Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (May 9, 2017)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Name: Choice: | Math | Stat |

- *Write your name in the blank space at the top of this question-paper, and also tick 'Math' or 'Stat' to indicate your first choice of one of the 2 PhD programs. Do the same on the attached answer-sheet.*

- *All questions carry 2 marks. Some of the questions have two parts of 1 mark each. There will be no partial credit.*

- *The answer to each question is a number (or a tuple of numbers), a set, or a function. Record only your answers on the answer-sheet as indicated. You are being given a separate work-sheet for doing rough work (this will not be collected).*

- *Use of calculators is not allowed. Please keep aside your notes and mobile phones, whose use during the examination is prohibited. Any candidate found to be adopting any unfair means will be disqualified.*

1. Let d_1, d_2, d_3, d_4, d_5 denote the dimensions of the subspaces of all real 29×29 matrices which are diagonal, upper triangular, trace zero, symmetric, and skew symmetric, respectively. Write down the 5-tuple $(d_1, d_2, d_3, d_4, d_5)$.

2. Find all real α for which the following quadratic form

$$Q(x_1, x_2, x_3) := x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3 + 2\alpha x_2x_3$$

is positive definite.

3. If $I \neq T \in M_4(\mathbb{C})$ has $(X-1)^4$ as its characteristic polynomial then what is the largest possible dimension of the centraliser of T in $M_4(\mathbb{C})$ (= the subspace of all matrices that commute with T)?

4. Let V denote the (complex) vector space of complex polynomials of degree at most 9 and consider the linear operator $T : V \to V$ defined by

$$T(a_0 + a_1X + \cdots + a_9X^9) =$$
$$a_0 + (a_2X + a_1X^2) + (a_4X^3 + a_5X^4 + a_3X^5) + (a_7X^6 + a_8X^7 + a_9X^8 + a_6X^9).$$

   (a) What is the trace of T^4?

   (b) What is the trace of T^2?

5. What is the signature of the symmetric bilinear form defined by the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \end{pmatrix}?$$

6. Let A be the 3×3 matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Determine all real numbers a for which the limit $\lim_{n\to\infty} a^n A^n$ exists and is non-zero. [For a sequence of 3×3 matrices $\{B_n\}$ and a 3×3 matrix B, $\lim_{n\to\infty} B_n = B$ means that, for all vectors $x \in \mathbb{R}^3$, we have $\lim_{n\to\infty} B_n x = Bx$ in $\mathbb{R}^3$.]

7. Let V denote the vector space consisting of all polynomials over $\mathbb{C}$ of degree at most 2017. Consider the linear operator $T : V \to V$ given by $T(f) = f'$, that is, T maps a polynomial f to its derivative f'. Write down all eigenvalues of T along with their algebraic and geometric multiplicities.

8. Let $A = (a_{ij})$ be the square matrix of size 2018 defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i+1 = j, \\ 1/3 & \text{if } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Let B be the leading principal minor of A of order 1009 (=the submatrix of A formed by the first 1009 rows and columns).

(a) What is the determinant of A?

(b) What is the rank of B?

9. For $\alpha \in (0,1)$, let the sequence $\{x_n\}$ be such that $x_0 = 0$, $x_1 = 1$ and $x_{n+1} = \alpha x_n + (1-\alpha)x_{n-1}$, $n \geq 1$. Find $\lim_{n\mapsto\infty} x_n$.

10. For what values of p is the series

$$\sum_{k=1}^{\infty} (-1)^k \, k^p \, \log k$$

(a) absolutely convergent?

(b) conditionally convergent?

11. Determine the radius of convergence of the following two power series.

(a) $\sum_{n=1}^{\infty} \frac{x^{6n+2}}{\left(1+\frac{1}{n}\right)^{n^2}}$.

(b) $\sum_{n=0}^{\infty} a_n (x-2017)^n$ with $a_n = \begin{cases} 1/2 & \text{if } n \text{ is even}, \\ 1/3 & \text{if } n \text{ is odd}. \end{cases}$

12. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given below:

$$f(t) = \begin{cases} |t/2| & t < -2, \\ |t+3/2| + 1/2 & -2 \leq t < -1, \\ |t^3| & -1 \leq t < 1, \\ |t-3/2|^2 + 3/4 & 1 \leq t < 2, \\ |t/2| & t \geq 2. \end{cases}$$

What is the number of connected components of the set $\{t \in \mathbb{R} : f \text{ is differentiable at } t\}$?

13. Determine the set of all points where the Taylor series of the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$

around the point $x = e$ converges to $f(x)$.

14. Consider the sequence of real-valued functions $\{f_n\}$ defined by

$$f_n(x) = \frac{1}{1+nx^2}.$$

Assuming the fact that $\{f_n\}$ converges uniformly to a function f find out all real numbers x for which

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

15. A bowl contains 5 strands of spaghetti. We select two ends at random and join them together. We repeatedly do this until there are no ends left. What is the expected number of loops in the bowl? Write the answer as a fraction.

16. The zero truncated random variable X_T has probability mass function,

$$P(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, \ldots$$

What is the mean of X_T when $X \sim \text{Poisson}(\lambda)$.

17. An electronic device has lifetime denoted by T. The device has value $V = 5$ if it fails before time $t = 3$; otherwise, it has value $V = 2T$. The probability density function of T is $f(t) = \frac{1}{1.5}e^{-t/(1.5)}, t > 0$. Determine $P(V \leq v)$ for

    (a) $0 \leq v < 6$.

    (b) $v \geq 8$.

18. If the random variable X has probability density function

$$f(x) = \begin{cases} \frac{x-1}{2} & \text{if } 1 < x < 3 \\ 0 & \text{otherwise,} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a uniform distribution.

19. Let X have density $f(x) = 3x^2$, $0 \leq x \leq 1$, and let $Y_i = e^{X_i^2}$. What is the constant m to which $(Y_1 Y_2 \ldots Y_n)^{1/n}$ converges with probability 1 as $n \to \infty$?

20. Let $\{Y_n : n \geq 1\}$ be a sequence of independent standard normal random variables and let

$$X_n = \frac{Y_{2n}}{Y_{2n-1}}, \quad n = 1, 2, \ldots$$

If $\overline{X_n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$, find

$$\lim_{n \to \infty} P\{\sqrt{n}\,(\overline{X_n} - 1) \leq 0\}.$$