QUIZ Marking Scheme: MA106, 2011

Note to the Students:
(1) You are entitled for a re-evaluation of your answer script if there is difference in actual evaluation and the model one given below. Those who do not turn up on the day of the distribution of the answer script lose this right.
(2) Please note that when you request for recheck, there is a good possibility that your new score may become less since complete recheck for the required question will be carried out. So, do not make unnecessary and unjust claims such as 'what I have written implies whatever was expected' or 'Please recheck Q.1' etc.
(3) For this, it is better if you study the model answers thoroughly before raising your complaints with the tutor.
(4) Discuss your complaint with the tutor. Only those complaints which he/she admits should be written for re-evaluation on the **last page** of the paper. Do not write anything inside the paper.
(5) Submit your answer script to the tutor on the spot.

Q.1 (Checked by Soham Mehta)

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & -7 \\ 2 & 5 & -3 & -1 \\ 4 & 1 & 6 & -9 \end{array}\right] \mapsto (R_2 - R_1/2, R_3 - R_1) \qquad [1]$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & -7 \\ 0 & 3 & -4 & 5/2 \\ 0 & -3 & 4 & -2 \end{array}\right] \mapsto (R_3 + R_2)$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & -7 \\ 0 & 3 & -4 & 5/2 \\ 0 & 0 & 0 & 1/2 \end{array}\right] \qquad [1]$$

The Gauss form of the matrix has a row of zeros [1]

whereas the extended matrix has a non zero entry in the same row. Therefore the system is inconsistent (i.e., there are no solutions). [1]

Note: One mark deducted in case:
(i) more than two mistakes in the GEM steps; (ii) the fourth column being incorrect; (iii) serious matrix notation errors.

(Errors of the type $0 \cdot z = 1/2$ or $0 \cdot x_3 = 1/2$ etc. are condoned.)

Q.2 (Checked by Indrasen)

Solution I Let $R_1, R_2, \ldots, R_n$ be the rows of A. Put

$$A_1 = \begin{pmatrix} R_1 \\ -R_2 \\ \vdots \\ (-1)^{i-1}R_i \\ \vdots \end{pmatrix} = (C_1, \ldots C_n).$$

Then

$$B = ((b_{ij})) = (((-1)^{i+j}a_{ij})) = (C_1, -C_2, \ldots, (-1)^{j-1}C_j, \ldots).$$

[2]

Put $k = \lfloor n/2 \rfloor$. [1]

Then $\det B = (-1)^k \det A_1 = (-1)^k(-1)^k \det A = \det A$. [1]

Solution II Put given an $n \times n$ matrix $A_n = ((a_{ij}))$, put $\tilde{A_n} = (((-1)^{i+j}a_{i+j})) = ((b_{ij})) = B$. Claim $\det \tilde{A_n} = \det A_n$. by induction on n. Clearly it is true for $n = 1$. Assume the statement is true for $n - 1$.

Expand along the 1st row to get $\det \tilde{A_n} = \sum_j (-1)^{1+j} b_{ij} Cof_{1j}(\tilde{A_n})$.

By induction $Cof_{1j}(\tilde{A_n}) = (-1)^{n-1}(-1)^{n-j}\widetilde{Cof_{1j}}(A_n) = (-1)^{1+j}Cof_{1j}(A_n)$.
Therefore $\det \tilde{A_n} = \sum_j (-1)^{1+j}(-1)^{1+j}(-1)^{1+j}a_{1j}Cof_{1j}(A_n) = \det A_n$.

Solution III $\det B = \sum_\sigma sgn(\sigma) b_\sigma = \sum_\sigma sgn(\sigma) \prod_i b_{i\sigma(i)}$ [2]

$= \sum_\sigma sgn(\sigma) \prod_i (-1)^{i+\sigma(i)} a_{i\sigma(i)}$ [1]
$= \sum_\sigma sgn(\sigma)(-1)^{n(n+1)} a_\sigma = \sum_\sigma sgn(\sigma) a_\sigma = \det A$. [1]

Q.3 (Checked by Suryateja and Soham Pradhan)

$p_{ij} = 1$ iff $(i, j) = (1, 2), (2, 1), (3, 7), (4, 6), (5, 5), (6, 4)$ or $(7, 3)$. Therefore, it follows that

$P = P_{(1,2)}P_{(4,6)}P_{(3,7)}$ [2]

$\det P = (-1)^3 = -1$. [2]

Q.4 (Checked by Suryateja and Soham Pradhan)

Put

$$A_n = \begin{bmatrix} 1 & & \cdots & 1 \\ n & 1 & \cdots & 1 \\ n & n-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ n & n-1 & \cdots & 1 \end{bmatrix}$$

Then

$$\det A_n = \begin{vmatrix} 1-n & 0_{1,n-1} \\ \star & A_{n-1} \end{vmatrix}$$

(by subtracting the II-row from the I-row) [2]

$= (-1)(n-1) \det A_{n-1} = \cdots = (-1)^{n-1}(n-1)!$. [2]

Q.5 (Checked by Naval Chopra and Kshitij))

Solution (I) α, β are any real numbers $\alpha \neq 0$. [1]

Taking $\alpha = 0$ we get $\{\beta v_n, v_2, \ldots, v_n\}$ L.D. for all possible β. Therefore $\alpha \neq 0$. [1]

Conversely, if $\alpha \neq 0$ then $\{v_1, \ldots, v_n\}$ is L.I. $\implies \{\alpha v_1, v_2, \ldots, v_n\}$ is L.I. [1]

$\implies \{\alpha v_1 + \beta v_n, v_2, \ldots, v_n\}$ is L.I. [1]

Solution (II) For linear independence of the required set, we must have
$\alpha_1(\alpha v_1 + \beta v_n) + \sum_{i \geq 2} \alpha_i v_i = 0$ iff $\alpha_1 \alpha = \alpha_2 = \cdots \alpha_1 \beta + \alpha_n = 0$. [1]

If $\alpha = 0$ then taking taking $\alpha_1 = 1, \alpha_2 = \ldots, \alpha_{n-1} = 0, \alpha_n = -\alpha_1 \beta$ gives L.D for all β. [1]

If $\alpha \neq 0$ then $\alpha_1 = 0$ and then $\alpha_2 = \ldots = \alpha_n = 0$ gives L. I. [1]

Conclusion: $\alpha \neq 0$ is the required condition. [1]

Solution (III)

$\{v_1, v_2, \ldots, v_n\}$ is L. I. iff $\det[v_1, \ldots, v_n] \neq 0$.

Given $\det[v_1, \ldots, v_n] \neq 0$,

$\det[\alpha v_1 + \beta v_2, v_2, \ldots, v_n] = \det[\alpha v_1, v_2, \ldots v_n] + \det[\beta v_n, v_2, \ldots, v_n] = \alpha \det[v_1, v_2, \ldots, v_n] \neq 0$
iff $\alpha = 0$. required condition: $\alpha \neq 0$. [4]

Q.6 (Checked by Naval Chopra and Kshitij))

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ [4]

No partial marking.

Q.7 (a) If f is linear then

$$4 = f(1,1,1) = f(1,0,0) + f(0,1,0) + f(0,0,1) = 1 + 2 + 3 = 5.???$$

which is absurd [2]

(b) Let V be a vector space. A subset W of V is said to be a subspace of V if it is a vector space [1]

with respect to the operations in V. [1]

OR

$W \subset V$ is a subspace if it is nonempty [1]

and satisfies the closure axioms. [1]

Q.8 (a) $A = A^T, B = B^T, \alpha, \beta \in \mathbf{C}$ implies $(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = \alpha A + \beta B$. Therefore, Sym_n satisfies the closure axioms. [1]

It is nonempty because $0_n \in Sym_n$. Therefore it is a subspace. [1]

(b) $\{E_{ii} \ : \ 1 \leq i \leq n\} \cup \{E_{ij} + E_{ji} \ : \ 1 \leq i < j \leq n\}$ is a basis for for Sym_n. [2]

Q.9 (a) Let $A = ((a_{ij}))$ be the $n \times n$ matrix. Then trace of S is defined by $tr\, A = \sum_i a_{ii}$. [2]

(b) $tr\,(AB) = \sum_i (AB)_{ii} = \sum_i (\sum_j a_{ij} b_{ji})$
$= \sum_j (\sum_i b_{ji} a_{ij}) = \sum_j (BA)_{jj} = tr\,(BA)$. [2]

Q.10 (a)(i) A subset S of a vector space is said to be linearly dependent if there exists $\alpha_1, \ldots, \alpha_k \in \mathbf{K}$, not all zero and $v_1, \ldots, v_k \in S$ such that $\sum_i \alpha_i v_i = 0$. [1]

(ii) A subset S of a vector space V is said to span V if for every $v \in V$, there exist $v_1, \ldots, v_k \in V$ and $\alpha_1, \ldots, \alpha_k \in \mathbf{K}$
such that $v = \sum_i \alpha_i v_i$. [1]

(b) Let S be linearly independent. Suppose $\sum_{i=1}^{k} \alpha_i f(v_i) = 0$ where, $v_i \in S$. Then $f(\sum_i \alpha_i v_i) = 0$. Because f is injective, this implies $\sum_i \alpha_i v_i = 0$. Since S is L. I. this implies $\alpha_1 = \ldots = \alpha_k = 0$. [1]

Conversely, let $f(S)$ be linearly independent. Suppose $\sum_i \alpha_i v_i = 0$ where $v_i \in S$. Then $f(\sum_i \alpha_i v_i) = 0$. Therefore $\sum_i \alpha_i f(v_i) = 0$ Therefore $\alpha_i = 0$ for all i. [1]

(Please note that if you interchange the proof of the statement and the converse, you get no marks)