Lecture 4

2.4 Operations on Matrices

The set of all matrices of a fixed size say $m \times n$, and with entries in $\mathbb{R}$ is denoted by $M_{m,n}(\mathbb{R})$ or by simply $M_{m,n}$ when the nature of the entries is understood. Certain algebraic operations of the reals induce similar operations on this space also, in an *entry-wise* fashion. Given two matrices $A = ((a_{ij})), B = ((b_{ij})) \in M_{m,n}$, we define $A + B \in M_{m,n}$ to be the matrix whose $(i,j)^{th}$ entry is $a_{ij} + b_{ij}$, i.e.,

$$A + B = ((a_{ij} + b_{ij})).$$

Given a scalar $\alpha \in \mathbb{R}$, we can define the matrix $\alpha A \in M_{m,n}$ to be the matrix whose $(i,j)^{th}$ entry is αa_{ij}. It is easily checked that the matrix $\mathbf{O}$ with all its entries zero is the zero element for the sum operation, the matrix $((-a_{ij}))$ is the negative of A etc. Also verify that scalar multiplication distributes over the addition. Earlier we have identified $\mathbb{R}^n$ with the set of all column vectors and hence with $M_{1,n}$. We now observe that this identification preserves the algebraic structure as well. The same is true of the identification of $\mathbb{R}^m$ with $M_{m,1}$.

The 'transpose' operation that we introduced earlier just for the convenience of printing acquires a wider meaning now. It identifies the space $M_{1,n}$ with $M_{n,1}$. We observe that this operation can be easily extended for all matrices as follows: A^T is the $n \times m$ matrix whose $(i,j)^{th}$ entry is a_{ji}. This way we get the *transpose operation:* $T : M_{m,n} \longrightarrow M_{n,m}$, by the formula $A \mapsto A^T$. (The functional notation would have been $T(A)$, but we would like to stick to classical notation. Other notations in practice are A^t and ${}^tA$ etc..) It is easily verified that $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$. This property of the transpose can be referred to as the linearity of the transpose. When we have functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad g : \mathbb{R}^m \longrightarrow \mathbb{R}^l$ we can talk about the composite $g \circ f$. We have also seen that if f and g are linear then so is $g \circ f$. A natural question to ask is: what is the relation between the matrix representations $\mathcal{M}_f =: A, \mathcal{M}_g =: B$ and $\mathcal{M}_{g\circ f}$? To determine this we have to compute the value of $(g\circ f)(\mathbf{e}_i)$.

$$(g \circ f)(\mathbf{e}_j) = g(\sum_i a_{ij}\mathbf{e}_i) = \sum_i a_{ij} g(\mathbf{e}_i) = \sum_i a_{ij}(\sum_k b_{ki}\mathbf{e}_k) = \sum_k (\sum_i b_{ki}a_{ij})\mathbf{e}_k.$$

Thus if $\mathcal{M}_{g\circ f} = C = ((c_{kj}))$ then we have

$$c_{kj} = \sum_{i=1}^{m} b_{ki}a_{ij} \tag{9}$$

Therefore we shall define the composite of B and A to be the $l \times n$ matrix C :

$$C = BA$$

where $C = ((c_{kj}))$ with c_{kj} given by (9). We shall merely list some basic properties of this multiplication. You are already familiar with most of them at least in the case when their size is small say $m, n, l \leq 3$. Other properties can be verified in a routine fashion:
1. *Associativity:* If A, B, C are matrices such that AB and BC are defined then $(AB)C$ and $A(BC)$ are also defined and are equal, $A(BC) = (AB)C$.
2. *Right and Left Distributivity:* If $A(B + C) = AB + AC$ provided at least one side is (and hence both sides are) defined. Similarly, we have $(B + C)A = BA + CA$, etc..

Let I_n denote the $n \times n$ matrix whose $(ij)^{th}$ entry is equal to δ_{ij}. (Recall $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. These are called Kronecker symbols.) The matrix I_n s called the *identity matrix.*

3. *Mutliplicative Identity* For any $A \in M_{m,n}, B \in M_{n,k}$ we have $AI_n = A$ and $I_nB = B$.

4.$(AB)^T = B^T A^T$.

5. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m,\ g : \mathbb{R}^m \longrightarrow \mathbb{R}^l$ be linear maps. Then we have, $\mathcal{M}_{g\circ f} = \mathcal{M}_g \mathcal{M}_f$.

The last mentioned property has been verified above. Indeed, we arrived at the definition of matrix multiplication through this property only.

Remark 1 Consider the space $M_{2,2}$. By writing the entries of successive rows one after another in a single row, we see that this is in one-to-one correspondence with the totality of ordered 4-tuples of real numbers. Moreover this correspondence preserves the two algebraic operations of taking sum and scalar product. Thus we can actually think of the space of 2×2 matrices with real entries as $\mathbb{R}^4$. Exactly in a similar fashion, $M_{m,n}$ can be thought of as $\mathbb{R}^{m\times n}$. Calculus on this space will involve 'mn variables'. Soon, we shall see the true meaning of the phrase *'number of variables',* the easiest way to understand the meaning of 'dimension.'

2.5 Invertible Transformations and Matrices

Definition 2 Any function $f : X \longrightarrow Y$ is said to be invertible, if there exists $g : Y \longrightarrow X$ such that $g \circ f = Id_X$ and $f \circ g = Id_Y$. Here for any space Z, Id_Z denotes the identity map of the space Z. The inverse of a function if it exists is unique and is denoted by f^{-1}.

We would like to define the invertible matrices also in this fashion. However, caution should be taken to assume that such a matrix is first of all a square matrix. Thus let A be a $n \times n$ matrix. Then A is called *invertible* if there exists another $n \times n$ matrix B such that $AB = BA = I_n$.

We call B an inverse of A and vice versa.

Remark 2

(i) Check that an inverse of a matrix is unique. [By taking a CAB: if C is another matrix such that $CA = AC = I_n$ then $C = CI_n = C(AB) = (CA)B = I_nB = B$.] Therefore we may call it *the inverse of* A and denote it by A^{-1}.

(ii) Check that if A_1, A_2 are invertible then so is A_1A_2. What is its inverse?

(iv) Clearly I_n, and $diag\,(a_1, a_2, \ldots, a_n)$ with $a_i \neq 0$ are invertible. In general, we do not yet know how to determine whether a given matrix is invertible. The study of invertible matrices is very important to us and we are going to spend some time on this soon. We shall first take up this below, for the case of matrices which are slight variations of the identity matrix, called elementary matrices.

(v) If $f_A, f_B : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are the linear transformations defined by $A, B = A^{-1}$ then it follows that $f_A \circ f_B = f_{AB} = Id$. Likewise $f_B \circ f_A = Id$. Even the converse holds. viz., if a linear transformation f is invertible, then its associated matrix $\mathcal{M}_f$ is invertible with its inverse corresponding to the transformation f^{-1}.

(vi) It is easily checked that an invertible map is one-one and onto.

(vii) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an invertible linear map, check that f^{-1} is linear.

Lecture 5

2.6 Elementary and Permutation Matrices

Consider a square matrix E_{ij} whose $(i,j)^{th}$ entry is 1 and all other entries are 0. We deliberately do not mention the size of this matrix, because we would like to use the same notation in all possible sizes, the actual size being understood by the context. If we multiply a matrix A by E_{ij} on the left, then what we get is a matrix whose i^{th} row is equal to the j^{th} row of A and all other rows are zero. In particular $E_{ij}E_{ij} = 0,\ (i \neq j)$. It follows that for any number α and $i \neq j$

$$(I + \alpha E_{ij})(I - \alpha E_{ij}) = I + \alpha E_{ij} - \alpha E_{ij} - \alpha^2 E_{ij}E_{ij} = I. \tag{10}$$

What happens if we take $i = j$ in the above discussion? Clearly (10) is no longer valid. On the other hand consider

$$(I + \alpha E_{ii})(I + \beta E_{ii}) = I + (\alpha + \beta + \alpha\beta)E_{ii} \tag{11}$$

This equation tells us that if we want the RHS to be equal to I then we must have $\alpha+\beta+\alpha\beta = 0$. Given any α we can solve for β iff $1+\alpha \neq 0$. Thus we have proved that $I+\alpha E_{ii}$ is invertible if $\alpha \neq -1$.

[Of course, this can be verified directly also: the matrix $I + \alpha E_{ii}$ is nothing but the diagonal matrix with all the diagonal entries equal to 1 except the ii^{th} one which is equal to $1 + \alpha$. Therefore, it is invertible iff $1 + \alpha \neq 0$.]

We can play this game a little further and consider $I+E_{ij}+E_{ji}-E_{ii}-E_{jj}$. Using similar method to the above check that this is also invertible. Now if we actually write down this matrix, this is nothing but the identity matrix after interchanging the i^{th} and j^{th} rows. These are called *transposition matrices.* The linear maps corresponding to them merely interchange the i^{th} and j^{th} coordinates ($\mathbf{e}_i$ and $\mathbf{e}_j$) and do not disturb anything else. We shall denote them simply by T_{ij}. To sum up we have:

Theorem 1 *The elementary matrices $I + \alpha E_{ij}$ $(i \neq j)$, $I + \alpha E_{ii}$, $(\alpha \neq -1)$ and $T_{ij} = I+E_{ij}+E_{ji}-E_{ii}-E_{jj}$ are all invertible with their respective inverses $I-\alpha E_{ij}, I-(\alpha/1+\alpha)E_{ii}$ and T_{ij}.*

These are called *elementary matrices.* The importance of them stems out from the fact that multiplication on the left by them with a matrix A have simple interpretation in terms of operations on the rows of A.

Another important class of matrices are the so called *permutation matrices.* They can be defined to be those square matrices which have all the entries in any given row (and column) equal to zero except one of them and this non zero entry is equal to 1. Thus for each $1 \leq i \leq n$ consider the i^{th} row. Look at the place where the entry 1 occurs. Call it $\sigma(i)$. This way we get a map $\sigma : N \longrightarrow N$ which is clearly a one-to-one mapping. Conversely, given a permutation $\sigma : N \longrightarrow N$, we define a matrix $P_\sigma = ((p_{ij}))$ by the formula

$$p_{ij} = \begin{cases} 0 & \text{if} \quad j \neq \sigma(i) \\ 1 & \text{if} \quad j = \sigma(i) \end{cases}$$

Can you now see that a permutation matrix is obtained by merely shuffling the rows of the identity matrix, or, by shuffling the columns? Check that they have another interesting property viz., $AA^t = A^tA = I_n$. In particular, they are invertible.
[**Proof:** If R_i denotes the i^{th} row of A and C_j denotes the j^{th} column of A^t, by definition of A^t, it follows that $C_j = R_j^t$. Since clearly, $R_iR_j^t = \delta_{ij}$ (the Kronecker symbols), it follows that $R_iC_j = \delta_{ij}$ which is the $(i,j)^{th}$ entry of I_n.]

2.7 Gauss Elimination

We discuss a widely used method called the Gauss elimination method to solve a system of m linear equations in n unknowns as given by (4). You are already familiar with how to solve such a system. Essentially, you carry out the following three types of operations on these equations which **do not alter the solutions** and convert the given system into simpler system by eliminating variables one by one.

(1) Interchanging two equations.

(2) Multiplying all the terms of an equation by a nonzero scalar.

(3) Adding to one equation a multiple of another equation.

Our aim is to put this method on a proper foundation and then reap a rich harvest that this method provides.

First of all we observe that there is no need to keep writing the name of the variables in the computations that follow. Only need is to keep track of which coefficient came from which variable. This we achieve by adopting the matrix equivalent formulation of the above system:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \tag{12}$$

The matrix $A = ((a_{ij}))$ is called the **coefficient matrix.** By a **solution** of (12) we mean any choice of $x_1, x_2, \dots, x_n$ which satisfies all the equations in it. If each $b_i = 0$ we say that the system is **homogeneous.** Otherwise it is called an **inhomogeneous system.** The matrix

$$\left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right] \tag{13}$$

is called the *augmented matrix.* Now the above three operations on the equations correspond to certain operations on the row of the augmented matrix. These are called elementary row operations.

We illustrate this method by three examples.

Example 5 A system with a unique solution: Consider the system :

$$\begin{aligned} 2x - 5y + 4z &= -3 \\ x - 2y + z &= 5 \\ x - 4y + 6z &= 10. \end{aligned}$$

The matrix form of this system is:

$$\left[\begin{array}{ccc|c} 2 & -5 & 4 & -3 \\ 1 & -2 & 1 & 5 \\ 1 & -4 & 6 & 10 \end{array}\right]. \tag{14}$$

The three basic operations mentioned above will be performed on the rows of the augmented matrix.

The first step is to make the $(1,1)^{th}$ entry equal to 1. We can do this by multiplying all the entries in the first row by $1/2$. Alternatively, we can interchange the first and the second rows. We get

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 2 & -5 & 4 & -3 \\ 1 & -4 & 6 & 10 \end{array}\right]. \tag{15}$$

The next step is to make all the entries in the first column zero except the (1,1)-entry. To do this, first we add -2 times the first row to the second row and add to the second row. Then we subtract the first row from the third row. This yields the matrix :

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & -1 & 2 & -13 \\ 0 & -2 & 5 & 5 \end{array}\right]. \tag{16}$$

This process from (15) to 16) is also called 'sweeping' a column. Here we are sweeping the first column below the first row. So far, in effect we have succeeded in eliminating the first variable.

Now we repeat the process for the smaller matrix.

$$\left[\begin{array}{cc|c} -1 & 2 & -13 \\ -2 & 5 & 5 \end{array}\right].$$

Multiply the top row by -1, to get

$$\left[\begin{array}{cc|c} 1 & -2 & 13 \\ -2 & 5 & 5 \end{array}\right]. \tag{17}$$

Now add twice the first row to the second to get

$$\left[\begin{array}{cc|c} 1 & -2 & 13 \\ 0 & 1 & 31 \end{array}\right]. \tag{18}$$

Put back the rows and columns that you have cut out earlier to get

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 5 \\ 0 & 1 & -2 & 13 \\ 0 & 0 & 1 & 31 \end{array}\right]. \tag{19}$$

The matrix in (19) represents the linear system

$$\begin{aligned} x - 2y + z &= 5 \\ y - 2z &= 13 \\ z &= 31. \end{aligned}$$

These can be solved successively by **backward substitution.**

$$z = 31 \quad y = 13 + 2z = 75 \quad x = 5 + 2y - z = 124.$$

Alternatively, we can continue the process called the **Gauss-Jordan Process**. Here at this stage, we first make sure that all diagonal entries are indeed 1 or 0. Then we sweep the columns for which the diagonal entry are 1 above these entries. This is carried out in the decreasing order of the column numbers.

So in (19), add twice third row to the second and then subtract the third row from the first. Next add twice the second row to the first. This gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 124 \\ 0 & 1 & 0 & 75 \\ 0 & 0 & 1 & 31 \end{array}\right] \tag{20}$$

The augmented matrix in (20) gives the desired solution $x = 124, y = 75$ and $z = 31$.

Notation 2.1 To save space further, we introduce notation for the elementary row operations. Let R_i denote the i^{th} row of a given matrix.

Operation	Notation
Multiply R_i by a scalar c	cR_i
Multiply R_j by a scalar c and add to R_i	$R_i + cR_j$
Interchange R_i and R_j	$R_i \leftrightarrow R_j$.

Example 6 A system with infinitely many solutions: Consider the following system of 3 equations in five unknowns:

$$\begin{aligned} x - 2y + z - u + v &= 5 \\ 2x - 5y + 4z + u - v &= -3 \\ x - 4y + 6z + 2u - v &= 10 \end{aligned} \tag{21}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 5 \\ 2 & -5 & 4 & 1 & -1 & -3 \\ 1 & -4 & 6 & 2 & -1 & 10 \end{array}\right] \tag{22}$$

We shall use the notation introduced above for the row operations.

$$\begin{array}{c} R_2 - 2R_1 \\ R_3 - R_1 \\ \longrightarrow \end{array} \quad \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 5 \\ 0 & -1 & 2 & 3 & -3 & -13 \\ 0 & -2 & 5 & 3 & -2 & 5 \end{array}\right]$$

$$\begin{array}{c} R_3 - 2R_2 \\ \longrightarrow \end{array} \quad \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 5 \\ 0 & -1 & 2 & 3 & -3 & -13 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right]$$

$$\begin{array}{c} -R_2 \\ \longrightarrow \end{array} \quad \left[\begin{array}{ccccc|c} 1 & -2 & 1 & -1 & 1 & 5 \\ 0 & 1 & -2 & -3 & 3 & 13 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right]$$

$$\begin{array}{c} R_1 + 2R_2 \\ \longrightarrow \end{array} \quad \left[\begin{array}{ccccc|c} 1 & 0 & -3 & -7 & 7 & 31 \\ 0 & 1 & -2 & -3 & 3 & 13 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right]$$

$$\begin{array}{c} R_2 + 2R_3 \\ R_1 + 3R_3 \\ \longrightarrow \end{array} \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -16 & 19 & 124 \\ 0 & 1 & 0 & -9 & 11 & 75 \\ 0 & 0 & 1 & -3 & 4 & 31 \end{array}\right]$$

The system of linear equations corresponding to the last augmented matrix is :

$$\begin{aligned} x &= 124 + 16u - 19v \\ y &= 75 + 9u - 11v \\ z &= 31 + 3u - 4v. \end{aligned}$$

Assign arbitrary value t_1 to u and t_2 to v to obtain values of x, y, z. We say that u and v are **independent variables** and x, y, z are **dependent variables**. Solve for x, y, z :

$$\begin{aligned} (x, y, z, u, v)^T &= (124 + 16t_1 - 19t_2, 75 + 9t_1 - 11t_2, 31 + 3t_1 - 4t_2, t_1, t_2)^T \\ &= (124, 75, 31, 0, 0)^T + t_1(16, 9, 3, 1, 0)^T + t_2(-19, -11, -4, 0, 1)^T. \end{aligned}$$

The above equation gives the general solution to the system. The vector $(124, 75, 31, 0, 0)$ is a particular solution of the inhomogeneous system. The vectors $\mathbf{v}_1 = (16, 9, 3, 1, 0)$ and $\mathbf{v}_2 = (-19, -11, -4, 0, 1)$ are solutions of the corresponding homogeneous system. These two solutions are linearly independent and every other solution of the homogeneous system is a linear combination of these two solutions. Finally every solution of the inhomogeneous system is the sum of a solution of the homogeneous system and one particular solution of the inhomogeneous system. This illustrates a general principle which we now state and prove .

Theorem 2 *Suppose $A\mathbf{x} = \mathbf{b}$ is a system of linear equations where $A = ((a_{ij}))$ is an $m \times n$ matrix and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t, \mathbf{b} = (b_1, b_2, \ldots, b_m)^t$. Suppose $\mathbf{c} = (c_1, c_2, \ldots, c_n)^t$ is a solution of $A\mathbf{x} = \mathbf{b}$ and S is the set of all solutions to the associated homogeneous system $A\mathbf{x} = 0$. Then the set of all solutions to $A\mathbf{x} = \mathbf{b}$ is*

$$\mathbf{c} + S := \{\mathbf{c} + \mathbf{v} | \mathbf{v} \in S\}.$$

Proof: Let $\mathbf{r} \in \mathbb{R}^n$ be a solution of $A\mathbf{x} = \mathbf{b}$. Then $A(\mathbf{r} - \mathbf{c}) = A\mathbf{r} - A\mathbf{c} = \mathbf{b} - \mathbf{b} = 0$. Hence $\mathbf{r} - \mathbf{c} \in S$. Thus $\mathbf{r} \in \mathbf{c} + S$. Conversely let $\mathbf{v} \in S$. Then $A(\mathbf{c} + \mathbf{v}) = A\mathbf{c} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Hence $\mathbf{c} + \mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{b}$. ♠

Example 7 A system with no solution: Consider the system

$$\begin{aligned} x - 5y + 4z &= 3 \\ x - 5y + 3z &= 6 \\ 2x - 10y + 13z &= 5 \end{aligned} \tag{23}$$

The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & 3 \\ 1 & -5 & 3 & 6 \\ 2 & -10 & 13 & 5 \end{array}\right] \tag{24}$$

Apply Gauss-Elimination Method to (24) to get

$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 14 \end{array}\right] \tag{25}$$

The bottom row corresponds to the equation $0.z = 14$. Hence the system (23) has no solutions.

Observe that we had plenty of freedom to choose the order of equations. This becomes a boon numerical methods. However, it is also necessary to curtail this freedom in order to bring some uniformity, viz., once a set of equations is presented to us in a particular order, we should change the order of the equations only if it is very much necessary. For the same reason, making leading entry equal to 1 is not completely necessary at the beginning. Neither there is any need to sweep the columns above a particular row. With these modification the above method becomes a well defined powerful algorithm thereby providing various advantages. We shall refer to it by Gauss Elimination Method(GEM).

However, we have also observed that the last two sets of operations have their own advantages. Since there is no ambiguity in them, we shall incorporate them in to a second stage (extended) algorithm called Gauss-Jordan Method (GJEM). Below we describe these two methods in algorithmic form. Let us begin with some definitions.