Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (Dec 1, 2016)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs. Do the same on the attached answer-sheet.*
- *All questions carry 2 marks. There will be no partial credit. Simplify all your answers. In particular, the answer should not be in the form of a sum or product.*
- *The answer to each question is a number or a set or a yes/no statement. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*

1. Let

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}.$$

(i) What is the minimal polynomial of A?

(ii) Is A diagonalizable over $\mathbb{R}$?

2. For each real number α, we define the bilinear form $F_\alpha : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by

$$F_\alpha\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big) = 2x_1x_2 + (\alpha+5)x_1y_2 + x_1y_3 + (\alpha+5)x_2y_1$$

$$-(2\alpha+4)x_2y_2 + 2x_2y_3 + x_3y_1 + 2x_3y_2 + 2x_3y_3.$$

Find the set of $\alpha \in \mathbb{R}$ such that F_α is positive definite.

3. What is the number of non-conjugate 6×6 complex matrices having the characteristic polynomial $(x-5)^6 = 0$?

4. Let S be a subspace of the vector space of all 11×11 real matrices such that (i) every matrix in S is symmetric and (ii) S is closed under matrix multiplication. What is the maximum possible dimension of S?

5. Let A be a 55×55 diagonal matrix with characteristic polynomial

$$(x-c_1)(x-c_2)^2(x-c_3)^3 \ldots (x-c_{10})^{10},$$

where $c_1, \ldots, c_{10}$ are all distinct. Let V be the vector space of all 55×55 matrices B such that $AB = BA$. What is the dimension of V?

6. Let A be the complex square matrix of size 2016 whose diagonal entries are all -2016 and off-diagonal entries are all 1. What are the eigenvalues of A and their geometric multiplicities?

7. Let V be a subspace of $\mathbb{R}^{13}$ of dimension 6, and W be a subspace of $\mathbb{R}^{31}$ of dimension 29. What is the dimension of the space of all linear maps from $\mathbb{R}^{13}$ to $\mathbb{R}^{31}$ whose kernel contains V and whose image is contained in W?

8. Let V (resp. W) be the real vector space of all polynomials in two commuting (resp. noncommuting) variables with real coefficients and of degree strictly less than 100. What are the dimensions of V and W?

9. Find the number of connected components of the set

$$\left\{x \in \mathbb{R} : x^3\left(x^2+5x-\frac{65}{3}\right) > 70x^2-300x-297\right\}$$

under the usual topology on $\mathbb{R}$.

10. Let $P_n(x)$ be the Taylor polynomial at $x=0$ for the exponential function e^x. Compute the least n such that $|e-P_n(1)|<10^{-5}$.

11. Find the set of values of the real number a for which $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\frac{1}{n}\right)^a$ converges.

12. Let $p(x)$ be a polynomial of degree 7 with real coefficients such that $p(\pi)=\sqrt{3}$ and

$$\int_{-\pi}^{\pi} x^k p(x)dx = 0 \quad \text{for } 0 \le k \le 6.$$

What are the values of $p(0)$ and $p(-\pi)$?

13. Let $f:[0,1]\to\mathbb{R}$ be defined by

$$f(x) = \frac{57^{(x^2+1)}+3}{e^{x^2}+1113337x^2+1113339x^{3/2}+1113341x+1}.$$

Find the value of $\lim_{n\to\infty}\left(\int_0^1 f(x)^n\,dx\right)^{\frac{1}{n}}$.

14. Consider $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^{1/3}$. Let $g(x)=\sum_{n=0}^{\infty}a_n(x-3/2)^n$, where $a_n=\frac{f^{(n)}(3/2)}{n!}$ for $n\ge 0$. What is the largest open set contained in $\{x \mid f(x)=g(x)\}$?

15. An urn contains 11 balls numbered $1, 2, \ldots, 11$. We remove 4 balls at random without replacement and add their numbers. Compute the mean of the total.

16. Let X, Y and Z be independent, identically distributed random variables, each having the Bernoulli distribution with parameter p, $0 < p < 1$. Put $T = X + Y + Z$ and $S = XYZ$.

    Find $P(T = 2|S = 0)$.

17. Let $\{X_n; n \geq 1\}$ be a sequence of identically and independently distributed random variables with uniform distribution on $(0, 1)$. Suppose $Y_n = (X_1 X_2 \ldots X_n)^{1/n}$, (i.e., Y_n is the geometric mean of $X_1, X_2, \ldots, X_n$). Find the number c such that Y_n converges to c with probability 1.

18. Let $\{X_n; n \geq 1\}$ be a sequence of identically and independently distributed random variables having Poisson distribution with mean 1. Let $\bar{X}_n = \frac{X_1 + \ldots + X_n}{n}$. Find the limit of $P(\bar{X}_n \geq 1)$ as n goes to ∞.

19. Suppose that the joint probability function of two random variables X and Y is

    $$f(x, y) = \frac{xy^{x-1}}{3}, \quad x = 1, 2, 3 \text{ and } 0 < y < 1.$$

    Find the variance of X.

20. Suppose that the random variables X and Y are independent and identically distributed and that the moment generating function (mgf) of each is

    $$\psi(t) = e^{t^2 + 3t}, \text{ for } -\infty < t < \infty.$$

    Find the mgf of $Z = 2X - 3Y + 4$ at $t = 1$.