HILBERT FUNCTIONS OF COHEN-MACAULAY MODULES

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To my parents

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This thesis is dedicated to my parents.

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ABSTRACT

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Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring, M a finite Cohen-Macaulay A-module of dimension r and let I be an ideal of definition for M. In Chapter 2 we define the notion of minimal multiplicity of Cohen-Macaulay modules with respect to I and show that if M has minimal multiplicity with respect to I then the associated graded module $G_I(M)$ is Cohen-Macaulay. In Chapter 3 we assume that A is Cohen-Macaulay, M is maximal Cohen-Macaulay and I is $\mathfrak{m}$-primary. We find a relation between the first Hilbert coefficient of M, A and $\operatorname{Syz}_1^A(M)$. Sharper results are found when $I = \mathfrak{m}$. Set $\chi_1(M) = e_1(M) - e_0(M) + \mu(M)$. We prove that

$\chi_1(A)\mu(M) \geq \chi_1(M) + \chi_1(\operatorname{Syz}_1^A(M))$.

When A is Gorenstein then M is the first syzygy of $S^A(M) = (\operatorname{Syz}_1^A(M^*))^*$. A relation between the second Hilbert coefficient of M, A and $S^A(M)$ is found when $G(M)$ is Cohen-Macaulay and $\operatorname{depth} G(A) \geq d - 1$. In Chapter 4 we study Hilbert functions of maximal Cohen-Macaulay modules over a hypersurface ring A. We show that if $d > 0$ then the Hilbert function of M with respect to $\mathfrak{m}$ is non-decreasing. If $A = Q/(f)$ for some regular local ring Q, we determine a lower bound for $e_0(M)$ and $e_1(M)$. Furthermore we analyze the case when equality holds and prove that in this case $G(M)$ is Cohen-Macaulay. Furthermore in this case we also determine the Hilbert function of M.

INTRODUCTION

This thesis is devoted to the study of Hilbert functions of finitely generated Cohen-Macaulay modules over a local ring. In this thesis all rings are commutative noetherian and all modules are assumed to be finite.

Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian local ring and M a finite A-module. Let I be an ideal of definition for M, i.e., $\mathfrak{m}^n M \subseteq IM$ for some $n > 0$. We let $\mu(M)$ denotes the minimal number of generators of M and $\lambda(M)$ its length. The Hilbert function of M with respect to I is the function

$$H^I(M, n) = \lambda(I^n M / I^{n+1} M) \text{ for all } n \geq 0$$

It is an important invariant of M that determines, among other things the dimension and the multiplicity of the module. When the ring A is Cohen-Macaulay then Sally ([16], [20], [20], [18], [19], [21]), Rossi and Valla ([13], [14], [15]), and others, show that the Hilbert function also contains information about $G_I(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}$, the associated graded ring of A with respect to I. A main theme of this thesis is to extend such results to the module case that is, to the associated graded module $G_I(M) = \bigoplus_{n \geq 0} I^n M / I^{n+1} M$ considered as a $G_I(A)$-module.

The formal power series

$$H^I_M(z) = \sum_{n \geq 0} H^I(M, n) z^n$$

is called the *Hilbert series* of M with respect to I. It is well known that it is a rational function of the form

$$H^I_M(z) = \frac{h^I_M(z)}{(1-z)^r}, \text{ where } r = \dim M \quad \text{and}$$
$$h^I_M(z) = h^I_0(M) + h^I_1(M) z + \cdots + h^I_s(M) z^s \in \mathbb{Z}[z]$$

We call the polynomial $h^I_M(z)$ the *h-polynomial* of M with respect to I. It follows from the definition that

$$h^I_0(M) = \lambda(M/IM) \quad \text{and} \quad h^I_1(M) = \lambda(IM/I^2M) - r\lambda(M/IM)$$

If f is a polynomial, then we use $f^{(i)}$ to denote its i-th derivative. The integers $e^I_i(M) = h^{(i)}_M(1)/i!$ for $i \geq 0$ are called the *Hilbert coefficients* of M. The number $e^I(M) = e^I_0(M)$ is known as the *multiplicity* of M with respect to I. When $I = \mathfrak{m}$ we drop the superscript $\mathfrak{m}$; for example, we write $e_i(M)$ instead of $e^{\mathfrak{m}}_i(M)$. When $M = A$ there has been a lot of research regarding Hilbert coefficients and their relation to depth $G_I(A)$. Relatively little is known (see [4]) for modules in general.

We describe the contents of various chapters.

1.Preliminaries

In this chapter we extend to modules some of the basic techniques used to study Hilbert functions of a ring, in particular: Valabrega and Valla's Theorem, Sally Descent, Singh's Equality. Sally Descent is useful for arguments by induction on M. If $x \in \mathfrak{m}$ then Balwant Singh's equality relates the Hilbert function of a module M and M/xM. The proofs of these results for the ring A extends to the module case.

The only theorem that needs a detailed proof in the case of modules is Valabrega and Valla's theorem. If j is the largest integer such that $a \in I^j$ then a^* denotes the image of a in I^j/I^{j+1}; this element is called the *initial form* of a. Valla and Valabrega's theorem gives a necessary and sufficient condition for a sequence of elements $x_1, \ldots, x_r$ to have the property that the sequence $x^*_1, \ldots, x^*_r$ is a $G_I(A)$-regular. We extend to modules, assuming that the initial forms of $x_1, \ldots, x_r$ have degree 1.

Theorem 1.2 *Let $(A, \mathfrak{m})$ be a local ring, M an A-module and I an ideal in A. Let $x_1, \ldots, x_r$ be an M-regular sequence with $x_i \in I \setminus I^2$ for $i = 1, \ldots, s$. Set $J = (x_1, \ldots, x_s)$. The following conditions are equivalent:*

i. *$x^*_1, \ldots, x^*_s$ is a $G_I(M)$-regular sequence.*

ii. *$I^iM \cap JM = JI^{i-1}M$ for all $i \geq 1$.*

We prove this theorem by first treating the case when $x_1^*, \ldots, x_s^*$ is a $G_I(A)$-regular sequence. In this case the result is proved by considering the trivial extension $R = A \ltimes M$ and applying Valla and Valabrega's theorem to the ideal $L = (I, M)$. To treat the general case notice that the A-algebra homomorphism $A[X_1 \ldots, X_s] \to A$ mapping X_i to x_i for $i = 1, \ldots, s$ induces a ring homomorphism from $A' = A[X_1, \ldots, X_s]_{(\mathfrak{m}, X_1, \ldots, X_s)}$ to A that turns M into a A'-module. If $Q = (I, X_1, \ldots, X_s)A'$ then

$$G_Q(M) = G_I(M) \text{ and } G_Q(A) = G_I(A)[X_1^*, \ldots, X_s^*].$$

So the previous case is applicable and we get the result.

Another basic technique used in the entire thesis is reduction modulo superficial sequence. An element $x \in I$ is called *superficial* for M with respect to I if there exists an integer $c > 0$ such that

$$(I^n M :_M x) \cap I^c M = I^{n-1} M \text{ for all } \; n > c.$$

It is well known that superficial elements exist when the residue field of A is infinite. A sequence $x_1, \ldots, x_s$ is called a *superficial sequence* for M with respect to I if $\overline{x_i}$ is superficial for $M/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, s$. Many important invariants behave well under reduction by superficial sequences. An application of Balwant Singh's equality is to show that if $x \in I$ is M-superficial with respect to I then $e_i(M/xM) = e_i(M)$ for $i = 0, \ldots, \dim M - 1$. Combined with Sally Descent this yields that, if a property $\mathcal{P}$ of Cohen-Macaulay modules involves $e_0, \ldots, e_i$ then generally the property $\mathcal{P}$ is true if and only if it is true for all Cohen-Macaulay modules of dimension $\leq i$.

2. Cohen-Macaulay modules of minimal multiplicity

In [1] Abhyankar proves the inequality $e_0(A) \geq 1 + \mu(A) - d$ for a Cohen-Macaulay local ring A. In [20] Sally describes the difference between the two sides. Namely, she proves that if $x_1, \ldots, x_d$ is a maximal superficial sequence then

$$e_0(A) = 1 + \mu(M) - d + \lambda(\mathfrak{m}^2/(x_1, \ldots, x_d)\mathfrak{m}).$$

We extend this result to modules.

Theorem 2.1 *Let A be a local ring, M a Cohen-Macaulay A-module of dimension r and I an ideal of definition of M. If $x_1, \ldots, x_r$ is a maximal M-superficial sequence with respect to I and $J = (x_1, \ldots, x_r)$ then the following equality holds:*

$$e_0^I(M) = h_0^I(M) + h_1^I(M) + \lambda(I^2M/JIM)$$

We call the equality above as the *Abhyankar-Sally equality*. We prove this theorem by first considering the case when $x_1, \ldots, x_r$ is an A-regular sequence. Here the proof is similar to the ring case. To treat the general case note that none of the terms in the *Abhyankar-Sally equality* changes when we complete the ring with respect to $\mathfrak{m}$. So we may assume that the A is complete. By Cohen's Structure theorem there exists a regular local Q such that $A = Q/L$. If K is the inverse image of I then $G_L(M) = G_I(M)$. For $i = 1, \ldots, r$ pick $y_i \in K$ which maps to x_i. As the $x_1, \ldots, x_r$ is M-regular sequence so is the sequence r, hence by the New Intersection theorem it is also a Q-regular sequence. This reduces the theorem to the previous case.

The previous theorem allows to make the following definition:

Definition Let A be a local ring, M a Cohen-Macaulay A-module of dimension r and I an ideal of definition of M. We say M has *minimal multiplicity with respect to* I if $e_0^I(M) = h_0^I(M) + h_1^I(M)$.

When $M = A$ this notion gives the usual definition of a ring of minimal multiplicity. In the process of studying rings of minimal multiplicity it is useful to also look at the first Hilbert coefficient. Northcott proved that an inequality $e_1^I(A) \geq e_0^I(A) - \lambda(A/I)$ holds for Cohen-Macaulay local ring (see [6]). Fillmore extended it to Cohen-Macaulay modules [2, p. 218] i.e.

$$e_1^I(M) \geq e_0^I(M) - \lambda(M/IM)$$

We give a different proof of this result. We show that if equality holds then M has minimal multiplicity with respect to I, this extends a theorem due to Huneke [6] and Ooishi [11]. We study the difference.

$$\chi_1^I(M) = e_1^I(M) - e_0^I(M) + \lambda(M/IM)$$

Set depth $G_I(M) = \text{grade}(\mathcal{M}, G(M))$ where $\mathcal{M} = \mathfrak{m}/I \oplus (\bigoplus_{n\geq 1} I^n/I^{n+1})$ is the irrelevant maximal ideal of $G_I(A)$.

Using the techniques developed in the previous chapter and the Abhyankar-Sally equality, we prove following theorem which characterizes modules of minimal multiplicity.

Theorem 2.4. *Let A be a local ring, M a Cohen-Macaulay A-module of dimension r and I an ideal of definition of M. If M has minimal multiplicity with respect to I then $G_I(M)$ is a Cohen-Macaulay $G_I(A)$-module. Furthermore, the following conditions are equivalent :*

(i) *M has minimal multiplicity with respect to I*

(ii) *For every maximal M-superficial sequence $x_1, \ldots, x_r$ there is an equality*

$I^2M = (x_1, \ldots, x_r)IM$

(iii) *For some maximal M-superficial sequence $x_1, \ldots, x_r$ there is an equality*

$I^2M = (x_1, \ldots, x_r)IM$

(iv) $\deg h^I_M(z) \leq 1$.

(v) $\chi^I_1(M) = 0$.

3. Hilbert coefficients of Syzygy Modules.

In this chapter we study a problem on Hilbert coefficients that does not arise when one restricts to rings. We assume that A is Cohen-Macaulay and I is an $\mathfrak{m}$-primary ideal. We let $\text{Syz}^A_1(M)$ denote the first syzygy module of M in a minimal free resolution of M over A. We first show that if M is maximal Cohen-Macaulay then

$$\mu(M)e^I_1(A) \geq e^I_1(M) + e^I_1(\text{Syz}^A_1(M))$$

Sharper results are obtained when $I = \mathfrak{m}$. It is easy to see that that the function $n \mapsto \text{Tor}^A_1(M, A/\mathfrak{m}^{n+1})$ is a polynomial function (see Proposition 3.1.1. We show that it has degree $d-1$ if M is non-free and use it to derive the following theorem.

Theorem 3.1.6 *Let A be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let M be a maximal Cohen-Macaulay A-module. Then :*

1. $\mu(M)e_1(A) - e_1(M) - e_1(\mathrm{Syz}_1^A(M)) \geq e_0(\mathrm{Syz}_1^A(M))$

2. $\mu(M)\chi_1(A) \geq \chi_1(M) + \chi_1(\mathrm{Syz}_1^A(M))$.

In the theorem below we establish similar inequalities for higher Hilbert coefficients of maximal Cohen-Macaulay modules over Gorenstein rings. For every A-module we set $M^* = \mathrm{Hom}_A(M, A)$. Note that if M is maximal Cohen-Macaulay then so is M^*. Also, $\mathrm{type}(M) = \mathrm{Ext}_A^d(k, M)$ denotes the Cohen-Macaulay type of M.

Theorem 3.3.1 *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Let M be a maximal Cohen-Macaulay A-module and set $S^A(M) = \left(\mathrm{Syz}_1^A(M^*)\right)^*$. Assume $\operatorname{depth} G(A) \geq d - 1$ and $G(M)$ is Cohen-Macaulay. Then the following hold:*

1. $\mathrm{type}(M)e_2(A) \geq e_2(M) + e_2(S^A(M))$.

2. $\mathrm{type}(M)e_i(A) \geq e_i(M)$ *for each* $i \geq 0$.

Notice that we have a an exact sequence $0 \to M \to G \to S^A(M) \to 0$ with $G \cong A^{\mathrm{type}\, M}$. Furthermore if A is a hypersurface and M has no free summands, then $S^A(M) = \mathrm{Syz}_1^A(M)$ and $\mu(M) = \mathrm{type}(M)$. By using Sally Descent and Balwant Singh's equality we reduce the proof of the theorem to case of dimension one.

The proof of the theorem above and that of one in the next chapter involves a study of the modules:

$$L_t(M) = \bigoplus_{n \geq 0} \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$$

for all $t \geq 0$. If $x_1, \ldots, x_s$ is a sequence of elements in $\mathfrak{m}$, then we give $L_t(M)$ a structure of a graded $A[X_1, \ldots, X_s]$-module. Each exact sequence of A-modules $0 \to N \to F \to M \to 0$ induces a long exact sequence of graded $A[X_1, \ldots, X_s]$-modules

$$\cdots \to L_{t+1}(M) \to L_t(N) \to L_t(F) \to L_t(M) \to \cdots \to L_0(M) \to 0$$

Theorem 3.3.1 is proved by showing that if $\dim M = 1$ and x is M-superficial, then $L_1(M)$ is a Cohen-Macaulay $A[X]$-module.

4. Hilbert function of Modules over hypersurfaces

If A is regular and M is a maximal Cohen-Macaulay A-module, then it is free. The next case is that of hypersurface rings. In this chapter we assume that A is a hypersurface ring. It is easy to see that if $\operatorname{depth} G(M)$ is positive, then the Hilbert function of M is non-decreasing. The main result of this chapter is

Theorem 4.1.1 *Let $(A, \mathfrak{m})$ be a hypersurface ring of positive dimension. If M is a maximal Cohen-Macaulay A-module, then the Hilbert function of M is non-decreasing.*

It suffices to treat the case when A is complete. So $A = Q/(f)$ for some regular local ring $(Q, \mathfrak{n})$. Note that if M is a maximal A -module then $\operatorname{projdim}_Q M = 1$. Let

$$0 \longrightarrow Q^{\mu(M)} \xrightarrow{\phi_M} Q^{\mu(M)} \longrightarrow M \longrightarrow 0$$

be a minimal presentation of M. If x, y is a $M \oplus Q$-superficial sequence then we consider $L_0(Q)$, $L_0(M)$ and $L_1(M)$ as $A[X, Y]$-modules. We show that X, Y is a $L_0(Q)$ and $L_1(M)$ regular sequence and then use the exact sequence

$$0 \to L_1(M) \to L_0(Q)^n \to L_0(Q)^n \to L_0(M) \to 0.$$

to obtain equations for the Hilbert function of M.

The next goal is to give lower bounds on both the multiplicity and first Hilbert coefficient of M when $A = Q/(f)$ for some regular local ring $(Q, \mathfrak{n})$.

If $0 \to Q^n \xrightarrow{\phi_M} Q^n \to M \to 0$ is a minimal presentation of M the following are invariants of M.

$$I_M = \text{ ideal generated by the entries of } \phi_M$$
$$i(M) = \max\{i \mid I_\phi \subseteq \mathfrak{n}^i\}.$$

Theorem 4.2.5 *Let $(Q, \mathfrak{n})$ be a regular local ring, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$, M a maximal Cohen-Macaulay A-module and $K = \operatorname{Syz}_1^A(M)$. Then*

1. $e(M) \geq \mu(M)i(M)$ *and* $e_1(M) \geq \mu(M)\binom{i(M)}{2}$.

2. *M is a free A-module if and only if $i(M) = e$.*

3. *If $i(M) = e - 1$ then $G(M)$ is Cohen-Macaulay*

4. *The following are equivalent:*

   i. $e(M) = \mu(M)i(M)$.

   ii. $e_1(M) = \mu(M)\binom{i(M)}{2}$.

   iii. *$G(M)$ is Cohen-Macaulay and*

$$h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1})$$

If these conditions hold and M is not free, then $G(K)$ is Cohen-Macaulay and

$$h_K(z) = \mu(M)(1 + z + \ldots + z^{e-i(M)-1}).$$

The theorem above is easy to show when $\dim A = 0$. To reduce the general case to $\dim 1$ we introduce a notion of superficial element x with respect to to a map ϕ_M where

$$0 \longrightarrow Q^{\mu(M)} \xrightarrow{\phi_M} Q^{\mu(M)} \longrightarrow M \longrightarrow 0$$

This is a $Q \oplus M$-superficial element such that $i(M/xM) = i(M)$. When $\dim M = 1$ then 1 and 2 follow from the case of $\dim 0$. A technical lemma which relates the Hilbert function of M and M/xM gives 3. To prove 4. we prove a lemma which is interesting in its own right.

Lemma 4.2.7 *Let $(Q, \mathfrak{n})$ be a regular local ring of dimension two, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$. If M a maximal Cohen-Macaulay A-module then*

$$h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M)z^i \quad \text{and } h_i(M) \geq 0 \text{ for all } i$$

Theorem 4.2.5 is applied to the case of Ulrich modules, that is, maximal Cohen-Macaulay modules that satisfy $e(M) = \mu(M)$. It is known [7] that Ulrich A-modules exist when A is a complete hypersurface ring. Using the previous theorem we get that if M is Ulrich, then $i(M) = 1$ and so $G(\mathrm{Syz}_1^A(M))$ is Cohen-Macaulay.

1. PRELIMINARIES

In this thesis all rings are commutative Noetherian and all modules are assumed finite. In this chapter we introduce notation and develop a few techniques which will be used throughout this thesis. In section 1 we define the basic notions. In Section 2 we extend to modules some of the basic techniques used to study Hilbert functions of a ring, in particular: Valabrega and Valla's Theorem, Sally Descent, Singh's Equality.

1.1 Notation

Let $(A, \mathfrak{m})$ be a local ring of dimension d with residue field $k = A/\mathfrak{m}$. Let M be a finite A-module. Let I be an ideal of definition of M i.e. $\mathfrak{m}^n M \subseteq IM$ for some $n > 0$. Let $G_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of A with respect to I and let $G_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$ be the associated graded module of M with respect to I, considered as a graded $G_I(A)$-module. We let $\lambda(N)$ denote the length of an A-module N and $\mu(N)$ number of its minimal generators. The function $H^I(M, n) = \lambda(I^nM/I^{n+1}M)$, defined for all $n \geq 0$, is called the Hilbert function of M. The formal power series

$$H^I_M(z) = \sum_{n\geq 0} H^I(M, n)z^n$$

is called the Hilbert series of M. It is easy to see that

$$\sum_{n\geq 0} \lambda(M/I^{n+1}M)z^n = \frac{H^I_M(z)}{(1-z)}$$

It is well known that it is a rational function of the form

$$H^I_M(z) = \frac{h^I_M(z)}{(1-z)^r}, \text{ where } \; r = \dim M \quad \text{and}$$
$$h^I_M(z) = h^I_0(M) + h^I_1(M)z + \cdots + h^I_s(M)z^s \in \mathbb{Z}[z]$$

We call the polynomial $h^I_M(z)$ the *h-polynomial* of M with respect to I. It follows from the definition that

$$h^I_0(M) = \lambda(M/IM) \quad \text{and} \quad h^I_1(M) = \lambda(IM/I^2M) - r\lambda(M/IM)$$

We set $\operatorname{depth} G_I(M) = \operatorname{grade}(\mathcal{M}, G_I(M))$ where $\mathcal{M} = \mathfrak{m}/I \oplus (\bigoplus_{n\geq 1} I^n/I^{n+1})$ is the irrelevant maximal ideal of $G_I(A)$. If m is a non-zero element of M and if j is the largest integer such that $m \in I^jM$, then we let m^* denote the image of m in $I^jM \setminus I^{j+1}M$. If L is a submodule of M , then L^* denotes the graded sub module of $G_I(M)$ generated by all l^* with $l \in L$. It is well known that $G_I(M/L) \cong G_I(M)/L^*$.

Remark 1.1.1. Let $x_1, ..., x_s$ be a sequence in A with $x_i \in I$ and set $J = (x_1, ..., x_s)$. For the ring $B = A/J$, the ideal $K = I/J$ and the B-module, $N = M/JM$ there is an equality $G_I(N) = G_K(N)$. Since $G_K(B) = G_I(A)/J^*$ it follows that

$$\operatorname{depth}_{G_I(A)} G_I(N) = \operatorname{depth}_{G_K(B)} G_K(N)$$

Remark 1.1.2. We let $R = A \ltimes M$ denote the *trivial extension ring* of A by M. As an abelian group R is the direct sum of A and M. The multiplication is defined by

$$(a, x)(b, y) = (ab, ay + bx) \quad \text{ for all } a, b \in A \text{ and } x, y \in M.$$

It is known that R is a local Noetherian ring with maximal ideal $\mathfrak{n} = (\mathfrak{m}, M)$ and that M is an ideal of R with $M^2 = 0$. Set $L = (I, M)$. There are equalities $L^i = (J^i, J^{i-1}M)$ for all $i \geq 1$, and so we have $G_L(R) = G_I(A) \ltimes G_I(M)(-1)$. Let $x_1, \ldots, x_s$ be a sequence in A. Set $y_i = (x_i, 0)$ for $i = 1, \ldots, s$. It is clear that $y_1, \ldots, y_s$ form an R-regular sequence if and only if $x_1, \ldots, x_s$ form both a A-regular sequence and an M -regular sequence.

Definition 1.1.3. A property $\mathcal{P}$ is called *sufficiently general* and an element $a \in I$ is called *sufficiently general element* of an ideal I if the following holds: Let I be an ideal in A, minimally generated by $y_1, \ldots, y_l$. An element $a \in I$ is said to be sufficiently general with respect to $\mathcal{P}$ if there exists a non-empty Zariski-open subset U of k^l such that whenever $a = \sum a_i y_i$ and the image of $(a_1, \ldots, a_l)$ in k^l lies in U, then a satisfies $\mathcal{P}$.

An element $x \in I$ is called *superficial* for M with respect to I if there exists an integer $c > 0$ such that

$$(I^n M \colon {}_M x) \cap I^c M = I^{n-1} M \text{ for all } \ n > c.$$

We omit the phrase " with respect to I" if the ideal is clear from the context. It is well known that if k is infinite then a sufficiently general element of I is superficial for M (see [23, p. 303]).

Remark 1.1.4. If the residue field is finite we resort to the standard trick to replace A by $A' = A[X]_S$, I by $I' = IA'$ and M by $M' = M \otimes_A A[X]_S$ where $S = A[X] \setminus \mathfrak{m}A[X]$. It is easy to check that the residue field of A' is $k(X)$, the field of rational functions over k. Furthermore

$$H^{I'}(M', n) = H^I(M, n) \quad \text{ for all } n \geq 0$$
$$\operatorname{depth}_{G_{I'}(A')} G(M') = \operatorname{depth}_{G_I(A)} G(M)$$
$$\operatorname{projdim}_{A'} M' = \operatorname{projdim}_A M$$

If A is Gorenstein then A' is also Gorenstein. If A has a canonical module ω_A then $\omega_{A'} \cong \omega_A \otimes A'$. If A is a hypersurface then A' is Cohen-Macaulay with embedding dimension one more than the dimension, and so A' is a hypersurface. Therefore, for many problems we may without loss of generality assume k is infinite.

Remark 1.1.5. (i) If $\operatorname{depth} M > 0$ then it is easy to see that every M-superficial element is also M-regular. See [5, 2.1] for the case $M = A$. The proof of the general case is similar,

(ii) If x is superficial and M-regular, then by using the Artin-Rees lemma for M and xM one gets $(I^n M \colon {}_M x) = I^{n-1} M$ for all $n \gg 0$. See [17, p. 8] for the case $M = A$. The proof of the general case is similar.

1.2 Basic Techniques

For $M = A$ the theorem below follows from a result of Valabrega and Valla (see [24, 2.7]). We use it to obtain a result for modules.

Theorem 1.2.1. *Let $(A,\mathfrak{m})$ be a local ring, M an A-module and I an ideal in A. Let $x_1,\dots,x_s$ be an M-regular sequence with $x_i \in I \setminus I^2$ for $i=1,\dots,s$. Set $J = (x_1,\dots,x_s)$. The following conditions are equivalent:*

i. *$x_1,\dots,x_s$ is a $G_I(M)$-regular sequence.*

ii. *$I^iM \cap JM = I^{i-1}JM$ for all $i \geq 1$.*

Proof. Assume first $x_1^*,\dots,x_s^*$ is a $G_I(A)$-regular sequence. It is easy to see that $x_1,x_2,...,x_s$ is an A-regular sequence. By Valabrega and Valla's theorem for the ring A we get that $I^{n+1}\cap J = I^nJ$ for all $n \geq 1$. Let $R = A \ltimes M$. Set $L = (I,M)$. We have $L^i = (I^i, I^{i-1}M)$. We set $y_i = (x_i, 0)$ for all i, $1 \leq i \leq s$. Note that $Q = (y_1,\dots,y_s)R = (J, JM)$ and that $y_1,\dots,y_s$ form an R-regular sequence. For all $i \geq 0$ we have the following equalities

$$L^{i+1}\cap Q = (I^{i+1}\cap J, I^iM \cap JM)$$

$$QL^i = (J,JM)(I^i, I^{i-1}M) = (JI^i, I^{i-1}JM + I^iJM) = (JI^i, I^{i-1}JM).$$

It follows that $L^{i+l}\cap Q = QL^i$ if and only if $I^i \cap JM = I^{i-1}JM$. Using Valabrega and Valla's theorem for the ring R we see that $y_1^*,\dots,y_s^*$ is a $G(R)$-regular sequence if and only if $L^{i+l}\cap Q = QL^i$. It follows that $x_1^*,\dots,x_s^*$ is a $G_I(M)$-regular sequence if and only if $I^nM\cap JM = I^{n-1}JM$ for all $n \geq 1$.

To treat the general case we set

$$A' = A[X_1,\dots,X_s]_{(\mathfrak{m},X_1,\dots,X_s)} \quad \text{and} \quad Q = (I, X_1,\dots,X_s)A'$$

Note that $G_Q(A') = G_I(A)[X_1^*,\dots,X_s^*]$. So $X_1^*,\dots,X_s^*$ is a $G_Q(A')$-regular sequence. The A-algebra homomorphism $A[X_1...,X_s] \to A$ mapping X_i to x_i for $i=1,\dots,s$ induces a ring homomorphism $A' \to A$ that turns M into a A'-module. One easily checks that $G_Q(M) = G_I(M)$ and $X_1,\dots,X_s$ is an M-sequence. Set $K = (X_1,\dots,X_s)A'$.

For every submodule N of M and for all $i \geq 1$ we also have :

$$Q^i = (I^i, I^{i-1}K, \ldots, IK^{i-1}, K^i)A'$$
$$Q^iN = I^iN$$
$$K^iN = J^iN$$
$$Q^iM \cap KM = I^iM \cap JM$$
$$Q^{i-1}KM = I^{i-1}JM$$

It follows that $Q^iM \cap KM = Q^{i-1}KM$ if and only if $I^iM \cap JM = I^{i-1}JM$. The sequence $X_1^*, \ldots, X_s^*$ is $G_Q(A')$-regular, so by the already settled case we get that it is a $G_Q(M)$-regular sequence if and only if $Q^iM \cap KM = Q^{i-l}KM$. Since $G_Q(M) = G_I(M)$, we conclude that the sequence $x_1^*, \ldots, x_s^*$ is $G_I(M)$-regular if and only if $I^iM \cap JM = I^{i-1}JM$ for all $i \geq 1$. □

A sequence $x_1, \ldots, x_s$ is called a *superficial sequence* for M with respect to I if $\overline{x_i}$ is superficial for $M/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, s$. A sequence $x_1, \ldots, x_s$ in A is called a *superficial M-regular sequence* with respect to I if $x_1, \ldots, x_s$ is both an M-regular sequence and a M-superficial sequence with respect to I. Again we will omit the phrase "with respect to I" if I is clear from the context. The next lemma is well known. We include a proof for completeness.

Lemma 1.2.2. *Let $(A, \mathfrak{m})$ be a local ring, M an A-module and I an ideal for definition of M. Let $x_1, \ldots, x_s \in I$ be a superficial M-regular sequence. We then have $(x_1, \ldots, x_s)M \cap I^iM = (x_1, \ldots, x_s)I^{i-1}M$ for all $i \gg 0$.*

Proof. We argue by induction on s. The case $s = 1$ follows from Remark 1.1.5(ii) We assume the result holds for $s = n-1$ and prove it for $s = n$. Set $J = (x_1, \ldots, , x_{n-1})$ and $N = M/JM$. Since x_n is both regular and superficial on N, we have

$$(I^iN :_N x_n) = I^{i-1}N \text{ for all } i \gg 0.$$

By induction hypothesis we have

$$JM \cap I^iM = JI^{i-1}M \text{ for all } i \gg 0.$$

We take i large enough so that both preceding conditions hold.

Choose t in $(x_1,\dots,x_n)M\cap I^iM$ and write it as $t=t_1+x_nl$ with $t_1\in JM$. The equality $\bar{t}=x_n\bar{l}$ implies $\bar{l}\in I^{i-1}N$. Then $l=t_2+l_1$ with $t_2\in JM$ and $l_1\in I^{i-1}M$. Therefore, we have $t=t_1+x_n(l_1+t_2)$, and so

$$t-x_nl_1=t_1+x_nt_2\in(x_1,\dots,x_{n-l})M\cap I^iM=(x_1,\dots,x_{n-1})M.$$

It follows that $t\in(x_1,\dots,x_n)I^{i-1}M$ as desired. □

We use the preceding lemma and Theorem 3 to prove the following result.

Theorem 1.2.3. *Let $(A,\mathfrak{m})$ be a local ring, M an A module and I an ideal of definition for M. Let $x_1,\dots,x_s$ be a superficial M-regular sequence and set $J=(x_1,\dots,x_s)$. The following conditions are equivalent:*

(i) *$x_1,\dots,x_s$ is a $G_I(M)$-regular sequence.*

(ii) *$(JM)^*=(x_1^*,\dots,x_s^*)G_I(M)$.*

(iii) *The canonical map*

$G_I(M)/(x_1^*,\dots,x_s^*)G_I(M)\longrightarrow G_I(M/JM)$

is an isomorphism.

Proof. By definition, for each $i\geq 0$ there is an equality

$$(JM)_i^*=\frac{JM\cap I^iM+I^{i+1}M}{I^{i+1}M}\qquad(*)$$

(i) $\Longrightarrow$ (ii). Theorem 3 implies $JM\cap I^iM=JI^{i-1}M$ for all $i\geq 1$. Therefore, the equality (*) becomes

$$(JM)_i^*=\frac{JI^{i-1}M+I^{i+1}M}{I^{i+1}M}=(x_1^*,\dots,x_s^*)G_I(M)_i$$

So we get $(JM)^*=(x_1^*,\dots,x_s^*)G_I(M)$.

(ii) $\Longrightarrow$ (iii). Using (*) we get

$$JM\cap I^iM=JI^{i-1}M\mod I^{i+1}M.\qquad(**)$$

By Lemma 1.2.2 there exists a least positive integer l such that $JM \cap I^iM = JI^{i-1}M$ for all $i \geq l$. If $l \neq 1$ then $JM \cap I^{l-1}M \supsetneqq JI^{l-2}M$. If $t \in JM \cap I^{l-1}M \setminus JI^{l-2}M$, then by (**) we get $t_1 \in JI^{l-2}M$ such that $t - t_1 \in JM \cap I^lM = JI^{l-1}M$. So we get $t \in JI^{l-2}M$ a contradiction. Therefore $l = 1$, so (i) holds.

(ii) $\Longrightarrow$ (iii) follows from the isomorphism $G_I(M/JM) \cong G_I(M)/(JM)^*$.

(iii) $\Longrightarrow$ (ii) because $(x_1, \dots, x_s)G_I(M) \subseteq ((x_1, \dots, x_s)M)^*$. □

The next theorem is useful for arguments by induction on the dimension of M . In the ring case the statement (2) below is referred to as Sally descent; we use the same name in the module case as well.

Theorem 1.2.4. *Let (A, m) be a local ring, M an A-module and I an ideal of definition for M. Let $x_1, \dots, x_s$ be a superficial M-regular sequence. If we let $J = (x_1, \dots, x_s), (B, n) = (A/J, m/J), N = M/JM$ and $K = I/J$ then*

1. $\operatorname{depth}_{G_I(A)} G_I(M) \geq s$ *if and only if* $x_1^*, \dots, x_s^*$ *is a* $G_I(M)$*-sequence.*

2. $\operatorname{depth}_{G_I(A)} G_I(M) \geq s+1$ *if and only if* $\operatorname{depth}_{G_K(B)} G_K(N) \geq 1$.

Proof. (1) If $x_1^*, \dots, x_s^*$ is a $G_I(M)$-sequence, then clearly $\operatorname{depth} G_I(M) \geq s$. Conversely, if $\operatorname{depth} G_I(M) \geq s$, then we prove by induction on s that $x_1^*, \dots, x_s^*$ is a $G_I(M)$-sequence. We first prove the assertion for $s = 1$. We can choose a homogeneous element $y \in G_I(A)$ such that y is $G_I(M)$-regular. Since x_1 is M-superficial and M-regular there exists an integer n_0 such that $(0\colon_{G_I(M)} x_1^*)_n = 0$ for all $n \geq n_0$. If $p \in G_I(M)$ is homogeneous with $x_1^*p = 0$ then $x_1^*y^sp = 0$, where s is chosen so that $c = s \deg y + \deg p \geq n_0$. Therefore we get $y^sp \in (0\colon_{G_I(M)} x_1)_c = 0$. This forces $p = 0$. Assume the assertion holds for $s = n-1$. So $x_1^*, \dots, x_{n-1}^*$ is $G_I(M)$-regular by the induction hypothesis. Set $J_1 = (x_1, \dots, x_{n-1})$ and $L = (x_1^*, \dots, x_{n-1}^*)$. Theorem 1.2.3 yields $G_I(M)/LG_I(M) = G_I(M/J_1M)$ and so we get $\operatorname{depth} G(M/J_1M) \geq 1$. Since x_s is (M/J_1M)-superficial, the result for $s = 1$ shows that x_n^* is regular on $G_I(M/J_1M) = G_I(M)/LG_I(M)$. Therefore, $x_1^*, \dots, x_n^*$ is a $G_I(M)$-regular sequence.
(2) We have $\operatorname{depth}_{G_I(A)} G_I(N) = \operatorname{depth}_{G_K(B)} G_I(N)$, see Remark 1.1.1. Assume first

$\operatorname{depth}_{G_I(A)} G_I(M) \geq s+1$. By (1) we see that $x_1^*, \ldots, x_s^*$ is a $G_I(M)$- regular sequence and from Theorem 1.2.3 we get

$$G_I(M)/(x_1^*, \ldots, x_s^*)G_I(M) \cong G_I(M/(x_1, \ldots, x_s)M).$$

Therefore, we get $\operatorname{depth} G_I(N) \geq 1$. Conversely if $\operatorname{depth} G_I(N) \geq 1$, then an argument similar to that for the case $M = A$, [5, 2.2] shows that$x_1^*, \ldots, x_s^*$ is a $G_I(M)$-regular sequence. Theorem 1.2.3 yields $G_I(M/JM) \cong G_I(M)/(x_1^*, \ldots, x_s^*)G_I(M)$. Therefore, $\operatorname{depth} G_I(M) \geq s+1$. □

The first part of the following theorem is proved by Singh [22, Th. 1] for $M = A$. The same proof applies to the general case. We give a different proof for completeness. We use $\preccurlyeq$ and $\succcurlyeq$ to denote coefficient-wise inequalities of formal power series.

Theorem 1.2.5. *Let $(A, \mathfrak{m})$ be a local ring, M an A-module and I an ideal of definition for M. If $x \in I \setminus I^2, N = M/xM$ and $K = I/xI$ then*

1. $H^I(M, n) = \lambda(N/K^{n+1}N) - \lambda\left((I^{n+1}M \colon {}_M x)/I^nM\right).$

2. *x^* is $G_I(M)$ -regular if and only if $I^{n+1}M \colon {}_M x = I^nM$ for all $n \geq 0$.*

3. *$H^I_M(z) \preccurlyeq H^K_N(z)/(1-z)$ with equality if and only if x^* is $G_I(M)$-regular.*

Proof. 1. Consider the exact sequence:

$$0 \longrightarrow \frac{(I^{n+1}M \colon {}_M x)}{I^nM} \longrightarrow \frac{M}{I^nM} \xrightarrow{x} \frac{M}{I^{n+1}M} \longrightarrow \frac{N}{K^{n+1}N} \longrightarrow 0$$

Computing lengths we get the required result.

2. First assume x^* is $G_I(M)$ -regular. Let $p \in (I^{n+1}M \colon {}_M x)$. If $p \notin I^nM$ then there exists $j < n$ such that $p \in I^jM \setminus I^{j+1}M$. So $x^*p^* = 0$. Since x^* is regular we get $p^* = 0$ which is a contradiction. Therefore $I^{n+1}M \colon {}_M x = I^nM$ for all $n \geq 0$.

Note that for all $n \geq 0$ we have an exact sequence

$$0 \longrightarrow \frac{(I^{n+1}M \colon {}_M x) \cap I^{n-1}M}{I^nM} \longrightarrow \frac{I^{n-1}M}{I^nM} \xrightarrow{x^*} \frac{I^nM}{I^{n+1}M}$$

So if $I^{n+1}M \colon {}_M x = I^nM$ for all $n \geq 0$ then x^* is $G_I(M)$-regular.

We get 3. from 1. and 2. □

Singh's theorem motivates the introduction of the following entities.

$$b_n^I(x, M) = \lambda\left((I^{n+1}M :\ _M x)/I^n M\right)$$
$$b_{x,M}^I(z) = \sum_{n\geq 0} b_n^I(x, M) z^n$$

Notice that $b_0^I(x, M) = 0$. We note two useful corollaries of the previous theorem.

Corollary 1.2.6. *Let $(A, \mathfrak{m})$ be a Noetherian ring, M an A-module of dimension r and I an ideal of definition for M. Let $x \in I$ be both M-superficial and M-regular. Set $B = A/(x)$. For the B-module $N = M/xM$ and the ideal $K = I/(x)$ in B the following hold :*

(1) $\dim N = \dim M - 1$ *and* $h_0^K(N) = h_0^I(M)$.

(2) $b_{x,M}^I(z)$ *is a polynomial.*

(3) $h_M^I(z) = h_N^K - (1-z)^r b_{x,M}^I(z)$.

(4) $h_1^I(M) = h_1^K(N)$ *if and only if* $I^2M \cap xM = xIM$.

(5) $e_i^K(N) = e_i^I(M)$ *for every* $i = 0, \ldots, r-1$.

(6) $e_r^I(M) = e_r^K(N) - (-1)^r \sum_{n\geq 1} b_n^I(x, M) z^n$.

(7) x^* *is* $G_I(M)$*-regular if and only if* $b_n^I(x, M) = 0$ *for all* $n \geq 0$.

(8) $e_r^I(M) = e_r^K(N)$ *if and only if* x^* *is* $G_I(M)$*-regular.*

Proof. (1) is clear. Since x is both M-superficial and M-regular, we have $b_n^I(x, M) = 0$ for $n \gg 0$, which gives (2). Theorem 1.2.5.1 gives (3). Notice that $b_1^I(x, M) = 0$ if and only if $I^2M \cap xM = xIM$ from this (4) is obvious. Both (5) and(6) follow from (3). The assertion of (7) is clear and (8) follows from (6) and (7). □

Corollary 1.2.7. *Let $(A, \mathfrak{m})$ be a local ring, M an A-module of dimension r, and I a ideal of definition for M. Let $x_1, \ldots, x_s$ be a superficial M-regular sequence. For $N = M/(x_1, \ldots, x_s)M$ and $K = I/(x_1, \ldots, x_s)$ the following hold*

(1) $h_0^K(N) = h_0^I(M)$ *and* $e_i^K(N) = e_i^I(M)$ *for every* $i = 0, \dots, r-s$.

(2) $h_M^I(z) = h_N^K(z)$ *if and only if* $x_1^*, \dots, x_s^*$ *is a* $G_I(M)$*-regular sequence.*

Proof. (1) The result follows from Corollary 1.2.6.1 and (4), by induction on s.

(2) Set $E = G_I(M)/(x_1^*, \dots, x_s^*)G_I(M)$. If $x_1^*, \dots, x_s^*$ is a $G_I(M)$-regular sequence then the h-polynomial of $G_I(M)$ is the same as the h-polynomial of E. By Theorem 1.2.3.3 we get $G_K(N) = E$ and so we have $h_M^I(z) = h_N^K(z)$.

Conversely if $h_M^I(z) = h_N^K(z)$ then we have $H_M^I(z) = H_N^K(z)/(1-z)^s$. We get the result by induction on s using Theorem 1.2.5.3. □

When $\dim M = 1$, a classical theorem on one dimensional Cohen- Macaulay local rings extends to modules.

Proposition 1.2.8. *Let M be a Cohen-Macaulay module of dimension one over a local ring A and I be an ideal of definition for M. Let $x \in I$ be an M -superficial element and set*

$\rho_j^I(M) = \lambda(I^{j+1}M/xI^jM)$ *for all* $j \geq 0$. *The following equalities then hold*

1. $\rho_0^I(M) = e_0^I(M) - h_0^I(M)$.

2. $H^I(M, j) = e_0^I(M) - \rho_j^I(M)$

3. *If* $\deg h_M^I(z) = s$ *then* $\rho_j^I(M) = 0$ *for all* $j \geq s$ *and*

$$h_M^I(z) = h_0^I(M) + \sum_{j=1}^{s}(\rho_{j-1}^I(M) - \rho_j^I(M))z^j$$

4. $e_i^I(M) = \sum_{j \geq i-1} \binom{j}{i-1} \rho_j^I(M)$

Proof. By corollary 1.2.6(4) we have $e_0^I(M) = e_0^K(N) = A(M/xM)$. From this (1) follows. For (2) the proof given in [26, 6.18] for $M = A$ applies to the general case. Finally notice that (2) implies (3) and that (3) implies (4). □

Remark 1.2.9. The previous Proposition and Corollary 1.2.7(2) implies that if M is Cohen-Macaulay and $\operatorname{depth} G_I(M) \geq \dim M - 1$ then $e_i^I(M) \geq 0$ for all $i \geq 0$.

2. COHEN-MACAULAY MODULES OF MINIMAL MULTIPLICITY

The first theorem in this chapter illustrates the reason why it is meaningful to talk about Cohen-Macaulay modules of minimal multiplicity. Next we give a proof of Northcott's inequality for modules. Finally we prove a theorem which characterizes modules of minimal multiplicity.

When $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d and $x_1, \ldots, x_d$ a maximal A-superficial sequence the Abhyankar-Sally equality (see [20, 2.4]) reads:

$$e_0(A) = 1 + \mu(\mathfrak{m}) - d + \lambda(\mathfrak{m}^2/(x_1, \ldots, x_d)\mathfrak{m})$$

Note that $h_0(A) = 1$ and $h_1(A) = \mu(\mathfrak{m}) - d$. Valla extended this identity to $\mathfrak{m}$-primary ideals (see [25, Lemma 1]). We extend it further to modules.

Theorem 2.1. *Let A be a local ring, M a Cohen-Macaulay A-module of dimension r and I an ideal of definition of M. Let $x_1, \ldots, x_r$ be a maximal M-superficial sequence. Set $J = (x_1, \ldots, x_s)$. We have the following equality*

$$e_0^I(M) = h_0^I(M) + h_1^I(M) + \lambda(I^2M/JIM)$$

Proof. First we assume that $x_1, \ldots, x_r$ is an A-regular sequence. Denote the Koszul complex on $x_1, \ldots, x_r$ as $K_A(X)$. Using this resolution of A/J and the fact $J \subseteq I$ we get

$$\lambda\left(\operatorname{Tor}_1^A(A/J, M/IM)\right) = r\lambda(M/IM).$$

Next we tensor the exact sequence

$$0 \longrightarrow IM \longrightarrow M \longrightarrow M/IM \longrightarrow 0$$

with A/J to get the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(A/J, M/IM)) \longrightarrow IM/JIM \longrightarrow M/JM \longrightarrow M/IM \longrightarrow 0$$

Computing lengths we get

$$e_0(M) = \lambda(M/IM) - r\lambda(M/IM) + \lambda(IM/JIM). \tag{2.1}$$

Finally from the sequence

$$0 \longrightarrow I^2M/JIM \longrightarrow IM/JIM \longrightarrow IM/I^2M \longrightarrow 0$$

we get

$$\lambda(IM/JIM) = \lambda(I^2M/JIM) + \lambda(IM/I^2M). \tag{2.2}$$

Putting (2) and (3) above together we get the desired equality. We now prove the result in general. Let $(\widehat{A}, \widehat{\mathfrak{m}})$ be the $\mathfrak{m}$-adic completion of A. Since $\widehat{A}$ is a faithfully flat A algebra we can assume that $A \subseteq \widehat{A}$. Set $\widehat{I} = I\widehat{A}$. Note that we have $G_{\widehat{I}}(\widehat{M}) \cong G_I(M)$. It is also clear that if x is M-superficial then it is also $\widehat{M}$-superficial. Furthermore $I^2M/JIM \cong I^2\widehat{M}/JI\widehat{M}$. Therefore we may assume as well that A is complete. By Cohen's Structure Theorem there exists a regular local ring $(Q, \mathfrak{n})$ such that $A = Q/L$ for some ideal L in Q. Set K to be the inverse image of the ideal I. For $i = 1, \ldots, r$ pick $y_i \in K$ such that $\overline{y_i} = x_i$. Since $x_1, \ldots, x_r$ is an M -regular sequence, we get that $y_1, \ldots, y_r$ is also an M-regular sequence. By the New Intersection theorem it follows that $y_1, \ldots, y_r$ form a Q-regular sequence (see [8, 6.2.3]). Note that $G_I(M) = G_K(M)$ and that $I^2M/JIM = K^2M/(y_1, \ldots, y_r)KM$ .The special case considered above gives the desired equality. □

We call the equality above the *Abhyankar-Sally equality.* The previous theorem allows to make the following definition:

Definition 2.2. Let A be a Noetherian local ring, M Cohen-Macaulay A-module and I an ideal of definition for M. We say M has minimal multiplicity with respect to I if $e_0^I(M) = h_0^I(M) + h_1^I(M)$.

When $M = A$ we have the usual definition of Cohen-Macaulay local ring with minimal multiplicity. In the process of studying rings of minimal multiplicity it is useful to also look at the first Hilbert coefficient. Northcott proved that an inequality $e_1^I(A) \geq e_0^I(A) - \lambda(A/I)$ holds for Cohen-Macaulay local ring (see [10]). Fillmore extended it to Cohen-Macaulay modules [4, p. 218] i.e.

$$e_1^I(M) \geq e_0^I(M) - \lambda(M/IM)$$

We include a proof for completeness.

Proposition 2.3. *Let $(A, \mathfrak{m})$ be a local ring, M a Cohen-Macaulay A-module of dimension r, and I a ideal of definition for M. Then $e_1^I(M) \geq e_0^I(M) - \lambda(M/IM)$.*

Proof. Notice that $\lambda(M/IM) = h_0^I(M)$. Using Corollary 1.2.7(1) it suffices to prove the proposition for dimensions 0 and 1. If $\dim M = 0$ then $h_i^I(M) = H^I(M, i) \geq 0$. Since

$$e_1^I(M) = \sum_{i \geq 1} i h_i^I(M) = e_0^I(M) - \lambda(M/IM) + \sum_{i \geq 2} (i-1) h_i^I(M)$$

we get $e_1^I(M) \geq e_0^I(M) - \lambda(M/IM)$.

When $\dim M = 1$ choose $x \in I$ a superficial M-superficial element. For $N = M/xM$, $K = I/(x)$ we have

$$e_1^I(M) \geq e_1^K(N) \geq e_0^K(N) - \lambda(N/KN) \geq e_0^I(M) - \lambda(M/IM).$$

The first inequality holds from 1.2.6(6) and the second one is the dimension zero case proved above. The equality is due to 1.2.6(1) and (4). □

We study the difference.

$$\chi_1^I(M) = e_1^I(M) - e_0^I(M) + \lambda(M/IM).$$

We now prove

Theorem 2.4. *Let A be a local ring, M a Cohen-Macaulay A-module of dimension r and I an ideal of definition of M. If M has minimal multiplicity with respect to I then $G_I(M)$ is a Cohen-Macaulay $G_I(A)$-module. Furthermore the following conditions are equivalent*

(i) *M has minimal multiplicity with respect to I*

(ii) *For every maximal M-superficial sequence $x_1, \ldots, x_r$ there is an equality*

$I^2M = (x_1, \ldots, x_r)IM$

(iii) *For some maximal M-superficial sequence $x_1, \ldots, x_r$ there is an equality*

$I^2M = (x_1, \ldots, x_r)IM$

(iv) $\deg h^I_M(z) \leq 1$.

(v) $\chi^I_1(M) = 0$.

Proof. We first show that if M has minimal multiplicity then $G_I(M)$ is Cohen-Macaulay. When $\dim M = 0$ there is nothing to prove. So assume $\dim M = r > 0$. By Remark 1.1.4 it suffices to consider the case when the residue field of A is infinite. Let $x_1, \ldots, x_r$ be a maximal M-superficial sequence. Then by the Abhyankar-Sally equality we get $I^2M = JIM$. Therefore $I^nM = JI^nM$ for all $n \geq 2$. Therefore we get $I^nM \cap JM = I^{n-1}JM$ for all $n \geq 1$. Since M is Cohen-Macaulay we have that $x_1, \ldots, x_r$ is also an M-regular sequence. Therefore by Theorem 1.2 we get $x_1^*, \ldots, x_r^*$ is a $G_I(M)$-regular sequence and so we get $G_I(M)$ is Cohen- Macaulay. The Abhyankar-Sally equality gives the equivalence of (i) and (ii). Clearly (ii) implies (iii). If we assume (iii), then by Theorem 1.2 we get that $G_I(M)$ is Cohen-Macaulay and so by Theorem 1.2.4(1) we get $x_1^*, \ldots, x_r^*$ is a $G_I(M)$-regular sequence and so by Corollary 1.2.7(2) we have $h^I_M(z) = h^{I/J}_{M/JM}(z)$. Clearly $\deg h^{I/J}_{M/JM}(z) \leq 1$ and so we have (iv). Clearly (iv) implies (v).

We now prove (v) $\implies$ (i). If $\dim M = 0$ then $h^I_i(M) = H^I(M, i) \geq 0$. Since

$$e^I_1(M) = \sum_{i \geq 1} ih^I_i(M) = e^I_0(M) - \lambda(M/IM) + \sum_{i \geq 2}(i-1)h^I_i(M)$$

So we get $h^I_i(M) = 0$ for all $i \geq 2$. Therefore $e^I_0(M) = h^I_0(M) + h^I_1(M)$ and so M has minimal multiplicity. If $\dim M = 1$ and x is M-superficial then set $\rho^I_j(M) = \lambda(I^{j+1}M/xI^jM)$. By Proposition 1.2.8 we have

$$e^I_1(M) = \sum_{j \geq 0} \rho^I_j(M) = e^I_0(M) - h^I_0(M) + \sum_{i \geq 1} \rho^I_j(M)$$

So we get $\rho_j^I(M) = 0$ for all $j \geq 1$. In particular $\rho_1^I(M) = 0$ and therefore we get $I^2M = xIM$ and this implies that M has minimal multiplicity. As proved earlier this implies that $G_I(M)$ is Cohen-Macaulay. When $\dim M = r > 1$ and $x_1, \ldots, x_r$ is a maximal M-superficial sequence we set $N = M/(x_1, \ldots, x_{r-1})M$. Clearly $\dim N = 1$. By Corollary 1.2.7(1) we have $e_0^I(N) = e_0^I(M)$ and $e_1^I(N) = e_1^I(M)$. Therefore by the one dimensional case we get that N has minimal multiplicity and $G_I(N)$ is Cohen-Macaulay. So by Sally Descent (see Theorem 1.2.4(2)), we see that $G_I(M)$ is Cohen-Macaulay. From Corollary 1.2.7(2) we have $h_M^I(z) = h_N^I(z)$, so it follows that M has minimal multiplicity. □

We give a few examples of modules with minimal multiplicity.

Example 2.5. If A has minimal multiplicity with respect to I then all maximal Cohen-Macaulay A-modules have minimal multiplicity with respect to I.

In all the remaining examples we assume $I = \mathfrak{m}$.

Definition 2.6. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring. A maximal Cohen-Macaulay A-module M is called *Ulrich* if $e(M) = \mu(M)$.

Clearly Ulrich modules have minimal multiplicity. The following Cohen-Macaulay rings have Ulrich modules:

(a) If $\dim A = 1$ then it is easy to see that $\mathfrak{m}^n$ is Ulrich for all $n \gg 0$.

(b) If $A = Q/(f_1, \ldots, f_s)$ is a strict complete intersection (i.e. $f_1^*, \ldots, f_s^*$ form a $G(Q)$-regular sequence. See [7].

Next we give example's of modules over a hypersurface ring such that M has minimal multiplicity but it is not Ulrich.

Example 2.7. Set $Q = k[[y_1, \ldots, y_{d+1}]]$ and $\mathfrak{n}$ to be the maximal ideal of Q. Let a, b, c, d in $\mathfrak{n}$ be such that $f = ad - bc \in \mathfrak{n}^3 \setminus \mathfrak{n}^4$. Set $A = Q/(f)$. Set

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Define M and K by the exact sequences

$$0 \to Q^2 \xrightarrow{\phi} Q^2 \to M \to 0 \quad \text{and} \quad 0 \to Q^2 \xrightarrow{\psi} Q^2 \to K \to 0$$

Note that M and K are maximal Cohen-Macaulay A-modules and $K = \mathrm{Syz}_1^A(M)$. We prove in 4.2.12 that M or K have minimal multiplicity. We also prove that M and K are not Ulrich modules.

For the next example we need the following lemma which is interesting in its own right.

Lemma 2.8. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay module with a canonical module ω_A. Then A has minimal multiplicity with respect to $\mathfrak{m}$ if and only if ω_A has minimal multiplicity.*

Proof. Since ω_A is a maximal Cohen-Macaulay A-module, it follows that if A has minimal multiplicity then ω_A has minimal multiplicity. To prove the converse, it suffices to consider the case when the residue field of A is infinite (see Remark 1.1.4). If $\dim A = 0$ then $\mathfrak{m}^2\omega_A = 0$ Since ω_A is a faithful A-module we get $\mathfrak{m}^2 = 0$. Therefore A has minimal multiplicity. When $\dim A = d > 0$ then let $x_1, \ldots, x_d$ be a superficial $A \oplus \omega_A$ sequence such that $(x_1, \ldots, x_d)$ is also a minimal reduction of $\mathfrak{m}$. Set $(B, \mathfrak{n}) = (A/J, \mathfrak{m}/J)$. Note that $\omega_B = \omega_A/J\omega_A$. If ω_A has minimal multiplicity then by Theorem 2.4 we get $\mathfrak{m}^2\omega_A = J\mathfrak{m}\omega_A$. So $\mathfrak{n}^2\omega_B = 0$ and as shown before this yields $\mathfrak{n}^2 = 0$. Therefore $\mathfrak{m}^2 \subseteq J$. Since $x_1, \ldots, x_d$ is analytically independent in $\mathfrak{m}$ (cf. [2, Remark 4.6.9]) we get $\mathfrak{m}^2 = J\mathfrak{m}$. So A has minimal multiplicity by Theorem 2.4. □

Remark 2.9. We assert that $e(\omega_A) = e(A)$. When $\dim A = 0$ it is true because $\lambda(\omega_A) = \lambda(A)$. When $\dim A > 0$ then we go mod a maximal superficial $A \oplus \omega_A$-sequence to get the result by the dim 0 case and Corollary 1.2.7(1).

Example 2.10. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay module such that $\operatorname{type} A = 2$ and $e(A) = \mu(M) - d + 2$. Then $K = \mathrm{Syz}_1^A(M)$ has minimal multiplicity. For proof see 3.1.8. For an example of a ring of the kind above see [26, p. 336].

3. HILBERT COEFFICIENTS OF SYZYGY MODULES

In this chapter we assume that A is Cohen-Macaulay and I is an $\mathfrak{m}$-primary ideal. We let $\mathrm{Syz}_1^A(M)$ denote the first syzygy module of M in a minimal free resolution of M over A. When M is maximal Cohen-Macaulay A-module then $\mathrm{Syz}_1^A(M)$ is zero (when M is free) or a maximal Cohen-Macaulay A-module. Since multiplicity is additive on exact sequences of modules of the same dimension we have that

$$\mu(M)e_0^I(A) = e_0^I(M) + e_0^I(\mathrm{Syz}_1^A(M)).$$

For $e_1^I(M)$ we consider the function $n \mapsto \lambda\left(\mathrm{Tor}_1^A(M, A/I^{n+1})\right)$. In section 1. we show that this function a polynomial for $n \gg 0$ of degree at most $d-1$. A similar looking, but different result is proved by Kodiyalam [8, Theorem 2]. Sharper results are obtained when $I = \mathfrak{m}$. Results concerning the second Hilbert coefficient is in section 3. In section 2 we describe a construction which will be used in section 3 and in the next chapter.

3.1 The First Hilbert Coefficient

Proposition 3.1.1. *Let A be a Cohen-Macaulay local ring of dimension d, I an $\mathfrak{m}$-primary ideal and let M be a maximal Cohen-Macaulay A-module. Set $K = \mathrm{Syz}_1^A(M)$. Then*

1. $\displaystyle\sum_{n\geq 0} \lambda\left(\mathrm{Tor}_1^A(M, A/I^{n+1})\right) z^n = \frac{h_K^I(z) - \mu(M)h_A^I(z) + h_M^I(z)}{(1-z)^{d+1}}$

2. *For $n \gg 0$ the function $n \mapsto \lambda\left(\mathrm{Tor}_1^A(M, A/I^{n+1})\right)$ is given by a polynomial $t_M^I(z)$ of the form:*

$$t_M^I(z) = \left(\mu(M)e_1^I(A) - e_1^I(M) - e_1^I(K)\right)\frac{z^{d-1}}{(d-1)!} + \text{ lower terms in } z$$

3. $e_1^I(A)\mu(M) \geq e_1^I(M) + e_1^I(K)$

Proof. Set $F = A^{\mu(M)}$ and $T_n = \lambda\left(\mathrm{Tor}_1^A(M, A/I^{n+1})\right)$. Notice that $h_F^I(z) = \mu(M)h_A^I(z)$. We tensor the exact sequence

$$o \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with A/I^{n+1} to get the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^A(M, A/I^{n+1}) \longrightarrow K/I^{n+1}K \longrightarrow F/I^{n+1}F \longrightarrow M/I^{n+1}M \longrightarrow 0$$

Computing lengths and summing over all n we get (1). Set $s(z) = h_K^I(z) - \mu(M)h_A^I(z) + h_M^I(z)$. Since multiplicity is additive on short exact sequences we have that $s(1) = e_0^I(K) - \mu(M)e_0^I(A) + e_0^I(M) = 0$. Write $s(z) = (1-z)l(z)$. This shows $\deg t_M^I(z) \leq d - 1$. We have

$$l(1) = -s'(1) = \mu(M)e_1^I(A) - e_1^I(M) - e_1^I(K).$$

Using this we get (2). Finally notice that (3) follows from (2) □

When $I = \mathfrak{m}$ we have more precise results.

Theorem 3.1.2. *Let A be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let M be a maximal Cohen-Macaulay A-module. The following conditions are equivalent:*

(i) *M is free.*

(ii) $t_M(z) = 0$.

(iii) $\deg t_M^I(z) < d - 1$.

(iv) $\mu(M)e_1^I(A) = e_1^I(M) + e_1^I(\mathrm{Syz}_1^A(M))$

The theorem above proves that if M is non-free maximal Cohen-Macaulay module then $\deg t_M^I(z) = d - 1$. Theorem 3.1.6 gives information on its leading coefficient. For both theorems we need the following lemma:

Lemma 3.1.3. *If A is a Cohen-Macaulay local ring of dimension one and M is a maximal Cohen-Macaulay A-module, then*

$$\lambda\left(\mathrm{Tor}_1^A(M, A/\mathfrak{m}^{n+1})\right) \geq e_0\left(\mathrm{Syz}_1^A(M)\right) \ \textit{for all} \ n \gg 0$$

Proof. Since $\dim A = 1$ Proposition 3.1.1 shows that $\lambda\left(\mathrm{Tor}_1^A(M, A/\mathfrak{m}^{n+1})\right)$ is constant (say equal to l) for all large n. Choose n_0 large enough such that for all $n \geq n_0$ we have

$$\lambda\left(\mathrm{Tor}_1^A(M, A/\mathfrak{m}^{n+1})\right) = l$$
$$H(M,n) = e_0(M)$$
$$H(A,n) = e_0(A)$$

Pick $n > n_0$ and consider the exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^{n+1}} \longrightarrow \frac{A}{\mathfrak{m}^n} \longrightarrow 0$$

We tensor it with M to get the exact sequence

$$\mathrm{Tor}_1^A(M, A/\mathfrak{m}^n) \longrightarrow M \otimes_A \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \longrightarrow \frac{M}{\mathfrak{m}^{n+1}M} \longrightarrow \frac{M}{\mathfrak{m}^n M} \longrightarrow 0 \qquad (*)$$

Since $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong k^e$ we have

$$M \otimes_A \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \cong \left(\frac{M}{\mathfrak{m}M}\right)^e$$

Computing lengths using the exact sequence (*) we have

$$\begin{aligned} l &\geq e\mu(M) - \lambda(M/\mathfrak{m}^{n+1}M) + \lambda(M/\mathfrak{m}^n M) \\ &= e\mu(M) - \lambda(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M) \\ &= e\mu(M) - e_0(M) \\ &= e_0\left(\mathrm{Syz}_1^A(M)\right) \end{aligned}$$

This establishes the assertion of the lemma. □

Proof of Theorem 3.1.2. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear. By Proposition 3.1.1 we get (iii) $\implies$ (iv). So it remains to (iv) $\implies$ (i).

We first assume $d = 1$. If $\mu(M)e_1(A) = e_1(M) + e_1(\mathrm{Syz}_1^A(M))$ then by Proposition 3.1.1 we get $\mathrm{Tor}_1^A(M, A/\mathfrak{m}^n) = 0$ for all $n \gg 0$. By Lemma 3.1.3 we get $e_0(\mathrm{Syz}_1^A(M)) = 0$. This implies $\mathrm{Syz}_1^A(M) = 0$. Therefore M is free.

When $\dim M > 1$ we prove it first assuming k is infinite. Let $x_1, \ldots, x_d$ be a superficial sequence for $M \oplus A \oplus \operatorname{Syz}_1^A(M)$. Set $J = (x_1, \ldots, x_{d-1})$, $B = A/J$, $M_1 = M/JM$ and $K_1 = \operatorname{Syz}_1^A(M)/J \operatorname{Syz}_1^A(M)$. Since $x_1, \ldots, x_{d-1}$ is regular on A and M we have $K_1 = \operatorname{Syz}_1^B(M_1)$. Also note that $\mu(M) = \mu(M_1)$. By Corollary 1.2.7(1) we have $e_1(B) = e_1(A)$, $e_1(M_1) = e_1(M)$ and

$$e_1(\operatorname{Syz}_1^B(M_1)) = e_1(K_1) = e_1(\operatorname{Syz}_1^A(M)).$$

Therefore by the dimension one case we get M_1 is free B-module. Since $\operatorname{Tor}_1^A(M,k) \cong \operatorname{Tor}_1^B(M_1,k) = 0$ we get that M is a free A-module. If k is finite then set $A' = A[X]_S$ and $M' = M \otimes_A A'$ where $S = A[X] \setminus \mathfrak{m}A[X]$. The residue field of A' is $k(X)$ which is infinite. Furthermore $H(M,n) = H(M',n)$ for all $n \geq 0$. In particular $\mu(M) = \mu(M')$. Since A' is flat over A we get $\operatorname{Syz}_1^{A'}(M') \cong \operatorname{Syz}_1^A(M) \otimes_A A'$. So we have

$$\mu(M')e_1(A') - e_1(M') - e_1(\operatorname{Syz}_1^{A'}(M')) = \mu(M)e_1(A) - e_1(M) - e_1(\operatorname{Syz}_1^A(M)) = 0$$

Therefore by the previous case M' is free. So $\operatorname{Syz}_1^A(M) \otimes_A A' = 0$ and since A' is faithfully flat over A we get $\operatorname{Syz}_1^A(M) = 0$ and so M is free. $\square$

Remark 3.1.4. The preceding theorem fails for arbitrary $\mathfrak{m}$-primary ideals. Let $(A, \mathfrak{m})$ be a non-regular Cohen-Macaulay local ring of positive dimension d, with an infinite residue field. Let M be a non-free maximal Cohen-Macaulay A-module. Let $x_1, \ldots, x_d$ be a regular sequence for M, A and $K = \operatorname{Syz}_1^A(M)$. Set $J = (x_1, \ldots, x_d)$. It is easy to see that

$$h_A^J(z) = e_0^J(A), \; h_M^J(z) = e_0^J(M), \; h_K^J(z) = e_0^J(K)$$

From Proposition 3.1.1 we get $t_M^J(z) = 0$.

Theorem 3.1.2 enables us to define the following:

Definition 3.1.5. If $(A, \mathfrak{m})$ is Cohen-Macaulay of dimension d and M is maximal non-free Cohen-Macaulay then by 3.1.2 there is an equality

$$\sum_{n \geq 0} \lambda\left(\operatorname{Tor}_1^A(M, A/\mathfrak{m}^{n+1})\right) z^n = \frac{l_M(z)}{(1-z)^d} \quad \text{for some } l_M(z) \in \mathbb{Z}[z] \text{ with } l_M(1) \neq 0 \tag{3.1}$$

and an equality

$$(1-z)l_M(z) = h_{\mathrm{Syz}_1^A(M)}(z) - \mu(M)h_A(z) + h_M(z) \tag{3.2}$$

Set $e^T(M) = l_M(1)$.

We now prove

Theorem 3.1.6. *Let A be a Cohen-Macaulay local ring of dimension $d \geq 1$ and let M be a maximal Cohen-Macaulay A-module. Then*

1. $\mu(M)e_1(A) - e_1(M) - e_1(\mathrm{Syz}_1^A(M)) \geq e_0(\mathrm{Syz}_1^A(M))$

2. $\mu(M)\chi_1(A) \geq \chi_1(M) + \chi_1(\mathrm{Syz}_1^A(M))$.

Proof. Note that by Corollary 1.2.7(1) it suffices to prove the result for dimension 1. Also clearly the result holds when M is free. Therefore assume M is not free. Using Proposition 3.1.1 we get that $t_M(z) = \mu(M)e_1(A) - e_1(M) - e_1(\mathrm{Syz}_1^A(M))$. Using Lemma 3.1.3 we get (1). To prove (2) set $e^T(M) = \mu(M)e_1(A) - e_1(M) - e_1(\mathrm{Syz}_1^A(M))$ and $K = \mathrm{Syz}_1^A(M)$. Since $\mu(M)e_0(A) = e_0(M) + e_0(K)$ we get

$$\begin{aligned}\mu(M)\chi_1(A) &= \chi_1(M) + e_1(K) - e_0(K) + e^T(M) \\ &= \chi_1(M) + \chi_1(K) + e^T(M) - \mu(K)\end{aligned}$$

Since $e^T(M) \geq e_0(K)$ (by (1)) and $e_0(K) \geq \mu(K)$ we get the required result. □

Remark 3.1.7. If equality holds in (2) then $\mathrm{Syz}_1^A(M)$ is Ulrich. Note that $\chi_1(A)\mu(M) = \chi_1(M) + \chi_1(K) + e^T(M) - \mu(K)$. So it follows that equality holds in (2) if and only if $e^T(M) = \mu(K)$. Since $e^T(M) \geq e(K)$ by 3.1.6(1) we get the required result.

We use Theorem 3.1.6 in the following example.

Example 3.1.8. Let $(A, \mathfrak{m})$ be a Cohen-Macaulay module such that type $A = 2$ and $e(A) = \mu(M) - d + 2$. Then $K = \mathrm{Syz}_1^A(M)$ has minimal multiplicity. By Lemma 2.8 we get that ω_A does not have minimal multiplicity. So by Theorem 2.4 $\chi_1(\omega_A) \geq 1$.

Note that $h_A(z) = 1 + h_1(A)z + z^2$ and type $A < h_1(A)$. So $e(A) \geq 5$. We also have an exact sequence $0 \to K \to A^2 \to M \to 0$.

We first prove that K is not Ulrich. By Remark 1.1.4 it suffices to consider the case when the residue field of A is infinite. Let $x_1, \ldots, x_d$ be a $K \oplus A \oplus \omega_A$-superficial sequence. Set $(B, \mathfrak{n}) = (A/J, \mathfrak{m}/J)$, $N = M/JM$ and $L = K/JK$. So we have an exact sequence of B-modules $0 \to L \to B^2 \to N \to 0$. By Remark 2.9 it follows that $e(K) = e(A)$. If K is Ulrich then $L \cong k^{e(K)} = k^{e(A)}$. Since $0 \to L \to B^2$, this yields that $4 = \operatorname{type} A^2 \geq e(A)$ which is a contradiction.

To prove that K has minimal multiplicity note that by Theorem 3.1.6 and Remark 3.1.7 we get that $2 = 2\chi_1(A) > \chi_1(\omega_A) + \chi_1(K)$. Since $\chi_1(\omega_A) > 0$ we get $\chi_1(K) = 0$. So by Theorem 2.4 we get that K has minimal multiplicity.

Since $\chi_1(\omega_A) = 1 = \chi_1(\omega_B)$ it follows that $e_1(\omega_A) = e_1(\omega_B)$. Consequently we get that $G(\omega_A)$ is Cohen-Macaulay.

Remark 3.1.9. Note that when $\dim A = 0$ we get that $l_M(z)$ has non-negative coefficients. This yields the following: When $G(A)$ and $G(M)$ is Cohen-Macaulay then $e_i(A)\mu(M) \geq e_i(M)$ for all $i \geq 0$. In Section 3.3 it is proved that if A is Gorenstein then it suffices to assume that $\operatorname{depth} G(A) \geq d - 1$.

3.2 Basic Construction

Remark 3.2.1. For each $n \geq 0$ and $t \geq 0$ set $L_t(M)_n = \operatorname{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$. For $t \geq 0$ let $L_t(M) = \bigoplus_{n \geq 0} L_t(M)_n$. If $x_1, \ldots, x_s$ is a sequence of elements in $\mathfrak{m}$, then we give $L_t(M)$ a structure of a graded $A[X_1, \ldots, X_s]$-module as follows:

For $i = 1, \ldots, s$ let $\xi_i : A/\mathfrak{m}^n \to A/\mathfrak{m}^{n+1}$ be the maps given by $\xi_i(a + \mathfrak{m}^n) = x_i a + \mathfrak{m}^{n+1}$. These homomorphisms induces homomorphisms

$$\operatorname{Tor}_t^A(\xi_i, M) : \operatorname{Tor}_t^A(A/\mathfrak{m}^n, M) \longrightarrow \operatorname{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$$

Thus, for $i = 1, \ldots, s$ and each t we obtain homogeneous maps of degree 1:

$$X_i : L_t(M) \longrightarrow L_t(M).$$

For $i, j = 1, \ldots, s$ the equalities $\xi_i \xi_j = \xi_j \xi_i$ yields equalities $X_i X_j = X_j X_i$. So $L_t(M)$ is a graded $A[X_1, \ldots, X_s]$-module for each $t \geq 0$.

Proposition 3.2.2. *Let $M, F,$ and K be finite A-modules and let $x_1, \ldots, x_s$ be a sequence of elements in $\mathfrak{m}$. If $L_t(M)$, $L_t(F)$ and $L_t(K)$ are given the $A[X_1, \ldots, X_s]$-module structure described in Remark 3.2.1 then*

1. *Every exact sequence of A-modules $0 \to K \to F \to M \to 0$ induces a long exact sequence of graded $A[X_1, \ldots, X_s]$-modules*

   $\cdots \to L_{t+1}(M) \to L_t(K) \to L_t(F) \to L_t(M) \to \cdots \to L_0(M) \to 0.$

2. *For $i = 1, \ldots, s$ there is an equality*

$$\ker\left(L_0(M)_{n-1} \xrightarrow{X_i} L_0(M)_n\right) = \frac{\mathfrak{m}^{n+1}M :_M x_i}{\mathfrak{m}^n M}$$

3. *If $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$ is such that x_i^* is $G(M)$-regular then X_i is $L_0(M)$-regular.*

4. *If F is free A-module and x_i is K-superficial for some i then*

   (a) $\ker\left(L_1(M) \xrightarrow{X_i} L_1(M)\right)_n = 0$ *for $n \gg 0$*

   (b) If x_i^* is $G(K)$-regular then X_i is $L_1(M)$-regular.

Proof. To prove part 1, set

$$\beta_{t,n} = \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, \beta) : \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, K) \longrightarrow \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, F)$$

$$\alpha_{t,n} = \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, \alpha) : \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, F) \longrightarrow \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$$

and consider the connecting homomorphisms

$$\delta_{t+1,n} : \mathrm{Tor}_{t+1}^A(A/\mathfrak{m}^{n+1}, M) \longrightarrow \mathrm{Tor}_t^A(A/\mathfrak{m}^{n+1}, K)$$

By a well known theorem in Homological algebra if $\mathbf{X}$ is a free resolution of K and $\mathbf{Z}$ is a free resolution of M then there exists a free resolution $\mathbf{Y}$ of F and an exact sequence of complexes of free A-modules $0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0$ whose homology sequence is the given exact sequence $0 \to K \to F \to M \to 0$. This yields for each n a commuting diagram of complexes with exact rows ;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{X} \otimes A/\mathfrak{m}^n & \longrightarrow & \mathbf{Y} \otimes A/\mathfrak{m}^n & \longrightarrow & \mathbf{Z} \otimes A/\mathfrak{m}^n & \longrightarrow & 0 \\
 & & \downarrow \xi_i & & \downarrow \xi_i & & \downarrow \xi_i & & \\
0 & \longrightarrow & \mathbf{X} \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & \mathbf{Y} \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & \mathbf{Z} \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & 0
\end{array}$$

In homology it induces the following commutative diagram :

$$\begin{array}{ccccccccc} \cdots \longrightarrow & L_{t+1}(M)_{n-1} & \xrightarrow{\delta_{t+1,n-1}} & L_t(K)_{n-1} & \xrightarrow{\beta_{t,n-1}} & L_t(F)_{n-1} & \xrightarrow{\alpha_{t,n-1}} & L_t(M)_{n-1} & \longrightarrow \cdots \\ & \downarrow X_i & & \downarrow X_i & & \downarrow X_i & & \downarrow X_i & \\ \cdots \longrightarrow & L_{t+1}(M)_n & \xrightarrow{\delta_{t+1,n}} & L_t(K)_n & \xrightarrow{\beta_{t,n}} & L_t(F)_n & \xrightarrow{\alpha_{t,n}} & L_t(M)_n & \longrightarrow \cdots \end{array}$$

This proves the desired assertion.

Remark 3.2.3. We will use the exact diagram above often. So when there is a reference to this remark, I mean to refer the commuting diagram above.

The second part is clear from the definition of the action X_i. Part 3. follows from 2. Note that if F is free, then $L_1(F) = 0$, so 1. gives an exact sequence of $A[X_1, \ldots, X_s]$-modules $0 \longrightarrow L_1(M) \longrightarrow L_0(K)$. Together with 2. and 3. this yields the assertions in 4,. $\square$

3.3 Second Hilbert coefficient

In the theorem below we establish similar inequalities for higher Hilbert coefficients of maximal Cohen-Macaulay modules over Gorenstein rings. For every A-module we set $M^* = \operatorname{Hom}_A(M, A)$. Note that if M is maximal Cohen-Macaulay then so is M^* cf.[2, 3.3.10.d] . Also, $\operatorname{type}(M) = \operatorname{Ext}_A^d(k, M)$ denotes the Cohen-Macaulay type of M.

Theorem 3.3.1. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring. Let M be a maximal Cohen-Macaulay A-module and set $S^A(M) = \left(\operatorname{Syz}_1^A(M^*)\right)^*$. If $G(M)$ is Cohen-Macaulay and* $\operatorname{depth} G(A) \geq d-1$ *then the following hold*

1. $\operatorname{type}(M) e_2(A) \geq e_2(M) + e_2(S^A(M))$.

2. $\operatorname{type}(M) e_i(A) \geq e_i(M)$ *for each* $i \geq 0$.

Note that if A is a hypersurface ring and if M has no free summands, then $S^A(M) = \operatorname{Syz}_1^A(M)$ and $\mu(M) = \operatorname{type}(M)$. Before proving Theorem 3.3.1 we need a few lemmas.

Lemma 3.3.2. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring with infinite residue field and let M be a maximal Cohen-Macaulay A-module. If x is A-regular then for the ring $B = A/(x)$ and the B-module $N = M/xM$ we have*

1. $\operatorname{Hom}_B(N, B) \cong M^*/xM^*$.
2. $\operatorname{Syz}_1^B(M^*/xM^*) \cong \operatorname{Syz}_1^A(M^*)/x \operatorname{Syz}_1^A(M^*)$.
3. $S^B(N) \cong S^A(M)/xS^A(M)$.

Proof. Since M is maximal Cohen-Macaulay x is also M-regular. To prove 1. we use the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow N \longrightarrow 0$$

to get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(N, A) \longrightarrow \operatorname{Hom}_A(M, A) \xrightarrow{x} \operatorname{Hom}_A(M, A) \\ \longrightarrow \operatorname{Ext}_A^1(N, A) \longrightarrow \operatorname{Ext}_A^1(M, A). \end{aligned}$$

Since M is maximal Cohen-Macaulay and A is Gorenstein we have $\operatorname{Ext}_A^1(M, A) = 0$ cf.[2, 3.3.10.d]. Using the isomorphisms

$$\operatorname{Hom}_A(N, A) = 0 \quad \text{and} \quad \operatorname{Ext}_A^1(N, A) \cong \operatorname{Hom}_B(N, B)$$

see [2, 3.1.16], we get the required result. Note that 2. holds since M^* is maximal Cohen-Macaulay. To prove 3. note that

$$\begin{aligned} S^A(M)/xS^A(M) &= \frac{\operatorname{Hom}_A\left(\operatorname{Syz}_1^A(M^*), A\right)}{x \operatorname{Hom}_A\left(\operatorname{Syz}_1^A(M^*), A\right)} \\ &\cong \operatorname{Hom}_B\left(\frac{\operatorname{Syz}_1^A(M^*)}{x \operatorname{Syz}_1^A(M^*)}, B\right) \\ &\cong \operatorname{Hom}_B\left(\operatorname{Syz}_1^B(M^*/xM^*), B\right) \\ &\cong \operatorname{Hom}_B\left(\operatorname{Syz}_1^B(\operatorname{Hom}_B(N, B)), B\right) \\ &= S^B(N) \end{aligned}$$

□

The next result helps in the dimension one case.

Lemma 3.3.3. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one, let M be a non-free maximal Cohen-Macaulay A-modules and let*

$$0 \longrightarrow E \longrightarrow F \longrightarrow M \longrightarrow 0$$

be an exact sequence with F a finite free A-module. Let x be $A \oplus M \oplus E$-superficial. If $L_1(M)$ is given the $A[X]$-module structure described in Remark 3.2.1 then we have

1. *$L_1(M)$ is a Noetherian $A[X]$-module of dimension one.*

2. *$(1-z)l_M(z) = h_E(z) - h_F(z) + h_M(z)$.*

3. *When $G(E)$ is Cohen-Macaulay then*

   a. *$L_1(M)$ is Cohen-Macaulay.*

   b. *$e_{i+1}(F) - e_{i+1}(M) - e_{i+1}(E) \geq 0$ for every $i \geq 0$.*

Proof. 1. Since $\dim A = 1$ and M is non-free, it follows from Lemma 3.1.3 that $\lambda(L_1(M)_n)$ is a non-zero constant for large n. Since $X \colon L_1(M)_n \to L_1(M)_{n+1}$ is injective for large n and since $\lambda(L_1(M)_n)$ is constant for large n, it follows that $L_1(M)_{n+1} = XL_1(M)_n$ for large n, say for all $n \geq s$. For each $i = 0, 1, \ldots, s$ choose a finite set $\mathcal{P}_i$ of generators of $L_1(M)_i$ as an A-module. It is easy to see that $\bigcup_{i=0}^{s} \mathcal{P}_i$ generates $L_1(M)$ over $A[X]$. Since $\lambda(L_1(M)/XL_1(M)) < \infty$ and $\lambda(L_1(M)) = \infty$ it follows that $L_1(M)$ has dimension one.

2. By Schanuel's lemma, [9, p. 158] we have $F \oplus \mathrm{Syz}_1^A(M) \cong E \oplus A^{\mu(M)}$. Therefore

$$(1-z)l_M(z) = h_{\mathrm{Syz}_1^A(M)}(z) - \mu(M)h_A(z) + h_M(z) = h_E(z) - h_F(z) + h_M(z)$$

3.a is clear from 1. and Proposition 3.2.2.3

3.b It follows from 1. that $l_M(z)$ is the h-polynomial of $L_1(M)$ considered as an $A[X]$-module. Set $e_i^T(M) = l_M^{(i)}(l)/i!$. Since $L_1(M)$ is Cohen-Macaulay all the coefficients of $l_M(z)$ is non-negative. Therefore $e_i^T(M) \geq 0$ for all $i \geq 0$. Using 2. we obtain

$$e_i^T(M) = e_{i+1}(F) - e_{i+1}(M) - e_{i+1}(E)$$

and so we get the required result. $\square$

Proof of Theorem 3.3.1. By Remark 1.1.4 it is sufficient to consider the case when the residue field of A is infinite. If M is free then M^* is free and so $S^A(M) = 0$. Since A is Gorenstein $\operatorname{type} A = 1$ and so $\operatorname{type} M = \mu(M)$. Therefore the result holds when M is free. Next we consider the case when M is non-free. Note that $\mu(M^*) = \operatorname{type}(M)$ cf.[2, 3.3.11.b] . Set $F = A^{\operatorname{type}(M)}$ and $L = \operatorname{Syz}_1^A(M^*)$. We have an exact sequence:

$$0 \longrightarrow L \longrightarrow F \longrightarrow M^* \longrightarrow 0.$$

Dualizing this sequence we get an exact sequence

$$0 \longrightarrow M \longrightarrow F^* \longrightarrow L^* \longrightarrow 0. \tag{3.3}$$

Note that $L^* = S^A(M)$ and $F^* = A^{\operatorname{type}(M)}$. When $\dim A = 1$, assertion 1. follows from Lemma 3.3.3. When $\dim M = 2$, let x be superficial for $M \oplus S^A(M) \oplus A$. Set $N = M/xM$ and $G = F/xF$. By Lemma 3.3.2 we get $S^B(N) = S^A(M)/xS^A(M)$. Since $G(M)$ is Cohen-Macaulay and $\operatorname{depth} G(A) \geq 1$ we have that $e_2(M) = e_2(N)$ and $e_2(F) = e_2(G)$. Furthermore it follows from Corollary 1.2.6.6 that $e_2(S^A(M)) \leq e_2(S^A(M)/xS^A(M)) = e_2(S^B(N))$. Therefore we have:

$$\begin{aligned} e_2(M) + e_2(S^A(M)) &\leq e_2(N) + e_2(S^A(M)/xS^A(M)) \\ &= e_2(N) + e_2(S^B(N)) \\ &\leq e_2(G) \\ &= e_2(F) \end{aligned}$$

Note that the second inequality above follows from the dimension one case.

When $\dim A > 2$ let $x_1, \ldots, x_{d-2}$ be a $M \oplus S^A(M) \oplus A$-superficial sequence. Set $J = (x_1, \ldots, x_{d-2}), N = M/JM, G = F/JF$. Using Lemma 3.3.2.3 inductively we get $S^B(N) = S^A(M)/JS^A(M)$. By Corollary 1.2.6.6 we get that $e_2(S^A(M)) = e_2(S^B(N)), e_2(M) = e_2(N)$ and $e_2(F) = e_2(G)$. So the result follows from the dimension 2 case.

To prove (ii) note that since $G(M)$ is Cohen-Macaulay and $\operatorname{depth} G(A) \geq d - 1$ it suffices to consider the case when $\dim A = 1$. Using exact sequence (3.3) we get $e(A) \operatorname{type}(M) \geq e(M)$. By Corollary 1.2.8.4 we get that $e_i(S^A(M)) \geq 0$ for all $i \geq 1$. So we get $e_{i+1}(A) \operatorname{type}(M) \geq e_{i+1}(M)$ by Lemma 3.3.3. $\square$

4. HILBERT FUNCTIONS OF MODULES OVER HYPERSURFACES

If A is regular and M is a maximal Cohen-Macaulay A-module, then it is free. The next case is that of hypersurface rings. In this chapter we assume that A is a hypersurface ring. In Section 1 we show that if M is a maximal A-module then the Hilbert function of M is non-decreasing. In Section 2 we assume $A = Q/(f)$, with Q-regular. When M is a maximal Cohen-Macaulay A-module we give lower bounds on $e(M)$ and $e_1(M)$ and study what happens when equality is achieved.

4.1 Monotonicity

The main theorem of this section is

Theorem 4.1.1. *Let $(A, \mathfrak{m})$ be a hypersurface ring of positive dimension. If M is a maximal Cohen-Macaulay A-module, then the Hilbert function of M is non-decreasing.*

The following remark will be used often.

Remark 4.1.2. Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series with non-negative coefficients. If the power series $g(z) = \sum_{n\geq 0} b_n z^n$ satisfies $g(z) = f(z)/(1-z)$, then $b_n = \sum_{i=0}^{n} a_i$, and so the sequence $\{b_n\}$ is nondecreasing.

The next proposition yields an easy criterion for monotonicity.

Proposition 4.1.3. *Let M be an A-module. If* $\operatorname{depth} G(M) \geq 1$ *then the Hilbert function of M is non-decreasing.*

Proof. Using Remark 1.1.4 we may assume that k, the residue field of A is infinite. Since k is infinite and $\operatorname{depth} G(M) \geq 1$ we may assume that there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, such that x^* is $G(M)$-regular. So it follows that the Hilbert function of M is non-decreasing. $\square$

Remark 4.1.4. If $\phi : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ is a surjective map of local rings and if M is a finite B-module then $\mathfrak{m}^n M = \mathfrak{n}^n M$ for all $n \geq 0$. Therefore $G_{\mathfrak{m}}(M) = G_{\mathfrak{n}}(M)$. The notation $G(M)$ will be used to denote this without any reference to the ring.

We deduce Theorem 4.1.1 from the following result.

Theorem 4.1.5. *Let $(Q, \mathfrak{m})$ be a Noetherian local ring and let M be a finite Q-module with* $\operatorname{projdim}_Q M \leq 1$. *If* $\operatorname{depth} G(Q) \geq 2$ *then the Hilbert function of M is non-decreasing.*

Proof. It is sufficient to consider the case when the residue field of Q is infinite (see Remark 1.1.4). Since $\operatorname{projdim}_Q M \leq 1$ we have a presentation of M

$$0 \longrightarrow Q^n \longrightarrow Q^m \longrightarrow M \longrightarrow 0. \tag{4.1}$$

with $0 \leq n \leq m$. Let x, y be elements in $\mathfrak{m} \setminus \mathfrak{m}^2$ such that x^*, y^* is a $G(Q)$-regular sequence. Let $L_0(Q)$, $L_0(M)$ and $L_1(M)$ be the $Q[X, Y]$-modules described in Remark 3.2.1.

The element X is $L_0(Q)$-regular by Proposition 3.2.2.3. Set $B = Q/(x)$ and note that

$$\frac{L_0(Q)}{XL_0(Q)} = \bigoplus_{n \geq 0} \frac{Q}{(x, \mathfrak{m}^{n+1})} = L_0(B).$$

Since $G(B) = G(Q)/x^*G(Q)$ we see that y^* is $G(B)$-regular. Proposition 3.2.2.3 shows that Y is $L_0(B)$-regular. Thus, the sequence X, Y is a $L_0(Q)$-regular sequence.

Using the exact sequence (4.1) and Proposition 3.2.2.1, we obtain an exact sequence of graded $Q[X, Y]$ modules

$$0 \longrightarrow L_1(M) \longrightarrow L_0(Q)^n \longrightarrow L_0(Q)^m \longrightarrow L_0(M) \longrightarrow 0. \tag{4.2}$$

Since X, Y is an $L_0(Q)$- regular sequence, we see that it is also $L_1(M)$-regular.

The regularity of X, Y imply equalities

$$\sum_{i\geq 0}\lambda(L_0(Q)_i)z^i = \frac{u(z)}{(1-z)^2} \quad \text{where} \quad u(z) = \sum_{i\geq 0}\lambda\left(\frac{L_0(Q)_i}{(X,Y)L_0(Q)_{i-1}}\right)z^i.$$

$$\sum_{i\geq 0}\lambda(L_1(M)_i)z^i = \frac{v(z)}{(1-z)^2} \quad \text{where} \quad v(z) = \sum_{i\geq 0}\lambda\left(\frac{L_1(M)_i}{(X,Y)L_1(M)_{i-1}}\right)z^i.$$

Using the exact sequence (4.2) we get

$$\sum_{i\geq 0}\lambda(L_0(M)_i)z^i = (m-n)\frac{u(z)}{(1-z)^2} + \frac{v(z)}{(1-z)^2}$$

The equality $H_M(z) = (1-z)\sum_{i\geq 0}\lambda(L_0(M)_i)z^i$ yields

$$H_M(z) = (m-n)u_2(z)/(1-z) + v_2(z)/(1-z).$$

Now Remark 4.1.2 shows that the Hilbert function of M is non-decreasing. □

We obtain Theorem 4.1.1 as a corollary to the previous theorem.

Proof of Theorem 4.1.1 . We may assume that A is complete and so $A \cong Q/(f)$ for some regular local ring $(Q,\mathfrak{n})$ and $f \in \mathfrak{n}^2$. Then $\operatorname{depth} M = \dim Q - 1$ and $\operatorname{projdim}_Q M = 1$. Using Theorem 4.1.5 it follows that the Hilbert function of M is non-decreasing. □

Since the Hilbert function is increasing if $\operatorname{depth} G(M) > 0$, we construct a maximal Cohen-Macaulay module M over a hypersurface ring A such that $\operatorname{depth} G(M) = 0$.

Example 4.1.6. Set $Q = k[[x,y]]$ and $\mathfrak{n} = (x,y)$. Define M by the exact sequence

$$0 \longrightarrow Q^2 \xrightarrow{\phi} Q^2 \longrightarrow M \longrightarrow 0$$

where

$$\phi = \begin{pmatrix} x & y \\ -y^2 & 0 \end{pmatrix}$$

Set $(A,\mathfrak{m}) = (Q/(y^3), \mathfrak{n}/(y^3))$. Since the determinant of ϕ annihilates M we get $y^3M = 0$. Note that M is a maximal Cohen-Macaulay A-module. Set $K = \operatorname{Syz}_1^A(M)$

and $E = M \oplus K$. Note that $G(Q) = k[x^*, y^*]$. Since $y^3 E = 0$, we have that if $P \in \operatorname{Ass}_{G(Q)}(G(E))$ then $P \supseteq (y^*)$. So we get that $x^* \notin P$ if P is a relevant associated prime of $G(E)$. Therefore x^* is an $E = M \oplus K$-superficial element. We first show that $G(M)$ is not Cohen-Macaulay. Assuming it is, then 1.2.4.1 shows that x^* is $G(M)$-regular. However if m_1, m_2 are the generators of M then $xm_1 = y^2 m_2 \in \mathfrak{n}^2 M$ and this implies $m_1 \in (\mathfrak{n}^2 M \colon {}_M x) = \mathfrak{n} M$, which is a contradiction.

We now show that $G(K)$ is Cohen-Macaulay. Note that M defines a matrix factorization (ϕ, ψ) where

$$\psi = \begin{pmatrix} 0 & -y \\ y^2 & x \end{pmatrix}$$

and $\phi\psi = \psi\phi = y^3 \varepsilon_3$. Since all the entries of ψ are in the maximal ideal we have

$$0 \longrightarrow G \xrightarrow{\psi} F \longrightarrow K \longrightarrow 0 \quad \text{where } F = G = Q^2 \tag{4.3}$$

Set $b_i(K) = \lambda((\mathfrak{n}^{i+1} K \colon {}_K x)/\mathfrak{n}^i K)$. Since $y^3 K = 0$ we get $\mathfrak{n}^3 K = x\mathfrak{n}^2 K$. Therefore $\mathfrak{n}^{i+1} K = x\mathfrak{n}^i K$ for all $i \geq 2$. So we get $b_i(K) = 0$ for all $i \geq 2$. Therefore to prove $G(K)$ is Cohen-Macaulay it is sufficient to show that $b_1(K) = 0$.

Let $p \in (\mathfrak{n}^2 K \colon {}_K x)$. We want to show that $p \in \mathfrak{n} K$. Let $u = (u_1, u_2)$ be the pre-image of p in F. Using the commutative diagram in Remark 3.2.3 we see that there exists $v = (v_1, v_2) \in G$ such that

$$xu + \mathfrak{n}^2 G = \psi(v) + \mathfrak{n}^2 F$$

Therefore we get

$$\begin{aligned} -yv_2 &= xu_1 \mod \mathfrak{n}^2 \\ y^2 v_1 + xv_2 &= xu_2 \mod \mathfrak{n}^2 \end{aligned}$$

From the first equation we obtain $u_1, v_2 \in \mathfrak{n}$. Substituting this in the second equation we get $xu_2 \in \mathfrak{n}^2$. Since x^* is $G(Q)$-regular we get $u_2 \in \mathfrak{n}$, hence $p \in \mathfrak{n} K$. This shows $b_1(K) = 0$, and as asserted before, this proves that $G(K)$ is Cohen-Macaulay.

This example also has yields an example of an maximal Cohen-Macaulay module P over a hypersurface such that $G(P)$ is Cohen-Macaulay but $G(\operatorname{Syz}_1^A(P))$ is not

Cohen-Macaulay. Simply take $P = K$ in the previous example. Then note that $G(\mathrm{Syz}_1^A(P)) = M$.

The next corollary partly overlaps with a result of Elias [3]: all equicharacteristic Cohen-Macaulay rings of dimension 1 and embedding dimension 3 have non-decreasing Hilbert functions.

Corollary 4.1.7. *If* $(A, \mathfrak{m})$ *be a complete intersection of positive dimension and codimension* 2 *then the Hilbert function of* A *is non-decreasing.*

Proof. We may assume that A is complete and hence $A = Q/(f, g)$ for a regular sequence f, g in a regular local ring $(Q, \mathfrak{q})$. Set $(R, \mathfrak{n}) = (Q/(f), \mathfrak{q}/(f))$. Then $G(R)$ is Cohen-Macaulay, $\dim A = \dim R - 1$ and $\operatorname{projdim}_R A = 1$. Therefore by Theorem 4.1.5 we get that the Hilbert function of A is non-decreasing. □

The following example from [26, p. 337] illustrates an example when $(A, \mathfrak{m})$ is a complete intersection of dimension 1 and codimension 2 such that $\operatorname{depth} G(A) = 0$.

Example 4.1.8. Set $A = k[[t^6, t^7, t^{15}]]$. We have $A = k[X, Y, Z]/(Y^3 - XZ, X^5 - Z^2)$. Furthermore $G(A) = k[X, Y, Z]/(XZ, Y^6, Y^3Z, Z^2)$ and $h_A(z) = 1 + 2z + z^2 + z^3 + z^5$.

4.2 Hilbert coefficients

In the rest of this paper ε_s denotes the $s \times s$ identity matrix. Let $(Q, \mathfrak{n})$ be a regular local ring, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$, M a maximal Cohen-Macaulay A-module and $K = \mathrm{Syz}_1^A(M)$. By a *matrix-factorization* of f we mean a pair (ϕ, ψ) of square-matrices with elements in Q such that

$$\phi\psi = \psi\phi = f\varepsilon.$$

If M is an A-module then $\operatorname{projdim}_Q M = 1$. Also a presentation of M

$$0 \longrightarrow Q^n \xrightarrow{\phi} Q^n \longrightarrow M \longrightarrow 0$$

yields a matrix factorization of f. See [27, p. 54] for details.

In the sequel (ϕ_M, ψ_M) will denote a matrix factorization of f such that

$$0 \longrightarrow Q^n \xrightarrow{\phi_M} Q^n \longrightarrow M \longrightarrow 0$$

is a minimal presentation of M. Note that

$$0 \longrightarrow Q^n \xrightarrow{\psi_M} Q^n \longrightarrow \mathrm{Syz}_1^A(M) \longrightarrow 0$$

is a not-necessarily minimal presentation of $\mathrm{Syz}_1^A(M)$.

If $\phi : Q^n \longrightarrow Q^m$ is a linear map then we set

$$\begin{aligned} I_\phi &= \text{ ideal generated by the entries of } \phi \\ i_\phi &= \max\{i \mid I_\phi \subseteq \mathfrak{n}^i\}. \end{aligned}$$

If M has the minimal presentations:

$0 \to Q^n \xrightarrow{\phi} Q^n \to M \to 0$ and $0 \to Q^n \xrightarrow{\phi'} Q^n \to M \to 0$, then it is well known that $I_\phi = I_{\phi'}$, $i_\phi = i_{\phi'}$ and $\det(\phi) = u \det(\phi')$ with u a unit. We set $i(M) = i_\phi$ and $\det(M) = (\det(\phi))$. For $g \in Q$, $g \neq 0$, set $v_Q(g) = \max\{i \mid g \in \mathfrak{n}^e\}$. For convenience set $v_Q(0) = \infty$. Note that $e(Q/(g)) = v_Q(g)$ for any $g \neq 0$. We first consider the case when $\dim A = 0$.

Remark 4.2.1. Let $(Q, \mathfrak{n})$ be a DVR, $v_Q(f) = e$, $A = Q/(f)$ and M a finite A-module. If $\mathfrak{n} = (y)$ then $f = uy^e$, where u is a unit. Therefore

$$M \cong \bigoplus_{i=1}^{\mu(M)} Q/(y^{a_i}) \quad \text{with } 1 \leq a_1 \leq \ldots \leq a_{\mu(M)} \leq e.$$

This yields a minimal presentation of M:

$$0 \to Q^n \xrightarrow{\psi} Q^n \to M \to 0 \text{ where } \psi_{ij} = \delta_{ij} y^{a_i}.$$

This yields

1. $i(M) = a_1$.

2. $h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1})+$ higher powers of z

3. $h_0(M) \geq h_1(M) \geq \ldots \geq h_s(M)$.

4. M is free if and only if $i(M) = e$.

5. If $K = \mathrm{Syz}_1^A(M)$ then
$$K \cong \bigoplus_{i=1}^{\mu(M)} Q/(y^{e-a_i})$$

6. $e(M) \geq \mu(M)i(M)$ and $e_1(M) \geq \mu(M)\binom{i(M)}{2}$.

7. $v_Q(\det \phi) \geq i(M)\mu(M)$ with equality if and only if $e(M) = i(M)\mu(M)$.

We note an immediate corollary to assertion 3. in the previous remark.

Corollary 4.2.2. *Let $(A, \mathfrak{m})$ be a hypersurface ring. Let M be a maximal Cohen-Macaulay A-module such that $G(M)$ is Cohen-Macaulay. If $h_M(z) = h_0(M) + h_1(M)z + \cdots + h_s(M)z^s$ is the h-polynomial of M then $h_0(M) \geq h_1(M) \geq \ldots \geq h_s(M)$.*

Proof. Set $d = \dim A$. We may assume that A is complete with infinite residue field, hence $A = Q/(f)$ for some regular local ring $(Q, \mathfrak{n})$. Consider M as a Q-module. Let $x_1, \ldots, x_d$ be a $Q \oplus M$-superficial sequence. Set $J = (x_1, \ldots, x_d)$, $(R, \mathfrak{q}) = (Q/J, \mathfrak{n}/J)$, $B = A/J$ and $N = M/JM$. Note that R is a DVR. Since $G(M)$ is Cohen-Macaulay we also have $h_M(z) = h_N(z)$ and so the result follows from Remark 4.2.1.3. □

To use the other assertions in Remark 4.2.1 we need the following definitions.

Definition 4.2.3. Let $(Q, \mathfrak{n})$ be a regular local ring, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$ and M a maximal Cohen-Macaulay A-module. Let $0 \to Q^n \xrightarrow{\phi} Q^n \to M \to 0$ be a minimal presentation of M. We say that $x \in \mathfrak{n}$ is ϕ-superficial if

1. x is $(Q \oplus M \oplus A)$-superficial.

2. If $\phi = (\phi_{ij})$ then $v_Q(\phi_{ij}) = v_{Q/xQ}(\overline{\phi_{ij}})$

3. $v_Q(\det(\phi)) = v_{Q/xQ}(\det(\overline{\phi}))$.

Since $e(Q/(g)) = v_Q(g)$ for any $g \neq 0$ it follows that if x is $Q \oplus M \oplus A \oplus \left(\bigoplus_{ij} Q/(\phi_{ij})\right) \oplus Q/\det(\phi)$ -superficial then it is ϕ-superficial. So ϕ-superficial elements exist if the residue field of Q is infinite.

If x is ϕ-superficial, then clearly $i(M) = i(M/xM)$. Also note that Q/xQ is regular and we have an exact sequence

$$0 \longrightarrow \left(\frac{Q}{xQ}\right)^n \xrightarrow{\phi\otimes_Q Q/xQ} \left(\frac{Q}{xQ}\right)^n \longrightarrow \frac{M}{xM} \longrightarrow 0$$

This enables the following definition:

Definition 4.2.4. Let $(Q, \mathfrak{n})$ be a regular local ring, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$ and M a maximal Cohen-Macaulay A-module Let $0 \to Q^n \xrightarrow{\phi} Q^n \to M \to 0$ be a minimal presentation of M. We say that $x_1, \ldots, x_r$ is a ϕ-superficial sequence if $\overline{x_i}$ is $(\phi \otimes_Q Q/(x_1, \ldots, x_{i-1}))$-superficial for $i = 1, \ldots, r$.

Theorem 4.2.5. *Let $(Q, \mathfrak{n})$ be a regular local ring, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$, M a maximal Cohen-Macaulay A-module and $K = \mathrm{Syz}_1^A(M)$. Then*

1. *$e(M) \geq \mu(M)i(M)$ and $e_1(M) \geq \mu(M)\binom{i(M)}{2}$.*
2. *M is a free A-module if and only if $i(M) = e$.*
3. *If $i(M) = e - 1$ then $G(M)$ is Cohen-Macaulay.*
4. *The following conditions are equivalent:*
   i. *$e(M) = \mu(M)i(M)$.*
   ii. *$e_1(M) = \mu(M)\binom{i(M)}{2}$.*
   iii. *$G(M)$ is Cohen-Macaulay and*
   $$h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1})$$

   If these conditions hold and M is not free, then $G(K)$ is Cohen-Macaulay and
   $$h_K(z) = \mu(M)(1 + z + \ldots + z^{e-i(M)-1}).$$

We need a few preliminaries before we prove this theorem.

Notation Let M be an A- module.If x is $A \oplus M$ superficial (or more generally it is superficial with respect to to an injective map $\theta : Q^n \to Q^n$) then set $(B, \mathfrak{n}) = (A/(x), \mathfrak{m}/(x))$ and $N = M/xM$.

Lemma 4.2.6. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$ with infinite residue field. Let M be a Cohen-Macaulay A-module of dimension 1 with a presentation $G \xrightarrow{\phi} F \longrightarrow M \longrightarrow 0$ such that all entries in ϕ are in $\mathfrak{m}^l$. If* $\operatorname{depth} G(A) \geq 1$ *and x is a $A \oplus M$-superficial element then then*

1. $(\mathfrak{m}^{i+1}M :_M x)/\mathfrak{m}^i M = 0$ *for* $i = 0, \ldots, l-1$.
2. *Furthermore if* $\mathfrak{m}^l M \subseteq xM$ *then* $\operatorname{depth} G(M) \geq 1$.

Proof. Set $b_i(M) = \lambda((\mathfrak{m}^{i+1}M :_M x)/\mathfrak{m}^i M)$. Since all the entries of ϕ are in $\mathfrak{m}^l$ we have that $\phi_{0,j-1} = \phi \otimes A/\mathfrak{m}^j = 0$ for $j = 1, \ldots, l$.

1. Note that $b_0(K) = 0$ for any A-module K. Fix i with $1 \leq i \leq l-1$. Let $p \in (\mathfrak{m}^{i+1}M :_M x)$. Let u be the pre-image of p in F. Using the commutative diagram 3.2.3 and since $\phi_{0,i} = 0$ and $\phi_{0,i+1} = 0$ we obtain that $xu \in \mathfrak{m}^{i+1}F$. Since x is A-superficial and depth $G(A) \geq 1$ we get by Theorem 1.2.4.1 that x^* is also $G(A)$-regular. Therefore $u \in \mathfrak{m}^i G$ and so $p \in \mathfrak{m}^i M$, and so $b_i(M) = 0$.

2. Since $\mathfrak{m}^l M \subseteq xM$ we get that $\mathfrak{n}^l N = 0$ and so $\sum_{i=0}^{l-1} H(N, i) = e(N) = e(M)$. Using Theorem 1.2.5.2 we get that

$$H(M, l-1) = \sum_{i=0}^{l-1} H(N, i) - b_{l-1}(M) = e(M)$$

Using Proposition 1.2.8.2 we obtain $\mathfrak{m}^l M = x\mathfrak{m}^{l-1}M$. So we have that $\mathfrak{m}^{i+1}M = x\mathfrak{m}^i M$ for all $i \geq l-1$. So we obtain that $b_i(M) = 0$ for al $i \geq l-1$. This combined with 1. yields that x^* is $G(M)$-regular. □

An interesting consequence of the lemma above is the following lemma which gives information about the Hilbert function of a maximal Cohen-Macaulay module over a hypersurface ring of dimension 1.

Lemma 4.2.7. *Let $(Q,\mathfrak{n})$ be a regular local ring of dimension two, $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$, $e \geq 2$, $A = Q/(f)$. If M a maximal Cohen-Macaulay A-module. then*

$$h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M)z^i \quad \text{and } h_i(M) \geq 0 \text{ for all } i$$

Proof. The Hilbert function of M is non-decreasing by Corollary 4.1.1. Since $\dim M = 1$ we have that the Hilbert series of M is equal to $h_M(z)/(1-z)$. This implies that all the coefficients of $h_M(z)$ are non-negative. Set $b_i(M) = \lambda((\mathfrak{m}^{i+1}M :_M x)/\mathfrak{m}^i M)$. Since

$$A^n \xrightarrow{\phi \otimes A} A^n \longrightarrow M \longrightarrow 0$$

is exact and all the entries of ϕ are in $i(M)$ we get by Lemma 4.2.6.1 that $b_i(M) = 0$ for $i = 0, \ldots, i(M) - 1$. This and Remark 4.2.1 yields that

$$h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M)$$

□

Next we get an upper bound on l such that $\mathfrak{m}^l M \subseteq xM$ holds.

Remark 4.2.8. If $\dim A = 1$ and x is A-superficial then note that since the ring B ha length $e_0(A)$ we get that $\mathfrak{m}^{e_0(A)} \subseteq (x)$. Therefore if $\dim A = 1$, M a maximal A-module and x is $A \oplus M$-superficial then $\mathfrak{m}^{e_0(A)} M \subseteq (x)M$

The next lemma deals with the case when M is a syzygy of a maximal Cohen-Macaulay A-module.

Lemma 4.2.9. *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay A-module of dimension $1$and let L be a maximal non free Cohen-Macaulay A-module and set $M = \operatorname{Syz}_1^A(L)$. If x is $(A \oplus M \oplus L)$-superficial then $\mathfrak{m}^{e_0(A)-1} M \subseteq (x)M$.*

Proof. Note that by Remark 4.2.8 we have $\mathfrak{m}^{e_0(A)} \subseteq (x)$. We also have an exact sequence : $0 \to M \to F \to L \to 0$ where F is a free A-module. Set $G = F/xF$ and $W = L/xL$. Going mod x we get $0 \to N \to G \to W \to 0$. Note that $N \subseteq \mathfrak{n}G$. Therefore $\mathfrak{n}^{e_0(A)-1} N \subseteq \mathfrak{n}^{e_0(A)} G = 0$. Therefore it follows that $\mathfrak{m}^{e_0(A)-1} M \subseteq xM$. □

Proof of Theorem 4.2.5. Clearly we may assume that the residue field of Q is infinite. Let $0 \to Q^n \xrightarrow{\phi} Q^n \to M \to 0$ be a minimal presentation of M over Q. If $\dim A \geq 2$, then choose $x_1, \ldots, x_d$ to be a maximal ϕ-superficial sequence. Set $J = (x_1, \ldots, x_{d-1})$. Since all the invariants considered in the theorem remain same modulo J it suffices to assume $\dim A \leq 1$. When $\dim A = 0$ then all the all the results follow easily by Remark 4.2.1.

Therefore assume that $\dim A = 1$. Let x be ϕ-superficial. Set $R = Q/xQ$, $N = M/xM$, $\overline{f} =$ the image of f in R and $B = A/xA = R/(\overline{f})$. Note that

a. $i(M) = i(N)$.

b. $e(M) = e(N)$ and $e_1(M) \geq e_1(N)$ (by Corollary 1.2.6.5 and 6)

c. R is a DVR with maximal ideal say $\mathfrak{q} = (y)$.

d. $v_R(f) = v_Q(f) = e$.

So 1. follows from Remark 4.2.1 and (c) above.

2. If $i(M) = e$ then note that since $i(M) = i(N)$ we get by Remark 4.2.1.4 that N is a free B module. So M is a free A-module. Clearly if M is free then $i(M) = e$.

3. Let $M = F \oplus L$ where F is a free A-module and L has no free summands. Note that $i(L) = i(M)$. Since $G(A)$ is Cohen-Macaulay it suffices to show $G(L)$ is Cohen-Macaulay. Since $L = \operatorname{Syz}_1^A\left(\operatorname{Syz}_1^A(L)\right)$ it follows from Lemma 4.2.9 that if x is a $A \oplus L \oplus N$-superficial element, then $\mathfrak{m}^{e_0(A)-1}L \subseteq xL$. Since $A^n \to A^n \to L \to 0$ is exact and $\operatorname{depth} G(A) = 1$ we get by Lemma 4.2.6.2 that $G(L)$ is Cohen-Macaulay.

4. By Proposition 4.2.7 we get that

$$h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M) z^i \quad \text{and } h_i(M) \geq 0 \text{ for all } i.$$

So it follows that (i) and (ii) are equivalent. The assertion (iii) $\Longrightarrow$ (ii) is clear.

(i) $\Longrightarrow$ (iii).. Note that $\mu(N) = \mu(M)$ and

$$h_N(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(N) z^i.$$

Also all the coefficients are non-negative. Therefore it follows that $e(M) = e(N) = i(M)\mu(M)$ holds if and only if

$$h_M(z) = h_N(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1})$$

Since $h_M(z) = h_N(z)$ we also get that $G(A)$ is Cohen-Macaulay (see Corollary1.2.7.1).

Note that since M is not free $i(M) < e$. We first assert that M has no free summands. Otherwise $M = F \oplus W$ where F is free. This yields $h_M(z) = h_F(z) + h_W(z)$. Since all the coefficients of $h_F(z)$ and $h_W(z)$ are non-negative we get that coefficient of z^{e-1} is non-zero. This contradicts (c). Therefore if (ϕ, ψ) is a matrix-factorization of M then we have a minimal presentation of K

$$0 \longrightarrow Q^n \xrightarrow{\psi} Q^n \longrightarrow K \longrightarrow 0.$$

Let x be both ϕ and ψ-superficial. Set $N = M/xM$. Since (a) holds then note that $N \cong \left(R/(y^{i(M)})\right)^{\mu(M)}$. Then

$$\operatorname{Syz}_1^B(N) \cong \left(R/(y^{e-i(M)})\right)^{\mu(M)}.$$

Since $\operatorname{Syz}_1^B(N) \cong K/xK$ we get that $i(K) = e - \mu(M)$ and $e(K) = e(K/xK) = \mu(K/xK)i(K) = \mu(K)i(K)$. Therefore by the equivalence of (i) and (iii) we get the required result. □

Remark 4.2.10. Theorem 4.2.5 can be applied to the case of Ulrich modules, that is, maximal Cohen-Macaulay modules that satisfy $e(M) = \mu(M)$. It is known [7] that Ulrich A-modules exist when A is a complete hypersurface ring. Using the previous theorem we get that if M is Ulrich, then $i(M) = 1$ and so $G(\operatorname{Syz}_1^A(M))$ is Cohen-Macaulay. Furthermore

$$h_{\operatorname{Syz}_1^A(M)} = \mu(M)(1 + z + \ldots + z^{e-2}).$$

Therefore every complete hypersurface ring has a maximal Cohen-Macaulay A- module such that $G(M)$ and $G(\operatorname{Syz}_1^A(M))$ is Cohen-Macaulay.

An easy way to test the hypothesis of the previous theorem in the equicharacteristic case is the following:

Proposition 4.2.11. *Let* $Q = k[[y_1, \ldots, y_{d+1}]]$*. Let* M *be a* Q*-module with a minimal presentation* $0 \to Q^n \xrightarrow{\phi} Q^n \to M \to 0$*. Set*

$$\phi = \sum_{i \geq i(M)} \phi_i \quad \textit{where } \phi_i \textit{ are forms of degree } i$$

Then $\det \phi_{i(M)} \neq 0$ *if and only if* $h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1})$*.*

Proof. Note that $\det \phi_{i(M)} \neq 0$ if and only if $v_Q(\det \phi) = i(M)\mu(M)$.

Let $f = \det \phi$. Note that M is a maximal $A = Q/(f)$-module. Let $x_1, \ldots, x_d$ be a maximal ϕ-superficial sequence. Set $J = (x_1, \ldots, x_d)$, $R = Q/J$, $\overline{f}$ = image of f in R, $N = M/JM$ and $\overline{\phi} = \phi \otimes Q/J$. Note that

$$i(N) = i(M) \quad \text{and} \quad v_R(\det \overline{\phi}) = v_Q(\det \phi).$$

If $\det \phi_{i(M)} \neq 0$ then $v_Q(\det \phi) = i(M)\mu(M)$. So $v_R(\det \overline{\phi}) = i(M)\mu(M)$. Therefore by Remark 4.2.1.7 we get that $e(N) = \mu(N)i(N)$. This yields that $e(M) = i(M)\mu(M)$ and so by Theorem 4.2.5.4 we get the required assertion. Conversely if $h_M(z) = \mu(M)(1 + z + \ldots + z^{i(M)-1})$ then by Theorem 4.2.5 $G(M)$ is Cohen-Macaulay. So $h_N(z) = h_M(z)$. So we get $e(N) = \mu(N)i(N)$. Therefore by Remark 4.2.1.7 we get $v_R(\det \overline{\phi}) = i(N)\mu(N) = i(M)\mu(M)$. So $v_Q(\det \phi) = i(M)\mu(M)$. □

We now give an application of the proposition proved above.

Example 4.2.12. Set $Q = k[[y_1, \ldots, y_{d+1}]]$ and $\mathfrak{n}$ to be the maximal ideal of Q. Let a, b, c, d be in $\mathfrak{n}$ be such that $f = ad - bc \neq 0$. Set $A = Q/(f)$. Set

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Define M and K by the exact sequences

$$0 \to Q^2 \xrightarrow{\phi} Q^2 \to M \to 0 \quad \text{and} \quad 0 \to Q^2 \xrightarrow{\psi} Q^2 \to K \to 0$$

Note that M and K are A-modules and $K = \operatorname{Syz}_1^A(M)$.

1. If $f \in \mathfrak{m}^2$ then M is an Ulrich A-module.

2. If $f \in \mathfrak{m}^3 \backslash \mathfrak{m}^4$ then M or K has minimal multiplicity. However both are not Ulrich.

To prove 1. note that $i(M) = 1$ and $\det(\phi_1) \neq 0$. So by Proposition 4.2.11 we get that $h_M(z) = \mu(M)$. So M is Ulrich.

2. We first show that M and K are not Ulrich. Note that since $f \in \mathfrak{n}^3 \setminus \mathfrak{n}^4$ we still have $i(M) = 1$. Since $f \in \mathfrak{n}^3$ we also get $\det(\phi_1) = 0$. So by Proposition 4.2.11 we get that $h_M(z) \neq \mu(M)$. So M is not Ulrich. Similar argument yields that K is not Ulrich. Note that $h_A(z) = 1 + z + z^2$. To prove that M or K has minimal multiplicity note that by Theorem 3.1.6 and Remark 3.1.7 we get that $2 = 2\chi_1(A) > \chi_1(M) + \chi_1(K)$. So we get $\chi_1(M) = 0$ or $\chi_1(K) = 0$. So by Theorem 2.4 we get that M or K has minimal multiplicity.

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