathsf{C}, \bullet)$.
- The group of invertible right A-module endomorphisms of $V \bullet A$ for an object V of C.

Both constructions are natural in A. There is a simple relationship between the two. There is a natural transformation from the first to the second precisely if V is a right H-comodule.

G.1. The functor of points of a commutative Hopf monoid

Let $(\mathsf{C}, \bullet)$ be a symmetric monoidal category. Then there is a canonical embedding

$$\mathcal{G} : \mathsf{Hopf}^{\mathsf{co}}(\mathsf{C})^{\mathrm{op}} \to \mathsf{Group}^{\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})} \tag{G.1}$$

of the category of commutative Hopf monoids in C to the category of functors from commutative monoids in C to groups. This is obtained as follows. Recall from (E.15) that $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})$ with $\bullet$ is a cocartesian monoidal category, and its category of cogroup objects is $\mathsf{Hopf}^{\mathsf{co}}(\mathsf{C})$. Define the functor $\mathcal{G}$ to be the specialization of the Yoneda embedding (F.3) to $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})^{\mathrm{op}}$.

Let us make this functor explicit. For any commutative Hopf monoid H, there is a functor

$$\mathcal{G}(H) : \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \to \mathsf{Group}, \quad A \mapsto \mathsf{Mon}^{\mathsf{co}}(H, A).$$

Thus, this functor sends a commutative monoid A to the set of all morphisms of (commutative) monoids from H to A. Any such morphism $H \to A$ is called a A-point of H, and $\mathsf{Mon}^{\mathsf{co}}(H, A)$ is the set of A-points of H. This set has a group structure given by convolution: For $f, g : H \to A$, their convolution product $f * g$ is given by

$$H \xrightarrow{\Delta} H \bullet H \xrightarrow{f \bullet g} A \bullet A \xrightarrow{\mu} A.$$

One can directly check that if f and g are morphisms of monoids, then so is $f * g$. The composite map $\iota\epsilon : H \to I \to A$ serves as the unit for this product. Each $f : H \to A$ has an inverse given by the composite $f\,\mathrm{s} : H \to H \to A$.

G.2. The functor of matrices

Let $(\mathsf{C}, \bullet)$ be a symmetric monoidal category. To every object V in C, we associate a functor

$$\mathcal{F}_V : \mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \to \mathsf{Monoid}.$$

Let us first understand how this functor is defined on objects. For any commutative monoid A in C, let $\mathcal{F}_V(A)$ be the set of all right A-module maps $V \bullet A \to V \bullet A$, or equivalently, the set of all morphisms $V \to V \bullet A$. (This is because $V \bullet A$ is the free right A-module on V.) The set $\mathcal{F}_V(A)$ is a monoid under composition: Suppose ρ and ρ' are morphisms of the form $V \to V \bullet A$. Then note that their composite is given by

$$A \xrightarrow{\rho} V \bullet A \xrightarrow{\rho' \bullet \mathrm{id}} V \bullet A \bullet A \xrightarrow{\mathrm{id} \bullet \mu} V \bullet A.$$

The identity element is the composite

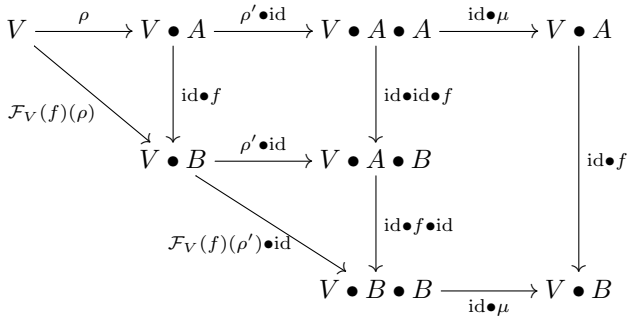
$$V \to V \bullet I \xrightarrow{\mathrm{id} \bullet \iota} V \bullet A.$$

If $A = I$, then $\mathcal{F}_V(A)$ is the monoid of endomorphisms of V.

Now let us understand how $\mathcal{F}_V$ is defined on morphisms. Suppose $f : A \to B$ is a morphism of (commutative) monoids in C. The map $\mathcal{F}_V(f) : \mathcal{F}_V(A) \to \mathcal{F}_V(B)$ is defined by:

$$(V \xrightarrow{\rho} V \bullet A) \longmapsto (V \xrightarrow{\rho} V \bullet A \xrightarrow{\mathrm{id} \bullet f} V \bullet B).$$

We need to check that $\mathcal{F}_V(f)$ is a monoid homomorphism. Since f respects units, we have $\mathcal{F}_V(\mathrm{id}) = \mathrm{id}$. To check that it respects composition:



Going across and down, we get $\mathcal{F}_V(f)(\rho'\rho)$, and going down and across, we get $\mathcal{F}_V(f)(\rho')\mathcal{F}_V(f)(\rho)$. The triangles commute by definition. The square commutes since both directions yield the morphism $\rho' \bullet f$. The pentagon commutes since $f : A \to B$ is a morphism of monoids.

It is clear that for $f : A \to B$ and $g : B \to C$,

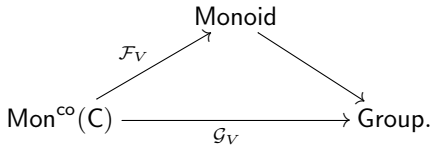
$$\mathcal{F}_V(g)\mathcal{F}_V(f) = \mathcal{F}_V(gf).$$

This completes the construction of the functor $\mathcal{F}_V$.

We always have the functor

$$\mathsf{Monoid} \to \mathsf{Group}$$

which sends a monoid (in Set) to its group of units. (A unit is an element of the monoid which has a left and right inverse.) Define



We say that a functor of the form $\mathsf{Mon}^{\mathsf{co}}(\mathsf{C}) \to \mathsf{Group}$ is *representable* if it is of the form $\mathcal{G}(H)$ for a commutative Hopf monoid H in C.

Proposition G.1. *Suppose $(\mathsf{C}, \bullet) = (\mathsf{Vec}, \otimes)$ and let V be a vector space of dimension n. Then $\mathcal{G}_V$ is representable, with the commutative Hopf algebra being*

$$\Bbbk[x_{11}, \ldots, x_{nn}, 1/\det]$$

for $n \geq 1$ and $\Bbbk$ for $n = 0$.

PROOF. The $n = 0$ case is clear. For $n \geq 1$, let H be the above Hopf algebra. Recall that the A-points of H are invertible $n \times n$ matrices with entries in A. This is precisely the group of A-linear endomorphisms of $A^n = V \otimes A$. □

Theorem G.2. *Suppose H is a commutative Hopf monoid in C, and V is an object in C. Then the following are equivalent.*

1. *A morphism $\mathcal{G}(H) \to \mathcal{G}_V$ in $\mathsf{Group}^{\mathsf{Mon}^{\mathsf{co}}(\mathsf{C})}$ (that is, a natural transformation from $\mathcal{G}(H)$ to $\mathcal{G}_V$).*
2. *A right H-comodule structure on V.*

PROOF. Suppose we are given a morphism $\mathcal{G}(H) \to \mathcal{G}_V$. In particular: For any commutative monoid A and a morphism $H \to A$ of monoids, we are given a morphism $V \to V \bullet A$. Now take $A := H$ and the identity morphism $H \to H$. This yields a morphism $V \to V \bullet H$. It is straightforward to check that this turns V into a right H-comodule.

Conversely, suppose we are given a right H-comodule structure on V. Then

$$(H \xrightarrow{f} A) \longmapsto (V \to V \bullet H \xrightarrow{id \bullet f} V \bullet A).$$

defines the required morphism $\mathcal{G}(H) \to \mathcal{G}_V$. It is straightforward to check that this is a morphism, and that the two constructions are inverse to each other. □

G.3. Problems

(1) Let $(\mathsf{C}, \bullet) = (\mathsf{Set}, \times)$, and let V be any set. Show that the functor $\mathcal{G}_V$ is not representable.

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