1. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be any linear map such that

$$f(1,1,0) = (0,0,1), \quad f(1,0,1) = (0,1,0), \quad f(0,1,1) = (1,0,0).$$

(a) Write the matrix representation of f with respect to the standard basis. [3]
(b) Is f injective? [2]
(c) Is f surjective? [1]

Solution:: $2f(1,1,1) = f(2,2,2) = f(1,1,0) + f(1,0,1) + f(0,1,1) = (1,1,1)$. [1] Therefore
$f(1,0,0) = f(1,1,1) - f(0,1,1) = (-1/2,,1/2,1/2)$.
$f(0,1,0) = f(1,1,1) - f(1,0,1) = (1/2,-1/2,1/2)$. And
$f(0,0,1) = f(1,1,1) - f(1,1,0) = (1/2,1/2,-1/2)$. [1]

The matrix of f is :

$$\begin{bmatrix} -1/2 & /1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

[1]

The determinant of the matrix is equal to $1/2$ [1]

and hence it is invertible. Therefore, the matrix f is both injective and surjective. [2]

There are many alternate answers possible for this question. In each case, partial marking comparable to the above is followed. For example, if you write down just the matrix then you will not get any mark. Similarly, simply asserting that the mpa is injective, surjective without justification, will not get any mark.

2. For a system of linear equations in the variables $x_1, \ldots, x_5$, the GJ form of the extended matrix is given to be:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

(a) Write down one solution of the (inhomogeneous) system. [2]

(b) Write down a basis for the space W of the set of all solutions of the associated homogeneous system. [2]

(c) Write down the solution of the inhomogeneous system which is near most to the point $(1, 1, 1, 1, 1)$. [2]

Solution: The GJ tells us that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ x_3 \\ x_5 \end{bmatrix} ------(*)$$

[1]

(a) A solution of the inhomogeneous system is got by replacing the last column in (*) by $(1, 0, 0)^t$ viz. $\mathbf{p} = (1, -3, 0, -1, 0)^t$. [1]

(Any other particular solution arrived correctly is also acceptable.)

(b) A basis for the solution space W of the homogeneous system is obtained by replacing the last column in (*) by $(0, 1, 0)^t$ and $(0, 0, 1)^t$ viz.,
$\{(0, 0, 1, 0, 0)^t, (-1, -1, 0, -1, 1)^t\}$ [2]

(One basic element is correct and the whole set is not a basis then you get 1 mark.)

(c) This is already an orthogonal basis. Therefore, the orthonormal basis is obtained by dividing each vector by its norm:
$\{(0, 0, 1, 0, 0), \frac{1}{2}(-1, -1, 0, -1, 1)\}$.

Given point $\mathbf{v} = (1, 1, 1, 1, 1)$. Therefore the projection $P_W(\mathbf{v} - \mathbf{p}) = P_W(0, 4, 1, 2, 1)$
$= (0, 0, 1, 0, 0) + \frac{5}{2}(1, 1, 0, 1, -1) = \frac{5}{2}(1, 1, \frac{2}{5}, 1, -1)$. [1]

The required solution is obtained by adding $\mathbf{p}$ to this viz.,
$P_W(\mathbf{v} - \mathbf{p}) + \mathbf{p} = \frac{1}{2}(7, -1, 2, 3, -5)$. [1]

Some numerical errors have been condoned and partial marking given.

3. Compute the inverse of the following matrix using GJEM. [6]

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Solution:

$$[A|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$(R_2 - 2R_1, R_3 - 3R_1, R_4 - 4R_1) \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right] \qquad [2]$$

$$(R_3 - 2R_2, R_4 - 3R_2) \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 2 & -3 & 0 & 1 \end{array}\right] \qquad [2]$$

$$(R_4 - 2R_3) \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \end{array}\right] = [I|A^{-1}] \qquad [2]$$

4. (a) Show that for any skew-Hermitian matrix A, if λ is a nonzero eigenvalue then λ is purely imaginary. [2]
(b) Define orthogonal and unitary matrices. [1+1]
(c) Show that eigenvaules of a unitary matrix are all of modulus 1. [2]

Solution: (a) A is skew-Hermitian implies that $A^* = -A$. Let $\lambda \neq 0$ be such that $A\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq 0$. Then $\bar{\lambda}\mathbf{v}^*\mathbf{v} = (\lambda\mathbf{v})^*\mathbf{v} = (A\mathbf{v})^*\mathbf{v} = \mathbf{v}^*A^*\mathbf{v} = -\mathbf{v}^*A\mathbf{v} = -\lambda\mathbf{v}^*\mathbf{v}$. This shows that $\bar{\lambda} = -\lambda$ and hence λ is purely imaginary. [2]

(Saying that if A is Skew Hermitian implies $\imath A$ is Hermitian and hence $\imath\lambda$ is real etc. –only one mark.)

(b) A square matrix A is called orthogonal if its entries are real
and $AA^t = I$. [1]
A square matrix A is unitary if its entries are complex
and $AA^* = I$. [1]

(After defining unitary matrix, one may also define an orthogonal matrix as a real unitary matrix.)

(c) If $AA^* = I$ then $A^*A = I$. Now if $A\mathbf{v} = \lambda\mathbf{v}, \mathbf{v} \neq 0$, we have
$\bar{\lambda}\lambda\mathbf{v}^*\mathbf{v} = (\lambda\mathbf{v})^*(\lambda\mathbf{v}) = (A\mathbf{v})^*(A\mathbf{v}) = \mathbf{v}^*(A^*A)\mathbf{v} = \mathbf{v}^*\mathbf{v} = \|\mathbf{v}\|^2 \neq 0$.
Therefore $\bar{\lambda}\lambda = 1$. [2]

5. Let $V = C[-\pi, \pi]$ be the vector space of all continuous real valued functions. Consider the inner product on V given by

$$\langle f,\, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)dt.$$

Apply Gram-Schmidt process to the set $\{1, x, x^2+1\}$ to get an orthonormal set. [6]

Solution: Put $\mathbf{u}_1 = 1, \mathbf{u}_2 = x, \mathbf{u}_3 = x^2+1$. $\langle \mathbf{u}_1,\, \mathbf{u}_1\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} dt = 1$. Therefore $\|\mathbf{u}\| = 1$. Therefore put $\mathbf{v}_1 = \mathbf{w}_1 = \mathbf{u}_1$. [1]

Next observe that $\langle \mathbf{w}_1,\, \mathbf{u}_2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} tdt = 0$. That is $\mathbf{w}_1 \perp \mathbf{u}_2$. Therefore put $\mathbf{w}_2 = \mathbf{u}_2 = x$. [2]

Now $\langle x,\, x\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} t^2dt = \frac{\pi^2}{3}$. Therefore put $\mathbf{v}_2 = \mathbf{w}_2/\|\mathbf{w}_2\| = \sqrt{3}x/\pi$. [1]

Now $\langle 1,\, x^2+1\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(t^2+1)dt = \pi^2/3+1$. and $\langle x,\, x^2+1\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(t^3+t)dt = 0$. Therefore $\mathbf{w}_3 = \mathbf{u}_3 - \langle \mathbf{w}_1,\, \mathbf{u}_3\rangle \mathbf{w}_1 = x^2+1-(\pi^2/3+1)\mathbf{w}_1 = x^2-\pi^2/3$. [1]

Finally, $\langle \mathbf{w}_3,\, \mathbf{w}_3\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(t^2-\pi^2/3)^2dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(t^4 - 2\pi^2/3t^2 + \pi^4/9)dt$
$= \pi^4/5 - (2\pi^2/3)(\pi^2/3) + \pi^4/9 = 4\pi^4/45$.
Therefore, put $\mathbf{v}_3 = \mathbf{w}_3/\|\mathbf{w}_3\| = (3\sqrt{5}/2\pi^2)(x^2-\pi^2/3)$.

The required orthonormal set is $\{1, \sqrt{3}x/\pi, (3\sqrt{5}/2\pi^2)(x^2-\pi^2/3)\}$. [2]

6. Given $A \in M_n(\mathbb{C})$, consider the endomorphisms $f, g : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by

$$f(B) = AB + BA^t; \quad g(B) = ABA^t, \quad B \in M_n(\mathbb{C}).$$

Let $B \in M_n(\mathbb{C})$ be a common eigenvector for f and g.
(a) Show that there are $a, b \in \mathbb{C}$ such that $(A^2 + aA + bI_n)B = 0$. [3]

(b) Show that A has an eigenvalue in $\mathbb{C}$. [3]

Solution: (a) Let $B \neq 0$ be such that $f(B) = \lambda B$ and $g(B) = \mu B$. Then
$AB + BA^t = \lambda B \qquad (1)$
$ABA^t = \mu B \qquad (2)$ [1]

Multiply (1) by A on the left and substitute from (2) to obtain

$$A^2B + \mu B = \lambda AB$$

which is the same as

$$(A^2 - \lambda A + \mu I)B = 0.$$

Here $a = -\lambda, b = \mu$. [2]

(b) Let $A^2 + aA + bI = (A - a_1 I)(A - a_2 I)$. [1]

Let $\mathbf{v}$ be a non zero column of B. Then we have $(A - a_1 I)(A - a_2 I)\mathbf{v} = 0$. [1] If $(A - a_2 I)\mathbf{v} = 0$ then a_2 is an eigenvalue; otherwise a_1 is. [1]

Alternatively,

$(A^2 + aA + bI)B = 0$ and $B \neq 0$ implies that $A^2 + aA + I$ is not invertible [1]

Therefore $det(A^2 + aA + bI) = 0$. [1]

Therefore either $det(A - a_1 I) = 0$ or $det(A - a_2) = 0$. and hence a_1 or a_2 is an eighenvalue [1]

7. (a) Let $A \in M_n(\mathbb{K})$. Consider the endomorphism $L_A : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ defined by

$$L_A(B) = AB.$$

Show that λ is an eigenvalue of A iff λ is an eigenvalue of L_A. [3]

(b) Define a normal matrix. [1]

(c) Give an example of a normal matrix which is not Hermitian. Justify your answer briefly. [2]

Solution: (a) Suppose λ is an eigenvalue of A. Let $\mathbf{v}$ be a non zero column vector such that $A\mathbf{v} = \lambda\mathbf{v}$. Take $B = [\mathbf{v}, \ldots, \mathbf{v}]$ which is then a non zero $n \times n$ matrix. Also we have $L_A(B) = AB = \lambda B$.

Therefore λ is an eigenvalue of L_A. [2]

Conversely, suppose λ is an eigenvalue of L_A. This means there is a non zero $n \times n$ matrix B such that $L_A(B) = \lambda B$. This just means $AB = \lambda B$. Put $\mathbf{v} = B^j$ to be any non zero column of B. Then we have $A\mathbf{v} = AB^j = \lambda B^j = \lambda\mathbf{v}$ and hence λ is an eigenvalue of A. [1]

(b) A square matrix A with complex entries is called normal iff $A^*A = AA^*$. [1]

(c) Take $A = \begin{bmatrix} 1 & \imath \\ \imath & 1 \end{bmatrix}$. [1]

Directly verify that $A^*A = AA^*$.

Since $A^* = \begin{bmatrix} 1 & -\imath \\ -\imath & 1 \end{bmatrix} \neq A$ it is not Hermitian. [1]

8. (a) State in each case whether true or false by circling the correct choice in the last column. [4]

(1) For every square matrix A over real numbers there exists a real, invertible square matrix C such that $C^{-1}AC$ is an upper triangular matrix	(TRUE/FALSE)
(2) Every real symmetric matrix is diagonalizable over reals.	(TRUE/FALSE)
(3) Eigenvalues of a normal matrix are real.	(TRUE/FALSE)
(4) A matrix is normal if its column vectors form an orthonormal basis.	(TRUE/FALSE)

Answer: (1) False [1]

(2) True [1]

(3) False [1]

(4) True [1]

(b) Compute the determinant of the following 8×8 matrix: [2]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t^2 \\ 0 & 0 & 0 & 0 & t^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t^4 & 0 & 0 \\ 0 & 0 & t^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^6 & 0 & 0 & 0 & 0 \\ t^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^8 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: The given matrix is the composite of four transpositions $T_{1,7}T_{2,8}T_{3,5}T_{4,6}$ and the diagonal matrix

$$diag(t^7, t^8, t^5, t^6, t^3, t^4, t, t^2).$$

therefore the determinant is $(-1)^4 t^{36} = t^{36}$. [2]

Any other valid method of computing is acceptable.

If the sign is ignored or wrong or correct but not justified then 1 mark is deleted.

Writing the determinant as $t_1 t_2 \cdots t_8$ carries no mark.

Some numerical errors such as t^{28} instead of t^{36} are ignored if the rest of the computations are correct.

9. Transform the following quadratic form

$$91x^2 - 24xy + 84y^2 = 25$$

into principal axis form and determine what conic it represents. [6]

Solution: The symmetric matrix associated to this quadratic form is

$$A = \begin{bmatrix} 91 & -12 \\ -12 & 84 \end{bmatrix}$$

[1]

Therefore $\chi_A(\lambda) = (91 - \lambda)(84 - \lambda) - 144 = (\lambda - 75)(\lambda - 100)$. [1]

A unit eigenvector corresponding to $\lambda = 75$ is given by

$$\begin{bmatrix} 16 & -12 \\ -12 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

say $(3/5, 4/5)$. Similarly, a unit eigenvector corresponding $\lambda = 100$ is given by $(4/5, -3/5)$. [1]

Therefore the transformation equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

[1]

The principal axis form is

$$75u^2 + 100v^2 = 25$$

which is equivalent

$$3u^2 + 4v^2 = 1$$

[1]

which represents an ellipse. [1]

10. (a) Given an $n \times n$ matrix $A = ((a_{ij}))$ such that $a_{ij} = 0$ for all $i \geq j$. Show that $A^n = 0$. [4]

(b) Prove or disprove the following statement: If the sum of all the columns of a 5×5 matrix A is a multiple of $(1, 1, 1, 1, 1)^t$ then A cannot be invertible. [2]

Solution: (a) We shall show that

$$A^n \mathbf{e}_i = 0, \quad 1 \leq i \leq n$$

from which it follows that $A^n = 0$.

In fact we shall prove that for each $1 \leq j \leq n$, $A^j \mathbf{e}_i = 0$ for $1 \leq i \leq j$. [1]

This will be done by induction on j. For $j = 1$, we have $A\mathbf{e}_1 = (a_{11}, a_{21,}, \ldots, a_{n1})^t = 0$ by hypothesis.

Assume the result for j. If $i < j + 1$ then $A^{j+1}\mathbf{e}_i = AA^j\mathbf{e}_i = A0 = 0$. [1]

Now

$$A^{j+1}\mathbf{e}_{j+1} = A^j(\sum_{i \leq j} a_{i,j+1}\mathbf{e}_i) = \sum_{i \leq j} a_{i,j+1}A^j\mathbf{e}_i = 0.$$

[1]

Proving that the first two columns (or the lastt two rows) last two rows of A^2 are zero carries carries 1 mark.

After that if you do not justify inductive step no mark awarded.

Alternative proofs such as the one given below are possible:

Let V_k denote the linear space of $\{e_1, \ldots e_k\}$. Then $A(V_1) = (0)$ and $A(V_i) \subset V_{i-1}$ for $2 \leq i \leq n$. Therefore

$$A^n(I\!R^n) \subset A^{n-1}(A(V_n)) \subset A^{n-1}(V_{n-1}) \subset \cdots \subset A(V_1) = (0).$$

Therefore $A^n = 0$. [4]

(b) This is false. Take $A = I_5$. [2]