MA106-Linear Algebra

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Vector Spaces

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Pattern recognition is one of the basic tenets of all sciences and more so of mathematics.

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1. (closure under addition) For every pair of elements $\mathbf{x}, \mathbf{y} \in V$ there is a unique element $\mathbf{x} + \mathbf{y} \in V$ called the sum of $\mathbf{x}$ and $\mathbf{y}$.

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We say V is a vector space if in addition to **the closure axioms** the operations of addition and scalar multiplication satisfy the following properties.

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3. $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \ : \ a_1, \ldots, a_n \in \mathbb{R}\}$. *We refer to* $\mathbb{R}^n$ *as* **n-dimensional euclidean space**.

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Examples of Vector Spaces

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Let S be any set and $F(S, \mathbb{K})$ denote the set of all functions from S to $\mathbb{K}$. Given any two functions $f_1, f_2 : S \longrightarrow \mathbb{K}$, and a scalar $\alpha \in \mathbb{K}$ we define $f_1 + f_2$ and αf_1 by

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If the sum and scalar multiplication are defined as in the above example then we know that V becomes a vector space. This is denoted by $C[a, b]$.

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Indeed, we have

$$\cdots \subset C^{r+1} \subset C^r \subset \cdots \subset C$$

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$$\mathbb{K}[t] = \{a_0 + a_1 t + \ldots + a_n t^n \ : \ a_0, a_1, \ldots, a_n \in \mathbb{K}, \quad n \geq 0\}$$

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Since W satisfies the closure axioms and since $x \in W$ it follows that $0 \in W$.

Subspaces

Definition

A nonempty subset W of a vector space V is called a **subspace** of V if it is a vector space under the operations in V.

Theorem

A nonempty subset W of a vector space V is a subspace of V if and only if W satisfies the two closure axioms.

Proof: If W is a subspace of V then it satisfies the closure axioms.
Conversely, suppose now that W satisfies the closure axioms.
We just need to prove the existence a 'zero' element in W and the 'negatives' of elements of W.
Let $x \in W$. (Remember $W \neq \emptyset$.)
Now consider the zero element $0 \in V$.
We have $0x = 0$.
Since W satisfies the closure axioms and since $x \in W$ it follows that $0 \in W$.
Next, If $y \in W$ then $(-1)y$ is the negative of y in V. But by closure axioms, $-y \in W$. Therefore, $-y$ is the negative of y in W as well.

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None of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ *are (real) subspaces of the vector space* $\mathbb{R}$.
Neither is the set $(-1, 1)$.
Indeed the only subspaces of $\mathbb{R}$ *are* $\{0\}$ *and the* $\mathbb{R}$ *itself.*

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