A Proof of Fundamental Theorem of Algebra Through Linear Algebra due to Derksen

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- MA106-Linear Algebra 2011

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- We present a proof of
  the Fundamental Theorem of Algebra (FTA)
  Every non constant polynomial in one variable with coefficients in $\mathbb{C}$ *has a root in* $\mathbb{C}$.

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  the Fundamental Theorem of Algebra (FTA)
  Every non constant polynomial in one variable with coefficients in $\mathbb{C}$ *has a root in* $\mathbb{C}$.
- through a sequence of easily do-able exercises.
  The proof uses elementary linear algebra
  except that we begin with
  the intermediate value theorem in proving that every odd degee real polynomial has a real root.

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- **Amer. Math. Monthly- 110,(2003), pp. 620-623.**

Sketch of the Proof

- In what follows $\mathbb{K}$ will denote any field. However, we need worry about only two cases here: $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

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- It should be noted that there is **no** purely algebraic proof of FTA which does not use IVT.
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- Indeed, all proofs of FTA use IVT explicitly or implicitly. The simple reason is that IVT is equivalent to any other axiom that is used in the construction of real numbers.
- From now onwards we only use linear algebra.

Sketch of the Proof

- **Companion Matrix** Let $p(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ be a monic polynomial of degree n. Its companion matrix C_p is defined to be the $n \times n$ matrix

$$C_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & \cdots & & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & & \cdots & \cdots & -a_1 \end{bmatrix}.$$

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- Ex.1. Show that the characteristic polynomial of C_p is p, i.e., $\det(C_p - tI_n) = (-1)^n p(t)$.

- Ex. 2. Show that every non constant polynomial $p(t) \in \mathbb{K}[t]$ of degree n has a root in $\mathbb{K}$ iff every endomorphism $\mathbb{K}^n \longrightarrow \mathbb{K}^n$ has an eigenvalue in $\mathbb{K}$

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- Recall that by an endomorphism of a vector space V we mean a linear transformation $V \longrightarrow V$.
- Ex. 3 Show that every $\mathbb{R}$-linear map $f : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1}$ has a real eigenvalue.

Sketch of the Proof

For positive integers r, let $S_1(\mathbb{K}, r)$ denote the following statement:
Every endomorphism $A : \mathbb{K}^n \longrightarrow \mathbb{K}^n$ has an eigenvector for all n not divisible by 2^r.
Also let $S_2(\mathbb{K}, r)$ denote the statement
Any two commuting endomorphisms $\mathbb{K}^n \longrightarrow \mathbb{K}^n$ have a common eigenvector for all n not divisible by 2^r.

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- (c) $S_1(\mathbb{C}, r) \Longrightarrow S_1(\mathbb{C}, r+1), r \geq 1.$
- Begin with IVT which is the same as $S_1(\mathbb{R}, 1)$. Putting $\mathbb{K} = \mathbb{R}$ in (a) we get $S_2(\mathbb{R}, 1)$.Now (b) gives $S_1(\mathbb{C}, 1)$ and repeated application of (c) gives $S_1(\mathbb{C}, k)$ for all $k \geq 1$.

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- Given n, write $n = 2^k \ell$ where ℓ is odd. Then $S_1(\mathbb{C}, k+1)$ implies that every polynomial of degree n has a root. And that completes the proof of FTA.

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- We shall prove this by induction on the dimension n of the vector space.
- For $n = 1$ this is clear, since any non zero vector is an eigenvector for all endomorphisms.
- So assume that the statement is true for smaller values of n and let A, B be two commuting $n \times n$ matrices over $\mathbb{K}$, and n is not divisible by 2^r.

$S_1(\mathbb{K}, r)$ implies $S_2(\mathbb{K}, r)$

- Let λ be an eigenvalue of A and put $V_1 = \mathcal{N}(A - \lambda I), V_2 = \mathcal{R}(A - \lambda I)$. Since B commutes with A, it follows that B commutes with $A - \lambda I$) also. Therefore B restricts to endomorphisms $\beta : V_1 \longrightarrow V_1$ and $\beta : V_2 \longrightarrow V_2$. (Refer: Tut. sheet 9.18.)

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$$\dim V_1 + \dim V_2 = n$$

and hence dim V_1 or dim V_2 is not divisible by 2^r.

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- If *dim* V_1 is not divisible by 2^r then $\beta : V_1 \longrightarrow V_1$ has an eigenvector which is also the eigenvector for A.
- If not, then dim $V_2 < n$ and not divisible by 2^r. Now we consider $\alpha, \beta : V_2 \longrightarrow V_2$ which are restrictions of A and B and hence are commuting endomorphisms.
  By induction we are through there is a common eigen vector $\mathbf{v} \in V_2 \subset V$ for α and β which is then a common eigen vector for A and B as well.

- Ex. 5 Show that the space $\mathrm{HERM}_n(\mathbb{C})$ of all complex Hermitian $n \times n$ matrices is a $\mathbb{R}$ vector space of dimension n^2.

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- Ex. 5 Show that the space $\mathrm{HERM}_n(\mathbb{C})$ of all complex Hermitian $n \times n$ matrices is a $\mathbb{R}$ vector space of dimension n^2.
  (Refer to Exercise 8.5 and a question in the Quiz.)
  Recall that a matrix A is hermitian if it is equal to its transpose conjugate A^*.
  and the next one is from 8.6.
- Ex. 6 Given $A \in M_n(\mathbb{C})$, the mappings

$$\alpha_A(B) = \frac{1}{2}(AB + BA^*); \quad \beta_A(B) = \frac{1}{2i}(AB - BA^*)$$

  define $\mathbb{R}$-linear maps $\mathrm{HERM}_n(\mathbb{C}) \longrightarrow \mathrm{HERM}_n(\mathbb{C})$.
  Show that α_A, β_A commute with each other.

- Answer:

$$
\begin{aligned}
& 4\imath\alpha_A \circ \beta_A(B) \\
= \; & 2\alpha_A(AB - BA^*) \\
= \; & [A(AB - BA^*) + (AB - BA^*)A^*] \\
= \; & [A(AB + BA^*) - (AB + BA^*)A^*] \\
= \; & 2\imath\beta_A(AB + BA^*) \\
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- It follows that $A\mathbf{u} = (\lambda + \imath\mu)\mathbf{u}$ and hence $\lambda + \imath\mu$ is an eigenvalue for A.

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- **Solution:** By Ex.6, given $A \in M_n(\mathbb{C})$, $\alpha_A, \beta_A : Herm_n(\mathbb{C}) \to Herm_n(\mathbb{C})$ are two commuting endomorphisms of the real vector space $Herm_n$ which is of dimension n^2, by ex. 5.

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- If n is odd, so is n^2.

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- If n is odd, so is n^2.
- By Ex.4, it follows that α_A, β_A have a common eigenvector.
- By Ex. 7, A has an eigenvalue.

- Ex. 9. Show that the space $\mathrm{Sym}_n(\mathbb{K})$ of symmetric $n \times n$ matrices forms a subspace of dimension $n(n+1)/2$ of $M_n(\mathbb{K})$. (Exercise 8.5)

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- Ex. 10 Given $A \in M_n(\mathbb{K})$, show that

$$\phi_A : B \mapsto \frac{1}{2}(AB + BA^t); \quad \psi_A : B \mapsto ABA^t$$

define two commuting endomorphisms of $\mathrm{Sym}_n(\mathbb{K})$. Show that if B is a common eigenvector of ϕ_A, ψ_A, then $(A^2 + aA + b\,I_n)B = 0$ for some $a, b \in \mathbb{K}$; further if $\mathbb{K} = \mathbb{C}$, conclude that A has an eigenvalue.

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- Multiply first relation by A on the left and use the second to obtain

$$A^2B + \mu B - \lambda AB = 0$$

which is the same as

$$(A^2 - \lambda A + \mu I_n)B = 0.$$

- For the last part, first observe that since B is an eigenvector there is at least one column vector $\mathbf{v}$ which is non zero. Therefore $(A^2 + aA + b\, I_n)\mathbf{v} = 0$.

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- Now write $A^2 + aA + bI_n = (A - \lambda_1 I_n)(A - \lambda_2 I_n)$. If $(A - \lambda_2 I_n)\mathbf{v} = 0$, then λ_2 is an eigenvalue of A and we are through.

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- Now write $A^2 + aA + bI_n = (A - \lambda_1 I_n)(A - \lambda_2 I_n)$. If $(A - \lambda_2 I_n)\mathbf{v} = 0$, then λ_2 is an eigenvalue of A and we are through.
- Otherwise put $\mathbf{u} = (A - \lambda_2 I_n)\mathbf{v} \neq 0$. Then $(A - \lambda_1 I_n)\mathbf{u} = 0$ and hence λ_1 is an eigenvalue of A.

- Ex. 11 Prove statement (c)
  $S_1(\mathbb{C}, r) \Longrightarrow S_1(\mathbb{C}, r+1)$. Hence conclude that $S_1(\mathbb{C}, r)$ is true for all $r \geq 1$.

- Ex. 11 Prove statement (c)
  $S_1(\mathbb{C}, r) \Longrightarrow S_1(\mathbb{C}, r+1)$. Hence conclude that $S_1(\mathbb{C}, r)$ is true for all $r \geq 1$.
- Answer:
  Let $n = 2^k \ell$ where $1 \leq k \leq r$ and ℓ is odd. Let $A \in M_n(\mathbb{C})$. Then ϕ_A, ψ_A are two mutually commuting operators on $Sym_n(\mathbb{C})$ which is of dimension $n(n+1)/2 = 2^{k-1}\ell(2^k\ell + 1)$ which is not divisible by 2^r.

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- By Ex. 10, with $\mathbb{K} = \mathbb{C}$ along with the induction hypothesis, it follows that A has an eigenvalue.

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- by Ex.2, it is enough to show that every $n \times n$ complex matrix has an eigenvalue.
- By Ex.8, this holds for odd n. This implies that $S_1(\mathbb{C}, 1)$. By Ex. 11 applied repeatedly, we get $S_1(\mathbb{C}, r)$ for all r.
  Given any polynomial of degree n, write $n = 2^r \ell$, where ℓ is odd. Then $S_1(\mathbb{C}, r+1)$ implies that every polynomial of degree n has a root.

THANK YOU

FOR YOUR ATTENTION