Lecture 13

5 Linear Transformations

5.1 Basic Definitions and Examples

We have already come across with the notion of linear transformations on euclidean spaces. We shall now see that this notion readily extends to the abstract set up of vector spaces along with many of its basic properties. In what follows let us denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$.

Definition 5.1 *Let V, W be any two vector spaces over $\mathbb{K}$. By a linear map (or a linear transformation $f : V \longrightarrow W$ we mean a function f satisfying*

$$f(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 f(\mathbf{v}_1) + \alpha_2 f(\mathbf{v}_2) \tag{45}$$

for all $\mathbf{v}_i \in V$ and $\alpha_i \in \mathbb{K}$.

Remark 5.1
(i) For any linear map $f : V \longrightarrow W$, we have $f(\sum_{i=1}^k \alpha_i \mathbf{v}_i) = \sum_{i=1}^k \alpha_i f(\mathbf{v}_i)$.
(ii) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a finite dimensional vector space, then every element of V is a linear combination of these $\mathbf{v}_i$, say $\mathbf{v} = \sum_{i=1}^k \alpha_i \mathbf{v}_i$. Therefore from (i) we get

$$f(\mathbf{v}) = \sum_{i=1}^{k} \alpha_i f(\mathbf{v}_i). \tag{46}$$

Thus, it follows that f is completely determined by its value on a basis of V. *This simply means that if f and g are two linear maps such that $f(\mathbf{v}_i) = g(\mathbf{v}_i)$ for all elements of a basis for V, then $f = g$.*
(iii) Formula (46) can be used to define a linear transformation on a vector space, viz., by choosing its values on a given basis elements and then using (46) to define it on other elements. Observe that , in defining f, we are free to choose the value of f on the elements of a given basis, as far as they all belong to the same vector space W. Thus if V has a basis consisting of k elements as above then for each ordered k tuple of elements of W we obtain a unique linear transformation $f : V \longrightarrow W$ and vise versa.

Definition 5.2 *A linear transformation $f : V \longrightarrow W$ is called an* isomorphism *if it is invertible, i.e., there exist $g : W \longrightarrow V$ such that $g \circ f = Id_V$ and $f \circ g = Id_W$. Observe that the inverse of f is unique if it exists. If there exists an isomorphism $f : V \longrightarrow W$ then we call V and W are isomorphic to each other.*

Remark 5.2 *It is worth recalling here that if $f : V \to W$ is a linear bijection, then $f^{-1} : W \to V$ is automatically linear. Here is a proof:*

$$f^{-1}(\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2) = \alpha_1 f^{-1}(\mathbf{w}_1) + \alpha_2 f(\mathbf{w}_2)$$

iff

$$f(f^{-1}(\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2)) = f(\alpha_1 f^{-1}(\mathbf{w}_1) + \alpha_2 f^{-1}(\mathbf{w}_2))$$

iff

$$\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 = \alpha_1 f \circ f^{-1}(\mathbf{w}_1) + \alpha_2 f \circ f^{-1}(\mathbf{w}_2)$$

which is obvious.

Remark 5.3

1. *$V \approx V$ for every vector space V.*
2. *$V \approx W$ implies $W \approx V$.*
3. *$V_1 \approx V_2$ and $V_2 \approx V_3$ implies $V_1 \approx V_3$. In other words, these these properties tell you that 'being isomorphic to' is an equivalence relation.*
4. *We have been using a particular isomorphism between $M_{m,n}(\mathbb{K})$ and $\mathbb{K}^{mn}$ all the time.*
5. *The map $A \mapsto A^T$ defines an isomorphism of $M_{m,n}(\mathbb{K})$ with $M_{n,m}(\mathbb{K})$.*
6. *As a vector space over $\mathbb{R}$, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$.*
7. *Given any two vector spaces, we would like to know whether they are isomorphic or not. For then the study of any linear algebraic property of one vector space is the same as that of the other. We shall soon see that this problem has a very satisfactory answer.*

Definition 5.3 *Let $f : V \longrightarrow W$ be a linear transformation. Put*

$$\mathcal{R}(f) := f(V) := \{f(v) \in W \ : \ v \in V\},\ \mathcal{N}(f) := \{v \in V \ : \ f(v) = 0\}.$$

One can easily check that $\mathcal{R}(f)$ and $\mathcal{N}(f)$ are both vector subspace of W and V respectively. They are respectively called the range *and the* null space *of f.*

Lemma 5.1 *Let $f : V \longrightarrow W$ be a linear transformation.*
(a) Suppose f is injective and $S \subset V$ is linearly independent. Then $f(S)$ is linearly independent.
(b) Suppose f is onto and S spans V. Then $f(S)$ spans W.
(c) Suppose S is a basis for V and f is an isomorphism then $f(S)$ is a basis for W.

Proof: (a) Let $\sum_{i=1}^k a_i f(\mathbf{v}_i) = 0$ where $\mathbf{v}_i \in S$. This is the same as saying $f(\sum_i a_i\mathbf{v}_i) = 0$. Since f is injective. This is the same as saying $\sum_i a_i\mathbf{v}_i = 0$ and since S is L. I., this is the same as saying $a_1 = \ldots = a_k = 0$.
(b) Given $\mathbf{w} \in W$. Pick $\mathbf{v} \in V$ such that $f(\mathbf{v}) = \mathbf{w}$. Now $L(S) = V$ implies that we can write $\mathbf{v} = \sum_{i=1}^n a_i\mathbf{v}_i$ with $\mathbf{v}_i \in S$. But then $\mathbf{w} = f(\mathbf{v}) = \sum_i a_i f(\mathbf{v}_i) \in L(f(S))$.
(c) Put (a) and (b) together. ♠

Theorem 5.1 *Let V and W be any two vector spaces of dimension n. Then V and W are isomorphic to each other and conversely.*

Proof: Pick bases A and B for V and W respectively. Then both A and B have same number of elements. Let $f : A \longrightarrow B$ be any bijection. Then by the above discussion f extends to a linear map $f : V \longrightarrow W$. If $g : B \longrightarrow A$ is the inverse of $f : A \longrightarrow B$ then g also extends to a linear map. Since $g \circ f = Id$ on A, it follows that $g \circ f = Id_V$ on the whole of V. Likewise $f \circ g = Id_W$.

Converse follows from part (c) of the above corollary. ♠

Remark 5.4 *Because of the above theorem a vector space of dimension n is isomorphic to $\mathbb{K}^n$. We have seen that the study of linear transformations on euclidean spaces can be converted into the study of matrices. It follows that the study of linear transformations on finite dimensional vector spaces can also be converted into the study of matrices.*

Exercises:

(1) Clearly a bijective linear transformation is invertible. Show that the inverse is also linear.

(2) Let V be a finite dimensional vector space and $f : V \longrightarrow V$ be a linear map. Prove that the following are equivalent:

(i) f is an isomorphism.

(ii) f is surjective.

(iii) f is injective.

(iv) there exist $g : V \longrightarrow V$ such that $g \circ f = Id_V$.

(v) there exists $h : V \longrightarrow V$ such that $f \circ h = Id_V$.

(3) Let A and B be any two $n \times n$ matrices and $AB = I_n$. Show that both A and B are invertible and they are inverses of each other.

[Proof: If f and g denote the corresponding linear transformations then we have $f \circ g = Id : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Therefore, g must be injective and f must be surjective. From the exercise (4) above, both f and g are invertible and $f \circ g = g \circ f = Id$. Hence $AB = I_n = BA$ which means $A = B^{-1}$.]

Example 5.1 *Suppose $f, g : V \to V$ are two commuting endomorphisms of a vector space V, i.e., $f \circ g = g \circ f$.*

Then

$$g(\mathcal{R}(f)) \subset \mathcal{R}(f) \quad \& \quad g(\mathcal{N}(f)) \subset \mathcal{N}(f).$$

(See exercise 9.18.) This property is easily proved. It is quite useful and we shall meet it again.

Example 5.2 An example from theory of differental equations

Consider the space $C^r = C^r[a, b]$ of all real valued functions which possess continuous derivatives of order $r \geq 1$. To each $f \in C^r$ consider its derivative $f' \in C^{r-1}$. This defines a function on $D : C^r \longrightarrow C^{r-1}$. From elementary calculus of one variable, we know that D is a linear map, i.e., $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$. Now define for each $f \in C^{r-1}$ consider $I(f) \in C^r$ defined by

$$\mathcal{I}(f)(x) = \int_a^x f(t)\, dt.$$

Then we see that $\mathcal{I}$ is also a linear map. Moreover, we have $D \circ \mathcal{I} = Id$. Can you say $\mathcal{I} \circ D = Id$? Is D a one-one map? Determine the set $\mathcal{N}(D) \subset C^r$. We can consider

operators similar to D for each r. Let us write

$$D^k := D \circ D \circ \cdots \circ D \quad (k \text{ factors })$$

to denote the composite of k such consecutive operators. Thus $D^k : C^r \longrightarrow C^{r-k}$ is the map defined by $D^k(f) = f^{(k)}$ the k^{th} derivative of f, $(r > k)$. Given real numbers $a_0, \ldots, a_k (= 1)$, consider $f = \sum_{i=0}^{k} a_i D^i$. Then show that $f : C^r \longrightarrow C^{r-k}$ is a linear map. Determining the zeros of this linear map is precisely the problem of solving the homogeneous linear differential equation of order k:

$$y^k + a_{k-1}y^{k-1} + \cdots + a_1 y + a_0 = 0.$$

You will study them in your next semester.

Exercise: On the vector space $\mathcal{P}[x]$ of all polynomials in one-variable, determine all linear maps $\phi : \mathcal{P}[x] \longrightarrow \mathcal{P}[x]$ having the property $\phi(fg) = f\phi(g) + g\phi(f)$ and $\phi(x) = 1$.

Lecture 14

5.2 Rank and Nullity

Definition 5.4 *Let $f : V \longrightarrow W$ be a linear transformation of finite dimensional vector spaces. By the* rank *of f we mean the dimension of the range of f. i.e., $rk(f) = \dim f(V) = \dim \mathcal{R}(f)$. By* nullity of f *we mean the dimension of the null space i.e., $n(f) = \dim \mathcal{N}(f)$.*

Exercise Go back to the exercise in which you are asked to prove five equivalent ways of saying when is a linear map an isomorphism. Can you now give an easier proof of say (ii) implies (i) or (iii) implies (i)?

Theorem 5.2 Rank and Nullity Theorem*: The rank and nullity of a linear transformation $f : V \longrightarrow W$ on a finite dimensional vector space V add up to the dimension of V :*

$$r(f) + n(f) = \dim V.$$

Proof: Suppose $\dim V = n$. Let $S = \{\mathbf{v}_1, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a basis of $\mathcal{N}(f)$. We can extend S to a basis $Sv' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-k}\}$ of V. We show that

$$T = \{f(\mathbf{w}_1), f(\mathbf{w}_2), \ldots, f(\mathbf{w}_{n-k})\}$$

is a basis of $\mathcal{R}(f)$.

Obsere that $f(S') = T \cup \{0\}$. By part (b) of the previous lemma it follows that T spans $f(V) = \mathcal{R}(f)$.

[Any $\mathbf{v} \in V$ can be expressed uniquely as

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \beta_1\mathbf{w}_1 + \cdots + \beta_{n-k}\mathbf{w}_{n-k}.$$

Hence

$$\begin{aligned} f(\mathbf{v}) &= \alpha_1 f(\mathbf{v}_1) + \cdots + \alpha_k f(\mathbf{v}_k) + \beta_1 f(\mathbf{w}_1) + \cdots + \beta_{n-k} f(\mathbf{w}_{n-k}) \\ &= \beta_1 f(\mathbf{w}_1) + \cdots + \beta_{n-k} f(\mathbf{w}_{n-k}). \end{aligned}$$

Hence T spans $\mathcal{R}\{.]$

Now, suppose

$$\beta_1 f(\mathbf{w}_1) + \cdots + \beta_{n-k} f(\mathbf{w}_{n-k}) = 0.$$

Then

$$f(\beta_1 \mathbf{w}_1 + \cdots + \beta_{n-k} \mathbf{w}_{n-k}) = 0.$$

Hence $\beta_1 \mathbf{w}_1 + \cdots + \beta_{n-k} \mathbf{w_{n-k}} \in \mathcal{N}(\mathbf{f})$. Hence there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \cdots + \beta_{n-k} \mathbf{w}_{n-k}.$$

By linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-k}\}$ we conclude that $\beta_1 = \beta_2 = \cdots = \beta_{n-k} = 0$. Hence T is L. I. Therefore it is a basis of $\mathcal{R}(f)$. ♠

5.3 Change of Basis

Given a linear map $f : V \longrightarrow W$ of finite dimensional vector spaces, we consider the question of associating some matrix to it. This requires us to choose ordered bases, one for V and another for W and then write the image of basis elements in V as linear combinations of basis elements in W. The coefficients then define a column vector and we place them side by side to obtain a matrix as before.

Unlike the euclidean spaces, arbitrary finite dimensional spaces do not come with a standard basis. This leads us to study the effect of choice of basis involved. We shall first carry out this in euclidean spaces. By general principles, this will then hold for all finite dimensional vector spaces also.

Question: Let A be an $n \times n$ matrix and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be any basis for $\mathbb{K}^n$. If f_A is the linear transformation corresponding to A with respect to the standard basis, what is the matrix of f_A with respect to the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$?

Answer: Let P be the matrix whose j^{th} column is the column vector $\mathbf{v}_j$, for each j. Then P is invertible. Let B the matrix that we are seeking. This means that $f_A(\mathbf{v}_j) = \sum_{i=1}^{n} b_{ij} \mathbf{v}_i$ for each $1 \leq j \leq n$. In terms of the matrices A, P, this equation can be expressed in form:

$$\begin{aligned} AP^j &= A\mathbf{v}_j = \textstyle\sum_j b_{ij} \mathbf{v}_i \\ &= [\mathbf{v}_1, \ldots, \mathbf{v}_n] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{bmatrix} \\ &= = [P^1, \ldots, P^n] B^j = PB^j, \quad 1 \leq j \leq n. \end{aligned}$$

This is the same as saying that $AP = PB$, i.e., $P^{-1}AP = B$.

Remark 5.5 *Thus we have solved the problem in the case of euclidean space when the old matrix is with respect to the standard matrix. In the general case, whatever basis with respect to which we have expressed a linear map to obtain a matrix A in the first place plays the role of the 'standard basis.' The new basis elements are then expressed in terms of 'standard basis' to obtain the invertible matrix P. The matrix with respect to this new basis of the same linear map is then given by $P^{-1}AP$.*

Example 5.3 *Consider the linear operator $d : \mathcal{P}(3) \longrightarrow \mathcal{P}(3)$ given by $d(p) = p'$. Write the matrix of d with respect the ordered basis X where $X = Q_1, Q_2, Q_3$ respectively as given below.*
(i) $Q_1 = \{1, x, x^2, x^3\}$.
(ii) $Q_2 = \{1, 1+x, 1+x^2, 1+x^3\}$.
(iii) $Q_3 = \{1, 1+x, (1+x)^2, (1+x)^3\}$.
Write down the transformation matrices of change of bases from (i) to (ii) and from (i) to (iii) and see how they are related.

Solution: (i) In this case the matrix of d is given by $A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(ii) Here the matrix is given by $A_2 = \begin{bmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. On the other hand, the matrix of the second basis w.r.t to the first basis is:

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One easily checks that $A_1 P = P A_2$. In the same manner we can carry out the rest of the exercise.

5.4 Similarity

We have been using matrix representations to study the linear maps. We have also seen that how a change of bases influences the matrix representation. Obviously, different matrices which represent the same linear map under different bases should share a lot of common properties. This leads us to the concept of similarities.

Definition 5.5 *Two $n \times n$ matrices A, B are said to be similar iff there exists an invertible matrix M such that $A = MBM^{-1}$. We shall use the notation $A \sim B$ for this.*

Remark 5.6
(1) Check the following statements:
(i) $A \sim A$;
(ii) $A \sim B \Longrightarrow B \sim A$;
(iii) $A \sim B, B \sim C \Longrightarrow A \sim C$.
(2) We have seen that the determinant of a square matrix has the property $\det AB = (\det A)(\det B)$. *From this it follows that if* $A \sim B$ *then* $\det A = \det B$. *Such properties are called* invariants. *Another such example is the* trace *of a square matrix which is defined as the sum of the diagonal entries. It now follows that these two matrix based notions are now available for linear maps $f : V \longrightarrow V$ where V is a finite dimensional vector space. Later we shall see more examples of such invariants.*

Lecture 15

Definition 5.6 *Let A be a $m \times n$ matrix. Consider the rows of A as elements of $\mathbb{R}^n$ and columns of A as elements of $\mathbb{R}^m$. Let $r(A)$ (respectively, $c(A)$) denote the linear span of the rows (of columns) of A. We define*
$\rho(A) :=$ the **row-rank** *of A = dimension of $r(A)$.*
$\kappa(A) :=$ the bf column-rank of A= dimension of $c(A)$.
$\delta(A) =$ The **determinantal rank** *of A (or simply* **rank** *of A) is the maximum r such that there exists a $r \times r$ submatrix of A which is invertible.*
$\gamma(A) =$ number of non zero rows in $G(A)$.

Our aim is to prove that all these numbers are equal to each other.

Remark 5.7
(i) Let A be an echelon matrix with r steps. Then $G(A) = A$ and $\gamma(A) = r$.

$$G(A) = \begin{bmatrix} \phi & * & * & * & * & * \\ 0 & \phi & * & * & * & * \\ 0 & 0 & 0 & \phi & * & * \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then it is easily seen that the first r rows are linearly independent. On the other hand the rest of the rows are identically zero. Hence the row rank $\rho(A)$ is equal to $r = \gamma(A)$.

(ii) Let now A be a reduced echelon matrix.

$$A = J(A) = \begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then it is easily seen that the determinantal rank $\delta(A)$ of A is equal to r. Further observe that the step columns are linearly independent. Moreover, it is easily seen that every other column is a linear combination of the step columns. Hence the step columns form a basis for $c(A)$. This means that column-rank of $\kappa(A)$ is also equal to r.

Thus we have proved that all the four types of rank functions $\gamma(A), \rho(A), \delta(A)$ and $\kappa(A)$ coincide for a reduced echelon matrix A. Below we shall prove that this is true for all matrices.
(iii) An $n \times n$ matrix is invertible iff its determinantal rank is equal to n.
(iv) Let $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation corresponding to A. Since $\{\mathbf{e}_i\}$ form a basis for $\mathbb{R}^n$ and $C_i = f(\mathbf{v}_i)$ are the columns of A, the range of f_A is equal to $c(A)$. Hence we have:

Theorem 5.3 *The rank of the linear map associated with a given matrix A is equal to the column rank of A.*

Theorem 5.4 *The rank of the linear map associated with a given matrix A is equal to the column rank $\kappa(A)$ of A.*

Theorem 5.5 *Let A be an $n \times n$ matrix. Then the following are equivalent.*
(i) *The determinantal rank of $A = n$.*
(ii) *The column rank of A is equal to n.*
(iii) *The row rank of A is equal to n.*
(iv) *The rows of A are independent.*
(v) *The columns of A are independent.*

Proof: (i) $\iff$ A is invertible $\iff$ the system $A\mathbf{x} = 0$ has a unique solution. $\iff$ columns of A are independent $\iff$ (v) $\iff$ (ii). By taking the transpose, we get the proof of the equivalence of other statements. ♠

Theorem 5.6 *An elementary row operation does not affect the row rank of a matrix.*

Proof: Let $S = \{R_1, \ldots, R_m\}$ denote the set of rows of A. Let S' denote the set of rows of $A' := EA$ where E is an elementary matrix. If $E = T_{i,j}$ a transposition, then $S = S'$. If $E = I + cE_{i,j}$ then $S' = \{R_1, \ldots, R_i + cR_j, \ldots, R_j, \ldots, R_n\}$. In all cases it is easily seen that the new set of rows S' is contained in $L(S)$. Therefore $L(S') \subset L(S)$. Since elementary row operations are reversible, it follows that $S \subseteq L(S')$ and hence $L(S) \subseteq L(S')$. Since the row rank is the same as the dimension of the span of rows we are done. ♠

Theorem 5.7 *An elementary row operation does not affect the column rank of A.*

Proof: If E is an elementary matrix, then observe that E defines an isomorphism of $c(A)$ with $c(EA)$. Hence column rank of $A = dim\, c(A) = \dim\, c(EA) =$ column rank of EA. ♠

Theorem 5.8 *The row rank and the column rank of a matrix are equal to each other.*

Proof: Let A be a matrix and $J(A)$ its associated reduced echelon form. The statement of the theorem is easily verified for $J(A)$. Since row-rank and column rank are not affected by elementary row operations, and $J(A)$ is obtained by a finite number of row operations on A, it follows that the row-rank and column rank of A are the same. ♠

Theorem 5.9 *Let A be a $m \times n$ matrix with its row rank equal to m. Then there exists a $m \times m$ submatrix of A which is invertible. In particular $\delta(A)$ is equal to m.*

Proof: Since we have shown that row-rank and column-rank are the same, it follows that there is an $m \times m$ submatrix B of A which has all its columns independent. By a previous theorem B is invertible. Hence $\delta(A) \geq m$. Since A has m rows $\delta(A) = m$. ♠

Theorem 5.10 *The row rank, the column rank and the rank of a matrix are equal to each other.*

Proof: We have already seen that the row rank and the column rank of A are equal. Let this common number be r.

Suppose there is a $d \times d$ submatrix $B = ((a_{i_p,j_q}))$ of $A = ((a_{ij}))$ which is invertible. From theorem 5.4, it follows that the rows of B are L. I. This immediately implies that the corresponding rows of A viz, $\{R_{i_1,}, \ldots, R_{i_d}\}$ are linearly independent. Therefore, $d \leq r$ the row-rank of A.

Moreover, as seen before, since $\rho(A) = r$ there are rows

$$\{R_{i_1,}, \ldots, R_{i_r}\}$$

of A which are independent. If C is the matrix formed out of these rows then $\rho(A) = r = \kappa(A)$. This means that there are columns

$$\{C_{j_1}, \ldots, C_{j_r}\}$$

of C which are independent. If M denotes the submatrix of C fomred by these columns then M is $r \times r$ matrix which are all its columns indepedent. By a previous result M is inveritble. Since M is also a submatrix of A, this proves $d \geq r$. ♠

Remark 5.8 *It follows that an elementary row operation does not affect the determinantal rank of a matrix. Alternatively, we can directly prove this fact and then appeal to GEM to get a proof that the determinantal rank is equal to the row-rank and column rank of a matrix. There are other approaches possible and different authors may prefer different approach.*

5.5 GEM applied to Null-space and Range

Here is an algorithm to write bases for the range and null-space of a linear transformation $f : V \to W$ of finite dimensional vector spaces.

Step 1 Given a linear liear transformation $f : V \to W$ of finite dimensional vector spaces fix bases $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for V and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for W and write the $m \times n$ matrix A of the linear map f.

Step 2 Obtain $J(A) = ((\alpha_{ij}))$, the Gauss-Jordan form of A.

Step 3 If $k_1, k_2, \ldots, k_r$ are the indices of the **step columns** in $J(A)$, then $\{f(v_{k_1}), \ldots, f(v_{k_r})\} = \{A^{(k_1)}, \ldots, A^{(k_r)}\}$ is a basis for the range.

Step 4 Let $l_1 < \cdots < l_s$ be the indices of the **non step columns**. Define the column vectors $\mathbf{u}_{l_1}, \ldots, \mathbf{u}_{l_s}$ by:

$$\mathbf{u}_{jl_i} = \begin{cases} 1 & \text{if } j = l_i; \\ -\alpha_{pl_i} & \text{if } j = k_p, 1 \leq p \leq r; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\mathbf{u}_{l_1}, \ldots, \mathbf{u}_{l_s}\}$ forms a basis for $\mathcal{N}(f)$. To see a proof of this, you have to simply recall how we had written the set of solutions of the $A\mathbf{x} = \mathbf{b}$ and apply it for the case when $\mathbf{b} = \mathbf{0}$.

Example 5.4 *Consider a linear map $f : \mathbb{R}^r \to \mathbb{R}^4$ whose matrix is A and whose Jordan form is:*

$$G(A) = \begin{bmatrix} \boxed{1} & 0 & 5 & 0 & 1 & 1 \\ 0 & \boxed{1} & 2 & 0 & 2 & 3 \\ 0 & 0 & \boxed{0} & \boxed{1} & -4 & -7 \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $\{f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_4)\} = \{A^{(1)}, A^{(2)}, A^{(4)}\}$ *is basis for* $\mathcal{R}(f)$ *where* $A^{(j)}$ *denotes the* j^{th} *column of* A.

Also $\{(-5, -2, 1, 0, 0, 0)^t, (-1, -2, 0, 4, 1, 0)^t, (-1, -3, 0, 7, 0, 1)^t\}$ *is a basis for* $\mathcal{N}(f)$.