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EXERCISE BANK

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1 Review of Vector Algebra

1. a) Show that:
   (i) $\|\mathbf{a}\times\mathbf{b}\|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2$; (ii) $\mathbf{a}\cdot\mathbf{b} \le \|\mathbf{a}\|\,\|\mathbf{b}\|$.
   b) Let $(\mathbf{abc})$ denote the scalar triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. Show that the modulus of $(\mathbf{abc})$ is equal to the volume of the parallelepiped spanned by the three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. Prove that
   $$(\mathbf{abc}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$
   Hence or otherwise, show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

2. Prove the identities:
   i) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.
   ii) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$. [Hint: First prove (i) for $\mathbf{i},\mathbf{j},\mathbf{k}$ in place of $\mathbf{a}$.]
   Now use the above identities to prove the **Jacobi** identity:
   iii) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}$.
   Also prove the following identities:
   iv) $(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = (\mathbf{abc})^2$.
   v) $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) = \mathbf{0}$.

3. Given non zero vectors $\mathbf{a},\mathbf{b}$ express $\mathbf{a}$ as $\mathbf{b}_1 + \mathbf{b}_2$ where $\mathbf{b}_1$ is parallel to $\mathbf{b}$ and $\mathbf{b}_2$ is perpendicular to $\mathbf{b}$.

4. If $\mathbf{a}$, $\mathbf{b}$ are fixed vectors with $\mathbf{a}\neq\mathbf{0}$, prove that every solution of $\mathbf{a}\times\mathbf{y} = \mathbf{b}$ is of the form $\mathbf{y} = \dfrac{\mathbf{b}\times\mathbf{a}}{|\mathbf{a}|^2} + c\,\mathbf{a}$ for some scalar c.

5. Let $\triangle = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.
   Prove that $|\triangle| \le k_1k_2k_3$ where $k_i = \sqrt{a_{i1}^2 + a_{i2}^2 + a_{i3}^2}$, $i = 1,2,3$.

6. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three non-coplanar vectors, prove that the only vector $\mathbf{r}$ satisfying
   $$\mathbf{r}\cdot\mathbf{a} = \mathbf{r}\cdot\mathbf{b} = \mathbf{r}\cdot\mathbf{c} = 1$$

is $\dfrac{\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}}{(\mathbf{a}\,\mathbf{b}\,\mathbf{c})}$.

7. A ray of light strikes a plane mirror and gets reflected. Given that the unit incident ray is $\mathbf{a}$ and the unit normal to the mirror is $\mathbf{n}$, find the reflected ray.

8. Let $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3, \mathbf{u} \neq \mathbf{0}$ be vectors such that whenever $\mathbf{w} \in \mathbb{R}^3$ and $\mathbf{w}$ is orthogonal to $\mathbf{u}, \mathbf{w}$ is orthogonal to $\mathbf{v}$. Show that there exists $\lambda \in \mathbb{R}$ such that $\mathbf{v} = \lambda\mathbf{u}$.

9. Using vector methods, show that the distance between two non parallel lines ℓ_1 and ℓ_2 is given by
$$d = \frac{|(\mathbf{v}_2 - \mathbf{v}_1)\cdot(\mathbf{a}_1 \times \mathbf{a}_2)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|},$$
where $\mathbf{v}_1, \mathbf{v}_2$ are any two points on ℓ_1 and ℓ_2 respectively, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the directions of ℓ_1 and ℓ_2.

   [Hint: Consider the plane through ℓ_2 which is parallel to ℓ_1. Show that the unit vector $\mathbf{a}_1 \times \mathbf{a}_2/\|\mathbf{a}_1 \times \mathbf{a}_2\|$ is normal to this plane; now project $\mathbf{v}_2 - \mathbf{v}_1$ onto the normal direction.]

2 Euclidean Space

1. Write down the composites of two of the following functions, whenever it makes sense.
   a) $f(x, y) = x^2 + y^2$ d) $\phi(x, y) = (y^2, x - y)$
   b) $g(x) = \sin x$ e) $\psi(x, y, z) = (xyz, \sin y, |z|)$.
   c) $h(x, y) = (y, -x)$ f) $\eta(x, y, z) = (e^x, \sin y, \cos(xy))$

2. Show that the standard distance function $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum(x_i - y_i)^2}$ on $\mathbb{R}^n$ satisfies
   (d1) *symmetry:* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$;
   (d2) *triangle inequality:* $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{w}) + d(\mathbf{w}, \mathbf{y})$;
   (d3) *positivity:* $d(\mathbf{x}, \mathbf{y}) \geq 0$ and $= 0$ if and only if $\mathbf{x} = \mathbf{y}$.
   Also show that δ and D defined below satisfy the metric properties (d1), (d2), (d3) in place of d.
$$\delta(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|; \quad D(\mathbf{x}, \mathbf{y}) = max\,\{|x_i - y_i| \ : \ 1 \leq i \leq n\}.$$

3. On the set of 64 squares of a chess board define the distance d from one square to another to be the least number of knight moves required. Check that this distance function satisfy (d1), (d2), (d3). Determine the diameter of the chess board with respect to this distance where diameter of a space is the supremum of all values of d.

4. Here is an example to show that it is not advisable to rely too much on intuition when dealing with higher dimensions. Consider a square of side 4 units and place four coins of unit radius one in each corner so as to touch two of the sides. Of course each coin touches two other coins. Now place a coin at the center of the square so as to touch all the four coins. What is the radius of the central coin?

   Do this inside a n-dimensional cube of side 4 units with n-dimensional balls of unit radius. You will need 2^n of them. For $n \leq 9$, the central ball which is kept so as to touch all the balls in the corner lies inside the cube. The surprise is that for $n > 9$ the central ball cannot fit inside the cube. Prove this by showing that the radius of the central ball is $\sqrt{n} - 1$.

3 Linear Maps on Euclidean Spaces

1. Show that if f is a linear map then $f(\sum_{i=1}^{k} \alpha_i \mathbf{x}_i) = \sum_{i=1}^{k} \alpha_i \mathbf{x}_i$.

2. Determine which of the following functions are linear.

   (a) $f(x) = x^2$. (b) $g(x) = 5x + 1$. (c) $h(x) = |x|$.

   (d) $p(x, y) = xy$. (e) $q(x, y) = (x, x)$. (f) $r(x, y) = (x, -y)$.

   (g) $s(x, y) = (2x, 3y)$. (h) $t(x, y, z) = (y, z, x)$.

3. Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $f(1,0,0) = f(0,1,0) = f(0,0,1) = f(1,1,1)$. What is $f(1,2,3)$?

4. Show that a linear map $g : \mathbb{R} \longrightarrow \mathbb{R}^3$ is determined by an ordered triple of real numbers.

5. Show that rotation through an angle θ at the origin defines a linear isomorphism of the plane.

6. Show that the reflection in a line passing through the origin defines a linear isomorphism of the plane.

7. By a rigid motion of $\mathbb{R}^n$ we mean a map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ which preserve the distance between any two points. For $n = 2, 3$ show that
   (a) Composite of rigid motions is a rigid motion.
   (b) A rigid motion is a one-one mapping.
   (c) Adding a vector $\mathbf{v}$ is a rigid motion (called *translation* by $\mathbf{v}$.)
   (d) Rotation through a fixed angle about a point is a rigid motion of the plane.
   (e) Rotation through a fixed angle about a line is a rigid motion of the 3-space.
   (f) Reflection in a line is a rigid motion of the plane.
   (g) Reflection in a plane is a rigid motion of the 3-space.

8. Let f be a rigid motion in the plane $\mathbb{R}^2$ which fixes the origin. Show that it is a rotation or the composite of a rotation and a reflection and hence is a linear map. Prove a similar statement for $\mathbb{R}^3$.

9. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function, $f = (f_1, \ldots, f_m)$. Show that f is linear iff each f_i is linear.

4 Matrix Representation

1. Examine whether the following transformations are linear or not. In case of linear transformations, write down their matrix representation:

(i) $T_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ such that $T(x_1, x_2, x_3, x_4)^t = (x_1, x_2)^t$.
(ii) $T_1 : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that $T(x_1, x_2, x_3, x_4)^t = (x_1, x_2, 0, 0)^t$
(iii) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ such that $T(x_1, x_2)^t = (3x_1 + x_2, 0, 0)^t$.
(iv) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $T(x_1, x_2, x_3)^t = (ax_1, bx_2, cx_3)^t$.

2. Describe the geometric meaning of the linear transformations defined by the following matrices:
(i) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ (v) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

5 Matrix Operations

1. Let A, B be matrices of type $n \times n$. Show that $A = B$ iff $\mathbf{e}_i^t A \mathbf{e}_j = \mathbf{e}_i^t B \mathbf{e}_j$ for all i, j where $\mathbf{e}_i$ are standard column vectors $= (0, \ldots, 0, 1, 0, \ldots, 0)^t$.

2. Given square matrices $A_1, \ldots, A_n$ of the same size, prove that:
(i) $(A_1 \cdots A_n)^t = A_n^t \cdots A_1^t$.
(ii) $(A_1 \cdots A_n)^{-1} = A_n^{-1} \cdots A_1^{-1}$.
(iii) $(A_1^t)^{-1} = (A^{-1})^t$.
(iv) $((A_1 \cdots A_n)^{-1})^t = (A_1^{-1})^t \cdots (A_n^{-1})^t$.
(v) If $A_1 A_2 = A_2 A_1$ then $A_1^k A_2^l = A_2^l A_1^k$ for positive integers k, l.

3. If A is an $m \times n$ matrix and B an $n \times p$ matrix, then show that the ith row of the product AB is equal to the product of the ith row vector A_i of A with B.

4. For any θ, let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Prove that for any two real numbers α, β, we have, $R_{\alpha+\beta} = R_\alpha R_\beta$.

5. A matrix is called *symmetric* if $A^t = A$. Let A and B be symmetric matrices of same size. Show that AB is a symmetric matrix iff $AB = BA$.

6. The (complex) *conjugate* $\overline{A}$, of a $m \times n$ matrix $A = ((a_{ij}))$ with complex entries is defined as the matrix $((\overline{a_{ij}}))$. Prove that $\overline{A + B} = \overline{A} + \overline{B}$, $\overline{\alpha A} = \overline{\alpha}\overline{A}$, and

$\overline{AB} = \overline{A}\ \overline{B}$, where the complex matrices A, B are of appropriate sizes and α is a complex number.

7. The *trace* of a square matrix $A = ((a_{ij}))$ is defined as the sum of its diagonal elements, i.e., $tr\, A := \sum_i a_{ii}$. Prove that if A, B are square matrices of the same order and α, β are scalars then
   (i) $tr\,(\alpha A + \beta B) = \alpha\, tr\,(A) + \beta\, tr\,(B)$; (ii) $tr(AB) = tr(BA)$;
   (iii) If A is invertible, then $tr\,(ABA^{-1}) = tr\,(B)$.

8. A square matrix A is called *nilpotent* if $A^m = 0$ for some positive integer m. Show that a $n \times n$ matrix $A = ((a_{ij}))$ in which $a_{ij} = 0$ if $i \geq j$ is nilpotent.

9. Show that the sum and the product of any two diagonal matrices of the same order are diagonal.

10. The *conjugate transpose* or (*Hermitian*) *adjoint* A^* of a complex $m \times n$ matrix A is defined as the transpose of its conjugate (or equivalently, the conjugate of its transpose). Prove that the properties of the adjoint are analogous to that of the transpose e.g. $(A + B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$. Note, however, that $(\alpha A)^* = \overline{\alpha} A^*$.

11. A square matrix A is called *Hermitian* if $A^* = A$ and *skew-Hermitian* if $A^* = -A$. Show that every square matrix can be uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.

12. If A and B are $n \times n$ Hermitian matrices and α and β are any real numbers, show that $C = \alpha A + \beta B$ is a Hermitian matrix.

13. If A and B are skew Hermitian matrices and α, β are real numbers, show that $\alpha A + \beta B$ is skew Hermitian. What happens if α and β are allowed to be complex numbers?

14. Let A, B be nilpotent matrices of the same order.
(i) Show by an example that $A+B, AB$ need not be nilpotent.
(ii) However, prove that this is the case if A and B commute with each other (i.e., if $AB = BA$). (Hint: In this case, show that the binomial theorem holds for expansion of $(A+B)^n$.)

15. A square matrix A over $\mathbb{C}$ is called **unitary** if $AA^* = Id = A^*A$. In addition, if A has real entries then it is called **orthogonal.** (This is the same as saying $AA^T = Id = A^TA$. Show that A, B are $n \times n$ unitary (orthogonal) matrices then AB is unitary (orthogonal).

16. A square matrix A with complex entries is called **normal** if $AA^* = A^*A$. Determine all 2×2 normal matrices. In particular, show that there are normal matrices which are neither unitary, Hermitian, skew-Hermitian, symmetric, nor skew-symmetric.

17. Given a polynomial $p(x)$ with real or complex coefficients, and a square matrix A, let p(A) denote the matrix obtained by 'substituting' A for the variable x. Thus if $p(x) = a_0 + a_1x \cdots a_kx^k$ then $p(A) = a_0I_n + a_1A + \cdots A_kA^k$. Given two polynomials p, q, show that
(i) $(p+q)(A) = p(A) + q(A)$
(ii) $(pq)(A) = p(A)q(A)$

18. Show that any diagonal matrix D whose only possible entries are 0 and 1 is idempotent, i.e., $D^2 = D$.

19. If $A^2 = A$, prove that $(A+I)^n = I + (2^n - 1)A$.

20. The matrix $A = \begin{bmatrix} a & i \\ i & b \end{bmatrix}$, where $i^2 = -1, a = \frac{1}{2}(1+\sqrt{5})$, and $b = \frac{1}{2}(1-\sqrt{5})$, has the property $A^2 = A$. Describe completely all 2×2 matrices A with complex entries such that $A^2 = A$.

21. A square matrix $A = ((a_{ij}))$ is called *upper triangular* if $a_{ij} = 0$ for all $j < i$. A *lower triangular* matrix is defined similarly (or equivalently, as the transpose of an upper triangular matrix). Prove that the sum as well as the product of two upper

triangular matrices (of equal orders) is upper triangular.

22. Show that $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$, for all $\lambda \in \mathbb{R}, n \geq 1$.

23. An $m \times n$ matrix, all whose entries are 1 is often denoted by $J_{m\times n}$ or simply by J if m, n are understood. Let $A = J_{n\times n}$ and $B = J_{n\times 1}$. Prove that $AB = nB, A^2B = n^2B, \ldots$ and, in general, for any polynomial $p(x) = a_0 + a_1x + \ldots + a_rx^r, p(A)B = p(n)B$.

24. If $I_n + A$ is invertible show that $(I_n + A)^{-1}$ and $I_n - A$ commute.

25. A *Markov* (or *stochastic*) matrix is a square matrix $((a_{ij}))$ such that $0 \leq a_{ij} \leq 1$, and $\sum_{j=1}^{n} a_{ij} = 1$, for all $i = 1, 2 \ldots, n$. Prove that the product of two Markov matrices is a Markov matrix.

26. Let $N_{n\times n}$ be an upper triangular matrix with diagonal entries zero. Show that $(I_n + N)^{-1} = I - N + N^2 - + \cdots + (-1)^{n-1}N^{n-1}$.

27. Show that if N is any nilpotent matrix then $I - N$ is invertible.

28. If A and B are square matrices, show that $I - AB$ is invertible iff $I - BA$ is invertible. [Hint: Start from $B(I - AB) = (I - BA)B$.]

29. Find the inverses of the permutation matrices:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

30. Let $N = \{1, 2, \ldots, n\}$. By a permutation on n letters we mean a bijective mapping $\sigma : N \longrightarrow N$. Given a permutation $\sigma : N \longrightarrow N$ define the permutation matrix P_σ to be the $n \times n$ matrix $((p_{ij}))$ where

$$p_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $P_{\sigma\circ\tau} = P_\tau P_\sigma$. Deduce that all permutation matrices are invertible and

$$P_\sigma^{-1} = P_{\sigma^{-1}} = P_\sigma^T.$$

31. Show that if A, B are $n \times n$ matrices and f_A, f_B are associated linear maps, then $f_A^{-1} = f_B$ iff $A^{-1} = B$.

32. For each (i, j), let $\mathbf{e}_{ij}$ denote $n \times n$ matrix whose $(i, j)^{th}$ entry is equal to 1 and all other entries are zero. Let A be any $n \times n$ matrix. Examine the effect of multiplication (i) on the left; (ii) on the right by $I_n + \alpha\mathbf{e}_{ij}$ on A.

6 Gauss Elimination

1. Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM

(i)
$$\begin{array}{rrrrrr} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = 8 \end{array}$$

(ii)
$$\begin{array}{rrrrr} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

(iii)
$$\begin{array}{rrrrr} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(iv)
$$\begin{array}{rrrrr} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

2. Prove that every invertible 2×2 matrix is a product of at most four elementary matrices.

3. Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, E_2, \ldots, E_n$ such that $E_n \ldots E_1 A$ is either the identity matrix or its bottom row is zero.

4. The nth **Hilbert matrix** H_n is defined as the $n \times n$ matrix whose $(i, j)^{\text{th}}$ entry is $\dfrac{1}{i + j - 1}$. Obtain H_3^{-1} by Gauss-Jordan elimination Method (GJEM).

5. Find the inverse of the following matrix using elementary row-operations:
$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}$.

6. Compute the last row of the inverse of the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & -17 & 1 & 2 \\ 4 & -17 & 8 & -5 \\ 0 & -5 & -2 & 1 \end{bmatrix}.$$

7 Determinant

1. Adopt the GEM to obtain an algorithm to determine the determinant of a square matrix A.

2. Compute the inverse of the following matrices using JGEM and the formula in terms of adjoint separately and compare the results.

$$\text{(i)} \begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix} \qquad \text{(ii)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

3. Solve the following systems by Cramer's rule:

$$\text{(i)} \quad \begin{aligned} -x + 3y - 2z &= 7 \\ 3x + y + 3z &= -3 \\ 2x + y + 2z &= -1 \end{aligned} \qquad \text{(ii)} \quad \begin{aligned} 4x + y - z &= 3 \\ 3x + 2y - 3z &= 1 \\ -x + y - 2z &= -2 \end{aligned}$$

4. Calculate the determinant of the following matrices :

$$\text{(a)} \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & \iddots & & & \\ 1 & & & & \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 2 & 3 & \dots & n \\ 3 & 3 & 3 & \dots & n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n & n & \dots & n \end{bmatrix}.$$

(c) $$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

5. (Vandermonde determinant)

   (a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

   (b) Prove an analogous formula for $n \times n$ matrices by using row operations to clear out the first column.

6. Write down in full the expansion formula for the determinant of any $n \times n$ matrix $A_n = ((a_{ij}))$, where $n = 3, 4$.

8 Vector Spaces, Subspaces, Bases and Dimension

1. Let $L(S)$ denote the subspace spanned by a subset S of a vector space V. Prove that if $S \subseteq T \subseteq V$ and T is a subspace of V, then $L(S) \subseteq T$. (That is $L(S)$ is the smallest subspace of V which contains S).

2. Given a set of n linearly independent vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V, show that for any scalar α, the set $\{\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{i-1}, \mathbf{v}_i + \alpha\mathbf{v}_j, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$ with $i \neq j$ is linearly independent.

3. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices with entries in $\mathbb{C}$. Show that $M_n(\mathbb{C})$ is a vector space over $\mathbb{C}$ and $\{E_{(i,j)}, 1 \leq i, j \leq n\}$, where $E_{(i,j)}$ denote $n \times n$ matrix with 1 at $(i,j)^{\text{th}}$ place and 0 elsewhere is a basis of it.

4. Examine whether the following sets of vectors constitute a vector space. If so, find the dimension and a basis of that vector space.

(a) The set of all (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ such that

(i) $x_4 = 0$; (ii) $x_1 \leq x_2$; (iii) $x_1^2 - x_2^2 = 0$; (iv) $x_1 = x_2 = x_3 = x_4$; (v) $x_1 x_2 = 0$.

(b) The set of all real functions of the form $a\cos x + b\sin x + c$, where a, b, c vary over all real numbers.

(c) Homogeneous polynomials in two variables of degree 3 together with the zero polynomial.

(d) The set of all $n \times n$ real matrices $((a_{ij}))$ which are:

(i) diagonal; (ii) upper triangular; (iii) having zero trace; (iv) symmetric; (v) anti-symmetric (i.e., those satisfying $A^t = -A$;) (vi) invertible.

(e) The set of all real polynomials of degree 5 together with the zero polynomial.

(f) The set of all complex polynomials of degree ≤ 5 with $p(0) = p(1)$ together with the zero polynomial.

(g) The real functions of the form $(ax + b)e^x, \quad a, b \in \mathbb{R}$.

5. Consider the following subsets of the space $M_n(\mathbb{C})$ of $n \times n$ matrices over complex numbers:

(a) The space $Sym_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \ : \ A = A^T\}$ of **symmetric matrices**.

(b) The space $Herm_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \ : \ A = A^*\}$ of **Hermitian matrices**.

(c) The space $Skew_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \ : \ A = -A^*\}$ of **skew-Hermitian Matrices**.

Show that each of them is a $\mathbb{R}$ vector subspace of $M_n(\mathbb{C})$ and compute their dimension by explicitly writing down a basis for each of them. Is any one of them a complex vector subspace?

6. Given $A \in M_n(\mathbb{C})$, show that the mappings

$$\alpha_A(B) = \frac{1}{2}(AB + BA^*); \quad \beta_A(B) = \frac{1}{2i}(AB - BA^*)$$

define $\mathbb{R}$-linear maps $\mathrm{HERM}_n(\mathbb{C}) \to \mathrm{HERM}_n(\mathbb{C})$.

Also show that α_A, β_A commute with each other.

7. Let $P_n[x]$ denote the vector space consisting of the zero polynomial and all real polynomials of degree $\leq n$, where n is fixed. Let S be a subset of all polynomials $p(x)$ in $P_n[x]$ satisfying the following conditions. Check whether S is a subspace; if so, compute dimension of S.

(i) $p(0) = 0$; (ii) p is an odd function; (iii) $p(0) = p''(0) = 0$.

8. Examine whether the following subsets of the set of real valued functions on $\mathbb{R}$ are linearly dependent or independent. Compute the dimension of the subspace spanned by each set
(a) $\{1+t, (1+t)^2\}$; (b) $\{x, |x|\}$.

9. Examine whether the following sets are linearly independent.
(a) $\{(a,b),(c,d)\} \subset \mathbb{R}^2$, with $ad - bc \neq 0$.
(b) $\{(1+i, 2i, 2), (1, 1+i, 1-i)\}$ in $\mathbb{C}^3$.
(c) For $\alpha_1, \ldots, \alpha_k$ distinct real numbers, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ where $\mathbf{v}_i = (1, \alpha_i, \alpha_i^2, \ldots, \alpha_i^{k-1})$.
(d) $\{e^{\alpha_1 x}, e^{\alpha_2 x}, \cdots, e^{\alpha_n x}\}$ for distinct real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.
(e) $\{1, \cos x, \cos 2x, \ldots, \cos nx\}$.
(f) $\{1, \sin x, \sin 2x, \ldots, \sin nx\}$.
(g) $\{e^x, xe^x, \ldots, x^n e^x\}$.

10. Let $\alpha_1, \alpha_2, \alpha_3$ be fixed real numbers. Show that the vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_4 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$ forms a subspace, which is spanned by $(1, 0, 0, \alpha_1)$, $(0, 1, 0, \alpha_2)$ and $(0, 0, 1, \alpha_3)$. Find the dimension of this subspace.

11. Show that the only possible subspaces of $\mathbb{R}^3$ are the zero space $\{0\}$, lines passing through origin, planes passing through origin and the whole space.

12. Let $C[a,b]$ be the space of all continuous real valued functions. Let $\phi : C[-1,1] \longrightarrow C[0,1]$ be the *restriction map* i.e., $\phi(f) = f|_{[0,1]}$.
(i) Show that $\{x, |x|\}$ is a linearly independent set in $C[-1,1]$.
(ii) Show that ϕ is a linear map.
(iii) Is $\{\phi(x), \phi(|x|)\}$ a linearly independent set in $C[0,1]$?
(iv) Is ϕ injective?
(v) Is ϕ surjective?

13. Let $V = C^\infty[0,1]$ denote the space of all smooth real valued functions on $[0,1]$. Define $\psi : V \longrightarrow \mathbb{R}^2$ by the formula $\psi(f) = (f'(0), f'(1))$. Determine all functions f which belongs to the null space of ψ and are convex upward throughout $[0,1]$.

9 Linear Maps

1. Let f be a bijective linear map. Show that the inverse is also linear.

2. Let $f : V \longrightarrow W$ be a linear transformation. Put

$$\mathcal{R}(f) := f(V) := \{f(v) \in W \ : \ v \in V\},\ \mathcal{N}(f) := \{v \in V \ : \ f(v) = 0\}.$$

(i) Show that $\mathcal{R}(f)$ and $\mathcal{N}(f)$ are both vector subspace of W and V respectively. (They are respectively called the *range* and the *null space* of f. The dimensions of these two spaces are respectively called the rank and nullity of f.)
(ii) $dim\, \mathcal{R}(f) \leq dim\, V$ where V is the domain of f.
(iii) Rank of $f+$ nullity of f is equal to $dim\, V$.

3. Let $T : U \longrightarrow V$ be a linear transformation. Show that
(i) T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$.
(ii) If $L(\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}) = U$ then $\mathcal{R}(T) = L(\{T(\mathbf{u}_1), T(\mathbf{u}_2), \cdots, T(\mathbf{u}_n)\})$.

4. Let $f : V \longrightarrow W$ be a linear transformation.
(a) Suppose f is injective and $S \subset V$ is linearly independent. Then show that $f(S)$ is linearly independent.
(b) Suppose f is onto and S spans V. Then show that $f(S)$ spans W.
(c) Suppose S is a basis for V and f is an isomorphism then show that $f(S)$ is a basis for W.

5. Let V be a finite dimensional vector space and $f : V \longrightarrow V$ be a linear map. Prove that the following are equivalent:
(i) f is an isomorphism.
(ii) f is injective.
(iii) f is surjective.
(iv) there exist $g : V \longrightarrow V$ such that $g \circ f = Id_V$.
(v) there exists $h : V \longrightarrow V$ such that $f \circ h = Id_V$.

6. Prove that linear independence is preserved under one-to-one linear transformations.

7. Let U , V be finite dimensional vector spaces having equal dimension. Prove that a linear transformation $T : U \longrightarrow V$ is onto only if it is one-to-one.

8. Consider the linear transformations $T_1 : U \longrightarrow V$ and $T_2 : V \longrightarrow W$. If T_2 is one-one then show that $rank(T_2 \circ T_1) = rank(T_1)$.

9. Solve the problem nos. 8 and 18 of section 5 using linear transformations.

10. Let A and B be any two $n \times n$ matrices and $AB = I_n$. Show that both A and B are invertible and they are inverses of each other.

11. Obtain the Gauss form of the following matrices. Use them to find rank and nullity of the matrix. Also write down a basis for the range. Finally obtain the Gauss-Jordan form and use to write down a basis for the null space.

(i) $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & 0 \\ 2 & -3 & 1 \\ 5 & 1 & 1 \end{bmatrix}$.

12. Define $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by $f((x_1, x_2, x_3, x_4, x_5)^t)$
$= (2x_3 - 2x_4 + x_5, 2x_2 - 8x_3 + 14x_4 - 5x_5, x_2 + 3x_3 + x_5)^t$.
Find bases for the null-space and the range of f, using GJEM.

13. Find the range and null-space of the following linear transformations. Also find the rank and nullity wherever applicable.
    (a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2)^t = (x_1 + x_2, x_1)^t$.
    (b) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2)^t = (x_1, x_1 + x_2, x_2)^t$.
    (c) $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^t = (x_1 - x_4, x_2 + x_3, x_3 - x_4)^t$.
    (d) $T : C(0, 1) \longrightarrow C(0, 1)$ defined by $T(f)(x) = f(x) \sin x$.
    (e) $T : C^1(0, 1) \longrightarrow C(0, 1)$ defined by $T(f)(x) = f'(x)e^x$.

14. Find a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that the set of all vectors satisfying $4x_1 - 3x_2 + x_3 = 0$ is – (i) the null-space of T, (ii) the range of T.

15. Show that each of the following linear transformations is nonsingular and find its inverse.
(i) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (5x_1, 3x_2)$
(ii) $T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_3 + x_2, x_3)$.

16. Let $\mathcal{P}[x]$ denote the space of all real polynomials in one variable. Let $V = \{p(x) \in \mathcal{P}[x] \ : \ p(0) = 0\}$. Prove that taking the derivative defines a one-to-one linear transformation from $D : V \longrightarrow \mathcal{P}$ and $D^{-1}(p)(x) = \int_0^x p(t)\, dt$.

17. Examine whether the following transformations are linear or not. In case of linear transformations, write down their matrix representation with respect to the standard bases.
(i) $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ such that $T(x_1 + iy_1, x_2 + iy_2) = (x_1, x_2)$.
(ii) $T : \mathbf{P}_3 \longrightarrow \mathbf{P}_3$ such that $T(a_0 + a_1x + a_2x^2 + a_3x^3) = -a_0 + 2a_1x + 3(a_0 - a_1)x^2$.
(iii) $T : \mathbf{P}_3 \longrightarrow \mathbf{P}_4$ such that $T(p(x)) = xp(x) + \int_0^x p(t)\, dt$.
(iv) $T : \mathbf{P}_3 \longrightarrow \mathbf{P}_3$ such that $T(p) = p'$.

18. Let $f, g : V \to V$ be two linear maps which commute with each other, i.e., $f \circ g = g \circ f$. Show that $f(\mathcal{R}(g)) \subset \mathcal{R}(g)$, and $f(\mathcal{N}(g)) \subset \mathcal{N}(g)$.

10 Rank of a Matrix

1. Let A be a $m \times n$ matrix and let f_A be the linear transformation from $\mathbb{R}^n \ \to \ \mathbb{R}^m$ induced by A. Show that column-rank (A) = dim (Range (f_A))=rk f_A.

2. Prove that under elementary row operations, the linear dependence and independence of columns and hence the column rank of a matrix are unaffected. Do similar assertions hold for linear dependence of rows and for the row rank ? Justify your answers. Finlay, does this apply to the rank of a matrix also?

3. Define three different types of 'rank' of a matrix and show that they are all equal.

4. Let A, B be $n \times n$ matrices. If A is invertible, then show that $rk\,(AB) = rk\,B = rk\,(BA)$.

5. Two $n \times n$ matrices A, B are said to be **similar** if there exists a non-singular matrix C (which will, in general, depend on both A and B) such than $B = C^{-1}AC$. Prove that:
   (i) Similarity is an equivalence relation.
   (ii) Similar matrices have equal traces. (A more fancy way to express this is to say that the trace is a similarity invariant.)
   (iii) Similar matrices have equal ranks.

6. Give an example of two square matrices A, B (of equal orders) such that $rk(A) = rk(B)$ but $rk(A^2) \neq rk(B^2)$.

7. Let A be a $m \times n$ matrix and B be a $n \times p$ matrix. Using linear transformations show that $rk(AB) \leq Min\{rk(A), rk(B)\}$.

8. Let A, B be $n \times n$ matrices. We say A is a left inverse of B and B is a right inverse of A if $AB = I_n$. Show that
   (i) If A has a left inverse then rk $(A) = n$ and hence A is invertible.
   (ii) If B has a right inverse then $rk\,(B) = n$ and hence B is invertible.

9. Let F be a finite field with q elements. If V is vector space of dimension n over F then show that $\#(V) = q^n$.

10. Let $GL(n, q)$ denote the group of all invertible $n \times n$ matrices with entries in a field with q elements. Show that $\#(GL(n, q)) = (q^n - 1))(q^n - q) \cdots (q^n - q^{n-1})$.

11. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ of dimension $n > 2$. Show that if U_j are some finitely many subspaces of V each of dimension $< n$ then $\cup_j U_j \neq V$.

11 Eigenvalues and Eigenvectors

1. Given a monic polynomial $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$, we define a $n \times n$ matrix

$$C(p) := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ -a_n & -a_{n-1} & \cdot & \cdots & -a_1 \end{bmatrix}$$

   $C(p)$ is called the companion matrix of p. Show that the characteristic polynomial of $C(p)$ is equal to $(-1)^n p$.

2. For a $n \times n$ matrix A over $\mathbb{C}$, let α_A and β_A be as in exercise 8.6. Suppose they have a common eigenvector. Then show that A has an eigenvalue in $\mathbb{C}$.

3. Use the fact that every linear map $f : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ has a real eigenvalue for all m to show that any two commuting linear maps $\alpha, \beta : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ have a common eigenvector. (Use induction and subspaces kernel and image of $\alpha - \lambda I_n$ where λ is an eigenvalue of α.)

4. Use the fact that every linear map $f : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m+1}$ has a real eigenvalue for all m to show that every $\mathbb{C}$-linear map $\mathbb{C}^{2n+1} \to \mathbb{C}^{2n+1}$ has an eigenvalue.

5. Given $A \in M_n(\mathbb{K})$, show that

$$\phi_A : B \mapsto \frac{1}{2}(AB + BA^t); \quad \psi_A : B \mapsto ABA^t$$

   define commuting endomorphisms of the space of all symmetric matrices $Sym_n(\mathbb{K})$. Show that if B is a common eigenvector of ϕ_A, ψ_A then $(A^2 + aA + b)B = 0$ for some $a, b \in \mathbb{K}$. Further if $\mathbb{K} = \mathbb{C}$, conclude that A has an eigenvalue.

6. Let $S_1(\mathbb{K}, r)$ be the statement that every linear map $f : \mathbb{K}^n \to \mathbb{K}^n$ has an eigenvalue for all n not divisible by 2^r. Let $S_2(\mathbb{K}, r)$ denote the statement that any two commuting linear maps $\mathbb{K}^n \to \mathbb{K}^n$ have a common eigenvector for all n not divisible by 2^r.
   (a) Show that $S_1(\mathbb{C}, r)$ for all r is equivalent to FTA.

(b) Show that intermediate value theorem implies $S_1(\mathbb{R}, 1)$.
(c) Show that $S_1(\mathbb{K}, r) \implies S_2(\mathbb{K}, r)$.
(d) Show that $S_2(\mathbb{R}, 1) \implies S_1(\mathbb{C}, 1)$.
(e) Show that $S_2(\mathbb{C}, r) \implies S_1(\mathbb{C}, r+1)$.
(f) Prove FTA.

7. Prove that
(a) the *trace* of a square matrix is equal to the sum of its *eigenvalues*.
(b) the determinant of a square matrix is equal to the product of its *eigenvalues*.

8. If the $n \times n$ matrices A and $\tilde{A}$ are **similar** (i.e. $\tilde{A} = P^{-1}AP$ for some nonsingular $n \times n$ matrix P) then show that
(i) A and $\tilde{A}$ have same eigenvalues;
(ii) if $\mathbf{v}$ is an eigenvector of A then $P^{-1}\mathbf{v}$ is an eigenvector of $\tilde{A}$.

9. Show that a $n \times n$ matrix A is diagonalizable iff $\mathbb{K}^n$ has a basis consisting of eigenvalues of A only.

10. Let A be $n \times n$ matrix. Suppose $\lambda_1, \cdots, \lambda_k$ are the only distinct *eigenvalues* of A. Show that A is diagonalizable if and only if the geometric multiplicity of λ_i is equal to its algebraic multiplicity $\forall i = 1, \cdots, k$.

11. Examine whether the following matrices can be diagonalised. If yes, find P such that $P^{-1}AP$ is diagonal.

(i) $\begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$.

12. Let a $n \times n$ matrix B have n distinct *eigenvalues*. Show that every $n \times n$ matrix A such that $AB = BA$ is diagonalizable.

13. Let A be $n \times n$ matrix. Prove that :
(i) 0 is an *eigenvalue* of A if and only if A is singular.

(ii) if λ is an *eigenvalue* of A then it is also an *eigenvalue* of A^t (where A^t denotes the transpose of A).
(iii) If x is an *eigenvector* of A corresponding to λ then x need not be an *eigenvector* of A^t corresponding to λ.

14. If λ is an *eigenvalue* of $A_{n\times n}$ then show that the polynomial $g(\lambda) = k_0 + k_1\lambda + \cdots + k_m\lambda^m$ is an *eigenvalue* of $g(A) = k_0 I_n + k_1 A + \cdots + k_m A^m$. Conversely, show that if μ is an *eigenvalue* of $g(A)$ then there exists λ such that $g(\lambda) = \mu$ and λ is an *eigenvalue* of A. (Hint : factorize $g(A) - \mu I_n$ using fundamental theorem of algebra in the proof of the converse).

12 Eigenvalues of Special Matrices

1. Let A be a square matrix. Show that:
   (i) the eigenvalues of A are real if A is hermitian or real symmetric.
   (ii) the eigenvalues of A are either 0 or purely imaginary if A is skew hermitian;
   (iii) the eigenvalues of A are of modulus equal to 1, if A is unitary;
   (iv) if A is a Markov matrix then it has one of its eigenvalues equal to 1;
   (v) A^tA has only non negative eigenvalues, if A is real.

2. Prove that the *eigenvectors* of a hermitian (or real symmetric) matrix corresponding to distinct *eigenvalues* are orthogonal.

3. Find all 2×2 matrices which are orthogonal and skew symmetric. Also find their eigenvalues.

4. Does there exist a 3×3 matrix which is orthogonal and skew symmetric?

5. Two $n \times n$ matrices A, B are said to be *congruent* (resp. *real congruent*) if there exists a unitary (resp. orthogonal) matrix P such that $P^{-1}AP = B$. Show that A is congruent to a diagonal matrix if there exists an orthonormal basis for $\mathbb{K}^n$ consisting of eigenvectors of A only.

6. Show that every $n \times n$ real symmetric matrix A is real congruent to a diagonal matrix.

7. Prove that a real symmetric matrix is *positive definite* if and only if its *eigenvalues* are positive.

8. By a symmetric quadratic form Q of n variables we mean a homogeneous degree 2 polynomial in n variables, say $Q(x) = \sum_{i \le j} \alpha_{ij} x_i x_j$. Given a quadratic form Q we associate a symmetric matrix $A_Q = (a_{ij})$ to it by taking $a_{ii} = \alpha_{ii}$ and $a_{ij} = \alpha_{ij}/2, i \neq j$. Show that $Q(x) = xA_Q x^t$, where $x = (x_1, \ldots, x_n)$. Write down the associated matrix or the quadratic form from the given data below:
(a) $Q_1(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 3x_3^3$; (b) $Q_2(x, y) = xy$.
(c) $Q_3(x, y, z) = xy + yz + zx$; (d) $Q_4(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.
(e) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & -5 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

9. Transform the following quadratic equations to principal axis form and find out what conics they represent:
(a) $41x_1^2 - 24x_1x_2 + 34x_2^2 = 0$ (b) $9x_1^2 - 6x_1x_2 + x_2^2 = 40$;
(c) $91x^2 - 24xy + 84y^2 = 25$. (d) $4xy + 3y^2 = 10$.

10. A quadratic form Q with real coefficient is called *negative definite* if $-Q$ is positive definite; it is called definite, if it is positive definite or negative definite; otherwise we call it *indefinite.* Show that Q is a definite real quadratic form iff the set of solutions $\{x \in \mathbb{R}^n \ : \ Q(x) = r\}$ is bounded for all $r \in \mathbb{R}$.

13 Inner Product Spaces, Gram-Schmidt Process

1. On the vector space $C^1[a, b]$ of continuously differentiable real valued functions, examine whether or not $\langle f, g \rangle$, defined below is an inner product in each case. Justify your answer.

(a) $\langle f, g \rangle = \int_a^b f'(t)g'(t)\, dt$

(b) $\langle f,\, g\rangle = \int_a^b \left(f(t)g(t) + f^{'}(t)g^{'}(t)\right) dt.$

(c) $\langle f,\, g\rangle = f(b)g(b)$

2. In the vector space $C[1, e]$, define $\langle f,\, g\rangle = \int_1^e \log x f(x)g(x)\, dx.$

   (a) if $f(x) = \sqrt{x}$, compute $\|f\| = \langle f,\, f\rangle^{1/2}$.

   (b) Find a linear polynomial $g(x) = ax + b$ that is orthogonal to $f(x) = 1$.

3. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be nonzero vectors in $\mathbb{R}^n$ such that $\langle \mathbf{v}_i,\, \mathbf{v}_j\rangle = 0$, whenever $i \neq j$. Prove that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are linearly independent.

4. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal basis for an inner product space V. Suppose $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i$ and $\mathbf{v} = \sum_{i=1}^{n} \beta_i \mathbf{v}_i \in V$. Prove that

   (a) $\alpha_i = \langle \mathbf{u},\, \mathbf{v}_i\rangle$ and $\beta_i = \langle \mathbf{v},\, \mathbf{v}_i\rangle$ for $i = 1, 2, \ldots, n$.

   (b) $\langle \mathbf{u},\, \mathbf{v}\rangle = \sum_{i=1}^{n} \alpha_i\beta_i$ and $\|\mathbf{u}\| = \left(\sum_{i=1}^{n} \alpha_i^2\right)^{1/2}$.

5. Let $C_0^1[a, b] = \{f \in C^1[a, b] : f(a) = 0\}$. Show that $\langle f,\, g\rangle = \int_a^b f^{'}(x)g^{'}(x)\, dx$ is an inner product on $C_0^1[a, b]$.

6. Prove that in a real inner product space V, the following are equivalent:
   (i) $\langle x,\, y\rangle = 0$;
   (ii) $\|x + y\| = \|x - y\|$;
   (iii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
   Describe this geometrically.

7. In the vector space $\mathcal{P}_n[t]$ of all real polynomials of degree $\leq n$, define

$$\langle f,\, g\rangle = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) g\left(\frac{j}{n}\right).$$

   (a) Prove that $\langle f,\, g\rangle$ is an inner product on P_n.

   (b) Compute $\langle f,\, g\rangle$, when $f(t) = t, g(t) = at + b$.

   (c) If $f(t) = t$, find all linear polynomials g orthogonal to f.

8. Write elements of $\mathbb{R}^n$ as column vectors of length n. Let A be an $n \times n$ real matrix and A^t be its transpose. For usual inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, prove that for any two $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\langle A\mathbf{u},\, \mathbf{v}\rangle = \langle \mathbf{u},\, A^t\mathbf{v}\rangle.$$

State and prove a similar result about $\mathbb{C}^n$.

9. In a real inner product space V, show that for any $x, y \in V$

$$\langle x+y,\, x-y\rangle = 0 \quad \text{iff } \|x\| = \|y\|.$$

10. Orthonormalize the following set of vectors in $\mathbb{R}^4$, using the Gram-Schmidt process.

(i) $\{(1,0,0,0), (1,1,0,0), (1,1,1,1)\}$;
(ii) $\{(1,-1,2,0), (1,0,1,0), (1,0,0,1)\}$.

11. Let V be an inner product space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an orthonormal set in V.

(a) Show that for all $\mathbf{v} \in V$, $\langle \mathbf{v},\, \mathbf{v}\rangle \geq \sum_{i=1}^n |\langle \mathbf{v},\, \mathbf{v}_i\rangle|^2$, (**Bessel's inequality**)

(b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is also a basis, show $\langle \mathbf{v},\, \mathbf{v}\rangle = \sum_{i=1}^n |\langle \mathbf{v},\, \mathbf{v}_i\rangle|^2$. (**Parseval's identity**)

12. Let A be an $n \times n$ square matrix with real (or complex) entries. Show that the following are equivalent.

(i) A is orthogonal (unitary).
(ii) A^t (resp. A^*) is orthogonal (unitary).
(iii) The column vectors of A form an orthonormal set.
(iv) The row vectors of A form an orthonormal set.