MA106-Linear Algebra

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For example consider the map $\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

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where $\lambda \in \mathbb{R}$. For $n = 1$ it is clear that a segment of length ℓ in $\mathbb{R}$ (no matter where it is situated in $\mathbb{R}$) is mapped to a segment of length $\ell\lambda$.

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A little more generally, if we consider the endomorphism which effects a scaling by λ_i in the i^{th} direction, then it is clear that the volume of an n-dimensional cube is changed by a factor of $\lambda_1 \cdots \lambda_n$.

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These considerations lead us to believe that each linear transformation effects a scaling in the volume. We must hunt for this scaling factor.

Determinants

Example

Consider $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *such that* $T(1,0) = (a,b) =: \mathbf{v}_1$ *and* $T(0,1) = (c,d) = \mathbf{v}_2$.

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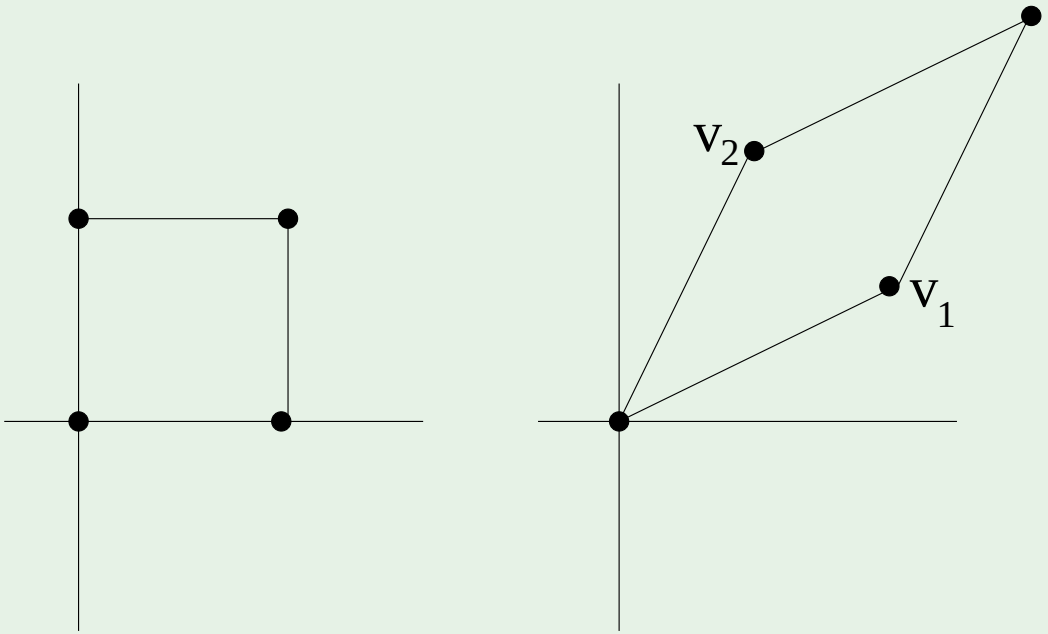
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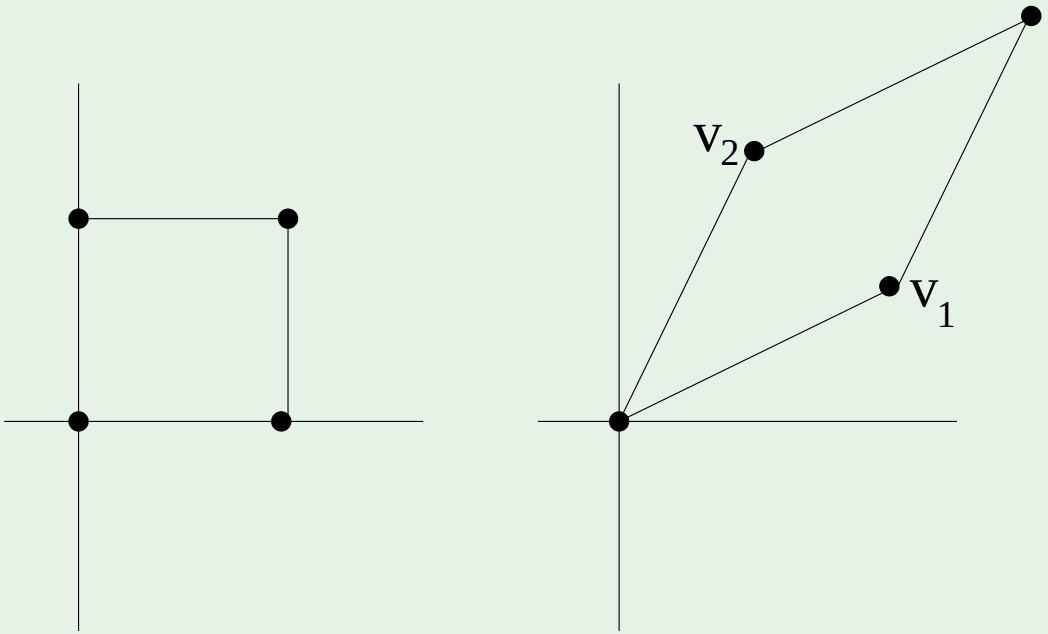
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In terms of vector product it is $|\mathbf{v}_1 \times \mathbf{v}_2|$.

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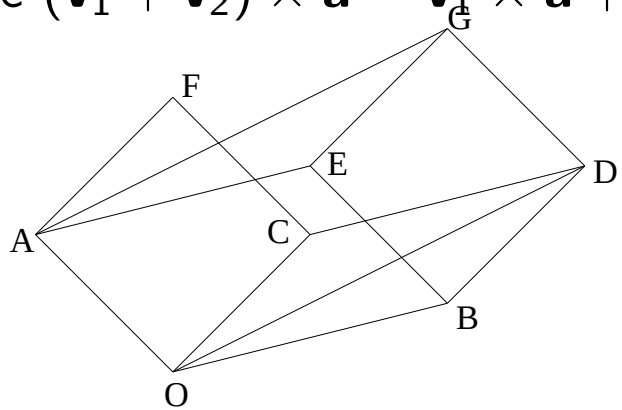
If we do not take the modulus, and only take the value $ad - bc$ it can thought of as a 'signed' area which is allowed to be negative. The vector product itself has the property viz.,

$$\mathbf{v}_1 \times \mathbf{v}_2 = -\mathbf{v}_2 \times \mathbf{v}_1.$$

We also have

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}.$$

The following simple diagram illustrates this and affirms that we are on the right track. Namely, we have $(\mathbf{v}_1 + \mathbf{v}_2) \times \mathbf{u} = \mathbf{v}_1 \times \mathbf{u} + \mathbf{v}_2 \times \mathbf{u}$.



OBEA+OCFA=OBEA+BDGE−OBD+AEG=ODGA

Geometric proof of the additivity of the area

(Caution: The above figure is not a 3-dimensional one!)

Determinants

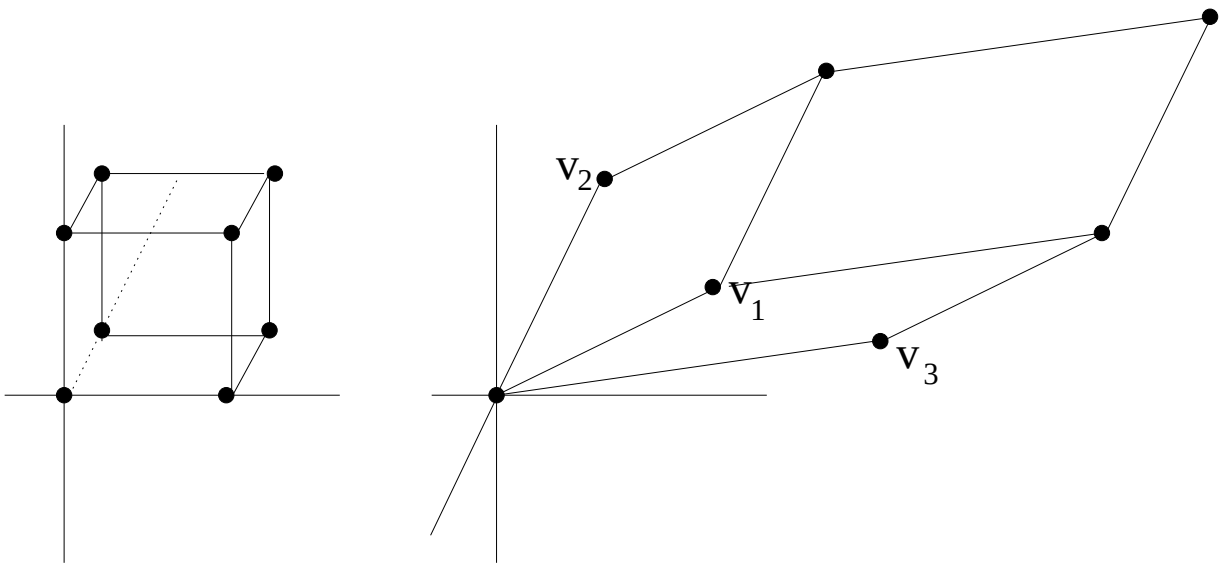
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Likewise, if $T' : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *is such that* $T'(\mathbf{e}_i) = \mathbf{v}_i$ *then the volume of the parallelepiped defined by the vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *is given by the modulus of the scalar triple product* $(\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3)$.

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In what follows we shall use the symbol $\mathbb{K}$ to denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complx numbers. (Indeed, for many of the concepts and results, we could have taken $\mathbb{K}$ to be just any field.)

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In what follows we shall use the symbol $\mathbb{K}$ to denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complx numbers. (Indeed, for many of the concepts and results, we could have taken $\mathbb{K}$ to be just any field.)
We are looking for a $\mathbb{K}$-valued function defined on the set of $n \times n$ matrices (for each n) say $f : M_n(\mathbb{K}) \longrightarrow \mathbb{K}$ which is likely to represent the volume of the n-dim. parallelepiped defined by the row vectors of a given matrix.

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- (iii) We have already observed that if we scale one of the vectors then the entire volume is also scaled by the same factor.
- Likewise the volume is 'additive, in each of the slots,

$$((\mathbf{u}_1 + \mathbf{u}_2)\mathbf{vw}) = (\mathbf{u}_1\mathbf{vw}) + (\mathbf{u}_2\mathbf{vw}).$$

Hunt for determinant

With these things in mind we shall now attack the problem systematically. Given a matrix $A \in M_n(\mathbb{K})$ say,

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix},$$

recall that by arranging the n rows of A one after another we can think of it as an ordered n-tuple of n-vectors: $A = (A_1, \ldots, A_n)$.

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recall that by arranging the n rows of A one after another we can think of it as an ordered n-tuple of n-vectors: $A = (A_1, \ldots, A_n)$. More generally, this method may be used to identify $M_{n,m}(\mathbb{K})$ with $\mathbb{K}^{mn}$. We are going to exploit it now.

Axioms for determinant

Definition

Let $1 \leq i \leq n$. A function $f : M_n(\mathbb{K}) \longrightarrow \mathbb{K}$ is called *linear in the i^{th} row* if, $f(A_1, \ldots, A_{i-1}, \alpha A + \beta B, A_{i+1}, \ldots, A_n) =$

$$\alpha f(A_1, \ldots, A_{i-1}, A, A_{i+1}, \ldots, A_n) + \beta f(A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_n) \qquad (1)$$

for all vectors $A_j, A, B \in \mathbb{K}^n$ and all scalars $\alpha, \beta \in \mathbb{K}$. A function f which is linear in i^{th} row for each $1 \leq i \leq n$ will be called **multi-linear.**

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For example, in this notation if $\sigma = Id$ then $E_\sigma = I_n$.
If $\sigma_1 = (1, 1, 2, 2), \sigma_2 = (3, 2, 1)$ say then

$$E_{\sigma_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad E_{\sigma_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Write down a few more examples to get familiar with this notation.

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Proof: Let us first illustrate this for $n = 2$. Write

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Thus the pattern in the general case is clear. By applying linearity in each slot at a time we obtain,

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Here the scalars a_σ are given by

$$a_\sigma = a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \tag{5}$$

The verification is straight forward. ♠

Definition

A map $f : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is called *alternating* if for every pair $(i, j), i \neq j$,

$$f(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -f(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n). \tag{6}$$

That is, if we interchange the place of any two rows then the sign of the value of the function is changed.

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To prove the converse, let us consider the 2×2 case.
We assume that f is multi-linear and has the property $f(A, A) = 0$ for all A. We must show that $f(B, C) = -f(C, B)$.

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$$\begin{aligned} 0 &= f(B + C, B + C) \\ &= = f(B, B + C) + f(C, B + C) \\ &= f(B, B) + f(B, C) + f(C, B) + f(C, C) \\ &= f(B, C) + f(C, B) \end{aligned}$$

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In the general case, given a matrix A consider another matrix B whose rows are that of A except in the i^{th} and j^{th} places we have $B_i = B_j = A_i + A_j$. Then $f(B) = 0$.

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On the other hand by multi-linearity, we have

$$\begin{aligned} & 0 = f(B) \\ = \; & f(A_1, \ldots, A_i, \ldots, A_i + A_j, \ldots, A_n) + f(A_1, \ldots, A_j, \ldots, A_i + A_j, \ldots, A_n) \\ = \; & f(A_1, \ldots, A_i, \ldots, A_i, \ldots, A_n) + f(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) \\ & + f(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n) + f(A_1, \ldots, A_j, \ldots, A_j, \ldots, A_n) \end{aligned}$$

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$$\begin{aligned} & 0 = f(B) \\ = \; & f(A_1, \ldots, A_i, \ldots, A_i + A_j, \ldots, A_n) + f(A_1, \ldots, A_j, \ldots, A_i + A_j, \ldots, A_n) \\ = \; & f(A_1, \ldots, A_i, \ldots, A_i, \ldots, A_n) + f(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) \\ & + f(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n) + f(A_1, \ldots, A_j, \ldots, A_j, \ldots, A_n) \end{aligned}$$

Hence $f(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -f(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$. ♠

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Theorem

Let f be a multi-linear alternating map on $M_n(\mathbb{K})$. Consider the elementary matrix $E = I + \alpha E_{ij}$. Then for all $A \in M_n$,

$$f(EA) = \begin{cases} f(A) & \text{if } i \neq j \\ (1+\alpha)f(A) & \text{if } i = j \end{cases}$$

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Therefore

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For $i = j$, we have $EA = (A_1, \ldots, A_{i-1}, (1+\alpha)A_i, A_{i+1}, \ldots, A_n)$. Therefore, $f(EA) = (1+\alpha)f(A)$, by linearity in the i^{th} slot. ♠

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Definition

Let $\Sigma(n)$ denote those elements $\sigma \in F(N, N)$ which are one-to-one. These are called permutations on n letters. If a permutation changes its value only on two indices then it is called a transposition.

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This means that $\tau_k \circ \cdots \tau_1 \circ \sigma = Id$. Hence $\sigma = \tau_1 \circ \cdots \circ \tau_k$. Thus we have proved:

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Every permutation is a (finite) composite of transpositions.

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Definition

Let $e(\sigma)$ denote the number of row changes in the GEM applied to E_σ to bring it to $G(E_\sigma) = I_n$. Let $sgn(\sigma) = (-1)^{e(\sigma)}$. We call this the signature of σ.

Axioms for determinant

Applying repeatedly the alternating property, it follows that

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Combining

$$(4), \qquad f(A) = \sum_\sigma a_\sigma f(E_\sigma)$$

and

$$(5), \qquad a_\sigma = a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

with (7), we get

$$f(A) = \left(\sum_{\sigma \in \Sigma(n)} sgn(\sigma) a_\sigma \right) f(I_n). \tag{8}$$

Existence

Theorem

For every $t \in \mathbb{R}$, there exists a unique $f_t : M_n(\mathbb{R}) \to \mathbb{R}$ which is row-wise multi-linear, alternating and such that $f_t(I_n) = t$. Moreover, for any such f_t, we have $f_t(E_\sigma) = sgn(\sigma)t$.

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To show that such a function exists, we just define f_t by the formula

$$f_t(A) = \left(\sum_{\sigma \in \Sigma} sgn(\sigma) a_\sigma \right) t. \tag{9}$$

To verify that f_t is multi-linear is straight forward.
Assuming that formula (9) has alternating property on the set $\{E_\sigma, \sigma \in \Sigma\}$, it follows easily that the same holds for all matrices. Therefore, it is enough to prove:

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Lemma

Let $\sigma, \tau \in \Sigma(n)$, where τ is a transposition. Then

$$sgn(\sigma) = -sgn(\tau \circ \sigma). \tag{10}$$

Existence

Proof: We prove it by induction on n. If $n = 2$ this is obvious. (For $n = 3$ this can be verified straight forward way, by considering various cases. Try this as an exercise before reading the rest of the proof which will help you to understand the proof better. However, this is not a logical necessity.)
Inductively, let the statement (10) hold for $n - 1$. Let $A = E_\sigma,\ B = E_{\tau\circ\sigma}$.

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Inductively, let the statement (10) hold for $n-1$. Let $A = E_\sigma,\ B = E_{\tau\circ\sigma}$.
Suppose τ interchanges i and j, $i \neq j$. Notice that B is obtained from A by interchanging the i^{th} and j^{th} row.
We look at the first step of GEM performed on both A and B cut down the first row and first column from both of them compare the results on either side, use induction to obtain the equality (10) for $n-1$ and finally relate it to (10) for n.

Existence

Suppose GEM applied to A interchanges 1^{st} row with k^{th} row. (We do not exclude the case when $k = 1$; it merely means that the first step of GEM does not do anything on A.)
The rest proof is best understood by glancing at the table below which is indeed a schematic representation of the proof.

Existence

Case 1: Let $\{1, k\} = \{i, j\}$. Here first step of GEM applied to A is the same as τ applied to A and hence both produce B. Clearly, the first step of GEM applied to B does nothing and all the remaining steps will coincide for both. Hence (10) gets verified here.

Case 2: Suppose the set $\{1, k, i, j\}$ has three elements. Observe that during the first step of GEM on A or B none of the other rows are affected. (Thus by simply ignoring these rows and the corresponding columns, this reduces to the case when $n = 3$ for which you have already verified (10). However, we shall write down a detailed proof.)

Existence

We consider first the subcase, where $k = 1$. Then the first step of GEM does not produce any change in A or B and we simply pass to the next stage. The equation (10) for n follows from the same for $n - 1$ by induction.

Existence

Next, suppose k is equal to one of the indices $\{i, j\}$ say for definiteness, $k = i$. This means that the non zero entry in the first column of A is in i^{th} place and hence the same for B is j^{th} place. Therefore the first step of GEM for A swaps rows $1 \leftrightarrow i$, where as the same for B swaps rows $1 \leftrightarrow j$. Therefore after cutting down the first row and first column, the two matrices are again related by the transposition $\tau' = (i-1, j-1)$. Again the equation (10) for n is obtained by multiplying by -1 the same equation for $n-1$.

Existence

Case 3: Suppose $\{1, k, i, j\}$ has four elements. Under this condition, it is clear that the first step of GEM on both A and B is to interchange the first and k^{th} rows. After this the two $(n-1) \times (n-1)$ matrices A', B' are again related by a transposition. Therefore, by induction, (10) is true for A', B' and hence after mutliplying it by -1 the same holds for A, B. ♠

Existence of determinant

Case	Stage	A		B	
(a)	Initial stage	Y . . . . . . R . . . . . . P	(1) (i) (j)	Y . . . . . . P . . . . . . R	(1) (i) (j)
	After step 1	Y . . . . . . R . . . . . . P	(1=k) (i) (j)	Y . . . . . . P . . . . . . R	(1=k) (i) (j)
(b)	Initial stage	Y . . . . . . R . . . . . . P	(1) (i=k) (j)	Y . . . . . . P . . . . . . R	(1) (i) (j)
	After Step 1	R . . . . . . Y . . . . . . P	(1) (i) (j)	R . . . . . . P . . . . . . Y	(1) (i) (j)

Axioms for determinant

Case 3		
Stage	A	B
Initial stage	Y ——— (1) . . . . . . R ——— (i) . . . . . . G ——— (k) . . . . . . P ——— (j)	Y ——— (1) . . . . . . P ——— (i) . . . . . . G ——— (k) . . . . . . R ——— (j) . . . . . .
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There exist a unique determinant function for each $n \geq 1$. This is given by the Expansion formula

$$\det A = \sum_{\sigma \in \Sigma(n)} sgn(\sigma) a_{\sigma} = \sum_{\sigma \in \Sigma(n)} sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \tag{11}$$

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We shall now establish other properties of $\det$ which we are familiar with.

Computing the determinant

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where c_i are the diagonal entries of $G(A)$. Now if the row rank of A is $< n$ then clearly some row of $G(A)$ is zero and hence one of the $c_i = 0$. Also, $\det G(A) = 0$ and since $\det A = \pm \det G(A)$, it follows that $\det A = 0$. So, the formula (13) is still valid.

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Let us compute the determinant of such a matrix. Let us apply GEM to M. Observe that unless A has become upper triangular, the process will never involve the rows numbered $> k$.

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- *(ii) Consider an $n \times n$ matrix M for which $a_{ij} = 0, k+1 \leq i \leq n, 1 \leq j \leq k$. Thus M can written in the form*

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \tag{14}$$

where A, B, D are respectively of type $k \times k, k \times (n-k), (n-k) \times (n-k)$. Such a matrix is called a block matrix.
Let us compute the determinant of such a matrix. Let us apply GEM to M. Observe that unless A has become upper triangular, the process will never involve the rows numbered $> k$. Suppose the number of steps in $G(A)$ is less than k. Then there will be a row of zeros in $G(A)$ and hence $G(M)$ will have a zero on the diagonal.

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Now suppose the number of steps in $G(A)$ is k.

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Now suppose the number of steps in $G(A)$ is k. Then GEM applied to M will first make A into an upper triangular matrix and the rest of the operation on M will not involve the first k rows at all.

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Now suppose the number of steps in $G(A)$ is k. Then GEM applied to M will first make A into an upper triangular matrix and the rest of the operation on M will not involve the first k rows at all. Therefore, these operations can be thought of as two separate sets of row operations on A and D which correspond to operations of GEM on them. Thus we conclude that the diagonal entries of $G(M)$ are those of $G(A)$ and $G(D)$ whereas the number of row swaps for M will be the sum of these numbers for A and D separately. Thus we have proved:

Theorem

For a block matrix

$$M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

we have $\det M = (\det A)(\det D)$.

Computing Determinants

Formula (10) has some other interesting applications.

Corollary

Let a permutation σ be the composite of s transpositions. Then $sgn(\sigma) = (-1)^s$.

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Proof: First part is clear. Now observe that for any transposition τ, we have $\tau^{-1} = \tau$. Therefore, if $\sigma = \tau_1 \circ \cdots \circ \tau_k$ then $\sigma^{-1} = \tau_k \circ \cdots \circ \tau_1$. Now apply the previous lemma. ♠

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Det A^T =Det A

Theorem

For any square matrix A we have $\det A^T = \det A$.

Proof: Putting $B = A^T$ and observing that $b_{ij} = a_{ji}$ from (11) it follows that

$$\det A^T = \sum_{\sigma} sgn(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}.$$

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$$a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} = a_{1\tau(1)} \cdots a_{n\tau(n)}.$$

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Remark

This result is extremely useful. For, all the properties of the determinant function that we formulated or verified for rows will now be available for columns also.

Product formula

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Proof: Fix B and consider the map $f(A) = \det(AB)$. Then $f(I_n) = \det B$. If we show that f is multi-linear and alternating then by the uniqueness of such maps the theorem would follow. Observe that the i^{th} row of AB is nothing but A_iB. Hence if two rows of A are equal then so are the corresponding two rows of AB. Hence $f(A) = \det(AB) = 0$. This proves alternating property. Now to check multi-linearity:

$$\begin{aligned} & f(A_1, \ldots, aA_i + a'A_i', \ldots, A_n) \\ = \; & \det(A_1B, \ldots, (aA_i + a'A_i')B, \ldots, A_nB) \\ = \; & a\det(A_1B, \ldots, A_iB, \ldots, A_nB) + a'\det(A_1B, \ldots, A_i'B, \ldots, A_nB) \\ = \; & a\,f(A_1, \ldots, A_i, \ldots, A_n) + a'f(A_1, \ldots, A_i', \ldots, A_n) \end{aligned}$$

This proves the multi-linearity of f. ♠

Laplace determinant

(**Skip the proofs**)

Definition

Let A_{ij} denote submatrix (minor) of A obtained by cutting down the i^{th} row and j^{th} column. We define the $(i,j)^{th}$ cofactor of A by $cof_{i,j}(A) = (-1)^{i+j} \det A_{ij}$.

For example

$$A_{32} = \begin{bmatrix} * & \times & * & * \\ * & \times & * & * \\ \times & \times & \times & \times \\ * & \times & * & * \end{bmatrix}$$

Expansion by minors

Remark

- (i) Consider a matrix $A = ((a_{ij}))$ in which $a_{1j} = a_{i1} = 0$ for all $i, j > 1$ and $a_{11} = 1$. Then check that $\det A = cof_{11}(A) = \det A_{11}$. (This is an easy application of GEM).

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  Likewise we now swap this column with the columns left to it and bring it to the first place. The determinant now changed its sign by $(-1)^{i+j-2} = (-1)^{i+j}$. Thus if we denote this new matrix by B then $\det B = (-1)^{i+j} \det A^{\#}_{ij}$. Also $B_{11} = A_{i,j}$ Hence by (i) above we have $\det B = \det B_{11} = \det A_{ij}$.

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Thus we have proved:

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For any square matrix A we have,

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The following theorem gives an inductive formula to obtain the determinant of a matrix in terms of its entries.

Expansion by minors

Theorem

Laplace Expansion Formula For Determinant: *For any $n \times n$ matrix A we have*

$$\det A = \sum_{j=1}^{n} a_{ij} cof_{ij} A, \ \forall \ 1 \leq i \leq n. \tag{16}$$

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Remark

Laplace formula can be used to give an inductive proof of the existence of the determinant. We shall not discuss this any further.

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$(adj\ A)A = A(adj\ A) = diag(\det A, \ldots, \det A)$

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In particular if $\det A \neq 0$, then it follows that A is invertible with the inverse given by

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Remark

Cramer's rule is not good for computational purposes as it involves computation of determinants which required huge number of operations, as compared to GEM and certain modifications of GEM.