MA106-Linear Algebra

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Anant R. Shastri

Department of Mathematics
Indian Institute of Technology, Bombay

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Linear Transformations

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Definition

Let V, W be any two vector spaces over $\mathbb{K}$. By a linear map (or a linear transformation) $f : V \longrightarrow W$ we mean a function f satisfying

$$f(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 f(\mathbf{v}_1) + \alpha_2 f(\mathbf{v}_2) \tag{1}$$

for all $\mathbf{v}_i \in V$ and $\alpha_i \in \mathbb{K}$.

Linear Transformations

Remark

(i) For any linear map $f : V \longrightarrow W$, we have $f(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i) = \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i)$.

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(i) For any linear map $f : V \longrightarrow W$, we have $f(\sum_{i=1}^{k} \alpha_i \mathbf{v}_i) = \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i)$.
(ii) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a finite dimensional vector space, then every element of V is a unique linear combination of these $\mathbf{v}_i$, say $\mathbf{v} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i$.

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Therefore from (i) we get

$$f(\mathbf{v}) = \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i). \tag{2}$$

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(ii) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a finite dimensional vector space, then every element of V is a unique linear combination of these $\mathbf{v}_i$, say $\mathbf{v} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i$. Therefore from (i) we get

$$f(\mathbf{v}) = \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i). \tag{2}$$

Thus, it follows that f *is completely determined by its value on a basis of* V. This simply means that if f and g are two linear maps such that $f(\mathbf{v}_i) = g(\mathbf{v}_i)$ for all elements of a basis for V, then $f = g$.

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(iii) Formula (2) can be used to define a linear transformation on a vector space, viz., by choosing its values on a given basis elements and then using (2) to define it on other elements.
Observe that, in defining f, we are free to choose the value of f on the elements of a given basis, as far as they all belong to the same vector space W. Thus if V has a basis consisting of k elements as above then for each ordered k tuple of elements of W we obtain a unique linear transformation $f : V \longrightarrow W$ and vise versa.

Isomorphisms

Definition

A linear transformation $f : V \longrightarrow W$ is called an *isomorphism* if it is invertible, i.e., there exist $g : W \longrightarrow V$ such that $g \circ f = Id_V$ and $f \circ g = Id_W$.

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Remark

It is worth recalling here that if $f : V \longrightarrow W$ is a linear bijection, then $f^{-1} : W \longrightarrow V$ is automatically linear:

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It is worth recalling here that if $f : V \to W$ is a linear bijection, then $f^{-1} : W \to V$ is automatically linear:

$$f^{-1}(\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2) = \alpha_1 f^{-1}(\mathbf{w}_1) + \alpha_2 f(\mathbf{w}_2)$$

iff

$$f(f^{-1}(\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2)) = f(\alpha_1 f^{-1}(\mathbf{w}_1) + \alpha_2 f^{-1}(\mathbf{w}_2))$$

iff

$$\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 = \alpha_1 f \circ f^{-1}(\mathbf{w}_1) + \alpha_2 f \circ f^{-1}(\mathbf{w}_2)$$

which is obvious.

Isomorphisms

Remark

1. $V \approx V$ for every vector space V.
2. $V \approx W$ implies $W \approx V$.
3. $V_1 \approx V_2$ and $V_2 \approx V_3$ implies $V_1 \approx V_3$. In other words, these these properties tell you that 'being isomorphic to' is an equivalence relation.
4. We have been using a particular isomorphism between $M_{m,n}(\mathbb{K})$ and $\mathbb{K}^{mn}$ all the time.
5. The map $A \mapsto A^T$ defines an isomorphism of $M_{m,n}(\mathbb{K})$ with $M_{n,m}(\mathbb{K})$.
6. As a vector space over $\mathbb{R}$, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$.
7. Given any two vector spaces, we would like to know whether they are isomorphic or not. For then the study of any linear algebraic property of one vector space is the same as that of the other. We shall soon see that this problem has a very satisfactory answer.

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One can easily check that $\mathcal{R}(f)$ and $\mathcal{N}(f)$ are both vector subspace of W and V respectively.
They are respectively called the *range* and the *null space* of f.

Linear Transformations

Lemma

Let $f : V \to W$ be a linear transformation. Then
(a) f is injective iff $\mathcal{N}(f) = (0)$.
(b) If $L\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = V$ then $L\{f(\mathbf{v}_1), \ldots, f(\mathbf{v}_k)\} = f(V) = \mathcal{R}(f)$.

Proof: of (a) Suppose $\mathcal{N}(f) = (0)$. Now if $\mathbf{v}_1, \mathbf{v}_2 \in V$, are such that $f(\mathbf{v}_1) = f(\mathbf{v}_2)$ then $f(\mathbf{v}_1 - \mathbf{v}_2) = 0$. This means that $\mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{N}(f) = (0)$. Hence $\mathbf{v}_1 = \mathbf{v}_2$. Conversely, if f is injective, then $\mathbf{v} \in \mathcal{N}(f)$ implies $f(\mathbf{v}) = 0 = f(0)$ and hence $\mathbf{v} = 0$.
(b) Given $\mathbf{w} \in f(V)$ pick $\mathbf{v} \in V$ such that $\mathbf{w} = f(\mathbf{v})$. Now write $\mathbf{v} = \sum_i \alpha_i \mathbf{v}_i$ and apply f on it. ♠

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Proof: (a) Let $\sum_{i=1}^{k} a_i f(\mathbf{v}_i) = 0$ where $\mathbf{v}_i \in S$. This is the same as saying $f(\sum_i a_i \mathbf{v}_i) = 0$.

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Proof: (a) Let $\sum_{i=1}^{k} a_i f(\mathbf{v}_i) = 0$ where $\mathbf{v}_i \in S$. This is the same as saying $f(\sum_i a_i \mathbf{v}_i) = 0$. Since f is injective, this is the same as saying that $\sum_i a_i \mathbf{v}_i = 0$ and since S is L. I., this is the same as saying $a_1 = \ldots = a_k = 0$.

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(b) Given $\mathbf{w} \in W$. Pick $\mathbf{v} \in V$ such that $f(\mathbf{v}) = \mathbf{w}$. Now $L(S) = V$ implies that we can write $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i$ with $\mathbf{v}_i \in S$.

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Proof: (a) Let $\sum_{i=1}^{k} a_i f(\mathbf{v}_i) = 0$ where $\mathbf{v}_i \in S$. This is the same as saying $f(\sum_i a_i \mathbf{v}_i) = 0$. Since f is injective, this is the same as saying that $\sum_i a_i \mathbf{v}_i = 0$ and since S is L. I., this is the same as saying $a_1 = \ldots = a_k = 0$.
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$\mathbf{w} = f(\mathbf{v}) = \sum_i a_i f(\mathbf{v}_i) \in L(f(S))$.
(c) Put (a) and (b) together. ♠

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Proof: Pick bases A and B for V and W respectively. Then both A and B have same number of elements. Let $f : A \longrightarrow B$ be any bijection.

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Proof: Pick bases A and B for V and W respectively. Then both A and B have same number of elements. Let $f : A \longrightarrow B$ be any bijection. Then by the above discussion f extends to a linear map $\hat{f} : V \longrightarrow W$.

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Proof: Pick bases A and B for V and W respectively. Then both A and B have same number of elements. Let $f : A \longrightarrow B$ be any bijection. Then by the above discussion f extends to a linear map $\hat{f} : V \longrightarrow W$. If $g : B \longrightarrow A$ is the inverse of $f : A \longrightarrow B$, then g also extends to a linear map $\hat{g} : W \to V$.

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Converse follows from part (c) of the above corollary. ♠

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3. We have seen that the study of linear transformations on euclidean spaces can be converted into the study of matrices. It follows that the study of linear transformations on finite dimensional vector spaces can also be converted into the study of matrices.

Rank and Nullity

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Proof: Suppose $\dim V = n$. Let $S = \{\mathbf{v}_1, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a basis of $\mathcal{N}(f)$. We can extend S to a basis $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-k}\}$ of V.

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$$T := \{f(\mathbf{w}_1), f(\mathbf{w}_2), \ldots, f(\mathbf{w}_{n-k})\}$$

is a basis of $\mathcal{R}(f)$.

Rank- Nullity

Observe that $f(S') = T$. By part (b) of the previous lemma, it follows that T spans $f(V) = \mathcal{R}(f)$.
Suppose

$$\beta_1 f(\mathbf{w}_1) + \cdots + \beta_{n-k} f(\mathbf{w}_{n-k}) = 0.$$

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$$f(\beta_1 \mathbf{w}_1 + \cdots + \beta_{n-k} \mathbf{w}_{n-k}) = 0.$$

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Then we see that $\mathcal{I}$ *is also a linear map. Moreover, we have* $D \circ \mathcal{I} = Id$. *Can you say* $\mathcal{I} \circ D = Id$? *Is* D *a one-one map? Determine the set* $\mathcal{N}(D) \subset C^r$.

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You will study them in your next Math course.

Commuting Endomorphisms

Example

Suppose $f, g : V \to V$ *are two commuting endomorphisms of a vector space* V, *i.e.,* $f \circ g = g \circ f$.
Then

$$g(\mathcal{R}(f)) \subset \mathcal{R}(f) \quad \& \quad g(\mathcal{N}(f)) \subset \mathcal{N}(f).$$

(See exercise 9.18.) This property is easily proved. It is quite useful and we shall meet it again.

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(i) f is an isomorphism.

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[Proof: If f and g denote the corresponding linear transformations then we have $f \circ g = Id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Therefore, g must be injective and f must be surjective. From the exercise above, both f and g are invertible and $f \circ g = g \circ f = Id$. Hence $AB = I_n = BA$ which means $A = B^{-1}$.]

Remark

This can also be proved directly by using GEM. First observe that $AB = Id_n$ that for every n-vector $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has a solution. This immediately implies that the no. of non zero rows in $G(A)$ is equal to m. Since we also have $m = n$, the GJEM gives the inverse of A and hence A is invertible. Of course, that B is actually the inverse of A follows easily as seen before.

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This is the same as saying that $AP = PB$, i.e., $P^{-1}AP = B$.

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The new basis elements are then expressed in terms of 'standard basis' to obtain the invertible matrix P. The matrix with respect to this new basis of the same linear map is then given by $P^{-1}AP$.

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(i) $Q_1 = \{1, x, x^2, x^3\}$.
(ii) $Q_2 = \{1, 1+x, 1+x^2, 1+x^3\}$.
(iii) $Q_3 = \{1, 1+x, (1+x)^2, (1+x)^3\}$.
Write down the transformation matrices of change of bases from (i) to (ii) and from (i) to (iii) and see how they are related.

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Solution: (i) In this case the matrix of d is given by $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

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One easily checks that $AP = PA_2$.

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One easily checks that $AP = PA_2$. In the same manner we can carry out the rest of the exercise.

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Definition

Two $n \times n$ matrices A, B are said to be similar iff there exists an invertible matrix M such that $A = MBM^{-1}$. We shall use the notation $A \sim B$ for this.

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$\gamma(A) =$ number of non zero rows in $G(A)$.

Our aim is to prove that all these numbers are equal to each other.

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$$G(A) = \begin{bmatrix} \phi & * & * & * & * & * \\ 0 & \phi & * & * & * & * \\ 0 & 0 & 0 & \phi & * & * \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Below we shall prove that this is true for all matrices.

Rank

(iii) An $n \times n$ matrix is invertible iff its determinantal rank is equal to n.
(iv) Let $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation corresponding to A. Since $\{\mathbf{e}_i\}$ form a basis for $\mathbb{R}^n$ and $C_i = f(\mathbf{v}_i)$ are the columns of A, the range of f_A is equal to $c(A)$. Hence we have:

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Theorem

The rank of the linear map associated with a given matrix A is equal to the column rank $\kappa(A)$ of A.

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(ii) *The column rank of A is equal to n.*

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(iv) *The rows of A are independent.*

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Proof: (i) $\iff$ A is invertible $\iff$ the system $A\mathbf{x} = 0$ has a unique solution.

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Proof: (i) $\iff$ A is invertible $\iff$ the system $A\mathbf{x} = 0$ has a unique solution.$\iff$ columns of A are independent $\iff$ (v) $\iff$ (*ii*). By taking the transpose, we get the proof of the equivalence of other statements. ♠

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Proof: If E is an elementary matrix, then observe that E defines an isomorphism of $c(A)$ with $c(EA)$. Hence column rank of $A = dim\, c(A) = \dim\, c(EA) =$ column rank of EA. ♠

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Proof: Let A be a matrix and $J(A)$ its associated reduced echelon form.The statement of the theorem is easily verified for $J(A)$. Since row-rak and column rank are not affected by elementary row operations, and $J(A)$ is obtained by a finite number of row operations on A, it follows that the row-rank and column rank of A are the same. ♠

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Proof: Since we have shown that row-rank and column-rank are the same, it follows that there is an $m \times m$ submatrix B of A which has all its columns independent. By a previous theorem B is invertible. Hence $\delta(A) \geq m$. Since A has m rows only $\delta(A) = m$. ♠

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Suppose there is a $d \times d$ submatrix $B = ((a_{i_p,j_q}))$ of $A = ((a_{ij}))$ which is invertible.

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Rank, Row-rank, and Column-rank

proof of theorem continued

Moreover, as seen before, since $\rho(A) = r$ there are rows

$$\{R_{i_1}, , \ldots, R_{i_r}\}$$

of A which are independent. If C is the matrix formed out of these rows then $\rho(A) = r = \kappa(A)$. This means that there are columns

$$\{C_{j_1}, \ldots, C_{j_r}\}$$

of C which are independent. If M denotes the submatrix of C fomred by these columns then M is $r \times r$ matrix which are all its columns indepedent. By a previous result M is inveritble. Since M is also a submatrix of A, this proves $d \geq r$. ♠

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It follows that an elementary row operation does not affect the determinantal rank of a matrix. Alternatively, we can try to directly prove this fact and then appeal to GEM to get a proof that the rank is equal to the row-rank and column rank of a matrix.

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We have seen that determinant is a similarity invariant. In the tutorial class, you must have seen that trace of a matrix is alos a similarity invariant. Here is another invariant.

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The rank of a matrix is a similarity invariant

Proof: Given two $n \times n$ matrices A, B we have to prove that

$$A \sim B \Longrightarrow r(A) = r(B).$$

Let M be an invertible matrix such that $B = MAM^{-1}$. Then
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Optional Exercise: On the vector space $\mathbb{K}[x]$ of all polynomials in one-variable, determine all linear maps $\phi : \mathbb{K}[x] \longrightarrow \mathbb{K}[x]$ having the property $\phi(fg) = f\phi(g) + g\phi(f)$ and $\phi(x) = 1$.

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GEM applied to Null-space and Range

Here is an algorithm to write bases for the range and null-space of a linear transformation $f : V \to W$ of finite dimensional vector spaces.

Step 1 Given a linear liear transformation $f : V \to W$ of finite dimensional vector spaces fix bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for W and write the $m \times n$ matrix A of the linear map f.

Step 2 Obtain $J(A) = ((\alpha_{ij}))$, the Gauss-Jordan form of A.

Step 3 If $k_1, k_2, \ldots, k_r$ are the indices of the **step columns** in $J(A)$, then $\{f(v_{k_1}), \ldots, f(v_{k_r})\} = \{A^{(k_1)}, \ldots, A^{(k_r)}\}$ is a basis for the range.

Step 4 Let $l_1 < \cdots < l_s$ be the indices of the **non step columns**. Define the column vectors $\mathbf{u}_{l_1}, \ldots, \mathbf{u}_{l_s}$ by:

$$\mathbf{u}_{jl_i} = \begin{cases} 1 & \text{if } j = l_i; \\ -\alpha_{pl_i} & \text{if } j = k_p, 1 \leq p \leq r; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\mathbf{u}_{l_1}, \ldots, \mathbf{u}_{l_s}\}$ forms a basis for $\mathcal{N}(f)$. To see a proof of this, you have to simply recall how we had written the set of solutions of the $A\mathbf{x} = \mathbf{b}$ and apply it for the case when $\mathbf{b} = \mathbf{0}$.

GEM applied to Null-space and Range

Example

Consider a linear map $f : \mathbb{R}^r \to \mathbb{R}^4$ *whose matrix is* A *and whose Jordan form is:*

$$G(A) = \begin{bmatrix} 1 & 0 & 5 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & -4 & -7 \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\{f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_4)\} = \{A^{(1)}, A^{(2)}, A^{(4)}\}$ *is basis for* $\mathcal{R}(f)$ *where* $A^{(j)}$ *denotes the* j^{th} *column of* A.
Also $\{(-5,-2,1,0,0,0)^t, (-1,-2,0,4,1,0)^t, (-1,-3,0,7,0,1)^t\}$ *is a basis for* $\mathcal{N}(f)$.