# Models for Qualitative and Quantitative Factors 

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From statistical point of view, there are vectors and matrices of various orders connected by the 'models' (I), (II), and (III) enunciated below.
(I) $Y_{n \times 1}=X_{n \times v} \tau_{v \times 1}+Z_{n \times c} \gamma_{c \times 1}+\epsilon_{n \times 1}$;
(II) $Y_{n \times 1}=B_{n \times b} \beta_{b \times 1}+X_{n \times v} \tau_{v \times 1}+Z_{n \times c} \gamma_{c \times 1}+\epsilon_{n \times 1}$;
(III) $Y_{n \times 1}=B_{n \times b} \beta_{b \times 1}+X_{n \times v}^{*} \tau_{v \times 1}+Z_{n \times c} \gamma_{c \times 1}+\epsilon_{n \times 1}$.

In the statistical literature on 'linear models', the above models are regarded as 'models with covariates' were the $\gamma$-parameters ascertain the effects of the 'covariates' embodied in the matrix $Z$.
In particular, we may identify the above models as arising out of CRD, RBD and BIBD respectively - all involving a number of covariates. In this talk, I will concentrate only on CRDs and RBDs. For a treatment of the BIBDs, I refer to Das et. al. (2002) and Liski et. al. (2002).

Consider the nonstochastic controllable covariates' model in the context of a general block design :

$$
\left\{Y, \quad \mu e_{N}+X_{1} \beta+X_{2} \tau+Z \gamma, \quad \sigma^{2} I\right\}
$$

where
$\mu \quad$ is intercept term
$\sigma^{2} \quad$ is the common variance of the observations
$\beta \quad$ is block effects vector of order $b \times 1$
$\tau \quad$ is treatment effects vector of order $v \times 1$
$\gamma \quad$ is covariate effects vector of order $c \times 1$
$Y \quad$ is observation vector of order $N \times 1, N=b v$
and $e_{N}$ is vector with each element equal to 1 of order $N \times 1$
$X_{1}$ and $X_{2}$ are incidence matrices of blocks and treatments respectively and $Z$ is the matrix of covariates. It is assumed without loss of generality that $\left|Z_{i j}\right| \leq 1$, for otherwise, location and scale transformations can be applied.
The information matrix for the above model is given by

$$
\Im=\left[\begin{array}{llll}
N & e^{\prime} X_{1} & e^{\prime} X_{2} & e^{\prime} Z \\
& X_{1}^{\prime} X_{1} & X_{1}^{\prime} X_{2} & X_{1}^{\prime} Z \\
& & X_{2}^{\prime} X_{2} & X_{2}^{\prime} Z \\
& & & Z^{\prime} Z
\end{array}\right]
$$

For orthogonal estimation of contrasts of block effects and contrasts of treatment effects as well as covariate effects, it is necessary and sufficient that

$$
\begin{equation*}
e^{\prime} Z=0, X_{1}^{\prime} Z=0 \text { and } X_{2}^{\prime} Z=0 \tag{1}
\end{equation*}
$$

Additionally, for the most efficient estimation of $\gamma$-components, we need

$$
\begin{equation*}
Z^{\prime} Z=N I_{c} \tag{2}
\end{equation*}
$$

Applied to the set-up of CRDs, the matrix $X_{1}$ is not present. In either case [CRD/RBD/ General Block Design] it turns out that the intercept parameter has NO special role to play when we seek most efficient orthogonal estimation of the parameters! That explains the absence of the intercept term in the specific models [(I),(II) and (III)] enunciated above.

Once again, for most efficient orthogonal estimation, the matrices of relevance and interest to us are:

## With reference to Model (I):

(a) $X_{n \times v}=((0,1))$ such that $X^{\prime} X=r \cdot I_{v \times v}$, $r$ being an integer and $I$ being an identity matrix;
(b) $Z_{n \times c}=((+1,-1))$ such that $Z^{\prime} Z=n \cdot I_{c \times c}$.
(c) $\quad X$ and $Z$ jointly satisfy $Z^{\prime} X=0$ (Null Matrix.) of order $(c \times v)$.

## With reference to Model (II):

(a) $B_{n \times b}=((0,1))$ such that $B^{\prime} B=v \cdot I_{b \times b}$, and $B^{\prime} X=J_{b \times v}=((1))$-matrix of all $1^{\prime} \mathrm{s}$;
(b) $X_{n \times v}=((0,1))$ such that $X^{\prime} X=b \cdot I_{v \times v}$;
(c) $Z_{n \times c}=((+1,-1))$ such that $Z^{\prime} Z=n \cdot I_{c \times c}$;
(d) $B, X$ and $Z$ jointly satisfy
(i) $Z^{\prime} B=0$;
(ii) $Z^{\prime} X=0$.

It is known (from statistical arguments : recall the expresions for error df under different models) that

$$
\begin{aligned}
c_{\max } & =n-v \text { in Model (I) } \\
& =(b-1)(v-1) \text { in Model (II) } \\
& =(n-b-v+1) \text { in a connected block design. }
\end{aligned}
$$

The combinatorial problems studied in Lopes Troya [1982], Das et. al. [2002], Liski et. al. [2002] and Rao et. al. [2002] deal with constructions of $Z$-matrices (having elements $+1,-1)$ satisfying the requirements displayed above under various models. The matrices $B, X$ and $X^{*}$ are assumed to be known and given.
The added complication is that rarely it has been possible to attain the upper bound(s) for "c", as indicated above.
Following Liski et. al. (2002) and Das et. al. (2002), we introduce $W$-matrices which, interestingly enough, put forward the combinatorial problem in a form easily comprehensible!
Let $W$ be a matrix formed exclusively of the elements +1 and -1 . Its order depends on the set-up under consideration. We will be dealing with a collection of such $W$-matrices satisfying one or more of the conditions listed below:
(C1) Each $W$-matrix has all column sums equal to 0;
(C2) Each $W$-matrix has all row sums equal to 0;
(C3) The grand total of all the entries in the Hadamard product of any two $W$-matrices (which is a matrix of the same order obtained by taking element-wise product), reduces to 0 .

In each set-up, the $Z$-matrices are recast as $W$-matrices. The idea is to break up each $n \times 1$ column vector of $Z$ into a $W$-matrix.
In case of Model (I) which refers to a CRD set-up, $W$-matrices are of order $r \times v$ and conditions to be satisfied are (C1) and (C3). There are as many $W$-matrices as the number of columns of $Z$. Without loss of generality we can assume $X$ to be of the form:

$$
X=I_{r \times r} \otimes e_{v}, \otimes \text { denoting the Kronecker product of matrices. }
$$

Here $e_{v}$ is a column - vector of order $v$, with each element equal to 1 .
Then the $i^{\text {th }}$ column of $W_{j}$ is formed of the $r$ elements in the $j^{\text {th }}$ column of $Z$ corresponding to the $r$ non-zero elements in the $i^{t h}$ column of $X$.
For Model (II) which refers to an RBD set-up, $W$-matrices are of order $b \times v$ and conditions to be satisfied are (C1), (C2) and (C3). Again, there are $c W$-matrices.
The inter-relations among the matrices $Z, X_{1}, X_{2}$ and $W$ are described in Liski et. al. (2002) and Das et. al. (2002).

For a CRD set-up, results available in Lopes Troya [1982] were rephrased in terms of $W$-matrices and slightly strengthened in Das et. al. [2002] and Liski et. al. [2001].
For RBD/BIBD set-up, initial results in terms of $W$-matrices are given in Das et. al. [2002] and Liski et. al. [2002].

At this stage a brief literature review is in order.
In the context of optimality study involving both qualitative and quantitative factors, Lopes Troya [1982a, 1982b] used the above 'covariates' model' in the set-up of Completely Randomized Designs (CRDs). Recently, Liski et. al. [2002] and Das et. al. [2002] studied some further aspects of the 'designing problem', covering the set-up of CRDs, Randomized Block Designs (RBDs) and Balanced Incomplete Block Designs (BIBDs). These later studies provided

1. a better understanding of the combinatorial problems and their complexity, after a nice and carefully revealing formulation;
2. some solutions to the combinatorial problems in various set-ups.

The above formulation in terms of W -matrices is due to Das et. al. (2002), subsequently incorporated in Liski et. al. (2002).
Orthogonal Arrays [OAs] introduced by Rao[1947] were generalized by Rao himself [1973] to Mixed Orthogonal Arrays (MOAs). There are various results on constructions of OAs and MOAs. We refer to the books of Hedayat, Sloane and Stufken [1999] and of Dey and Mukherjee [1999]. Also there is available a website of Sloane (www.research.att.com/ njas/) for ready reference and a catalogue of potential sources on OAs and MOAs.
A revisit to the 'designing problem' formulation and the available results (in the set-up of CRDs and RBDs) indicates that the above combinatorial problem identifies itself to that of existence of certain Mixed Orthogonal Arrays (MOAs). This observation and a host of constructional results in the set-up of CRDs and RBDs are due to Rao et. al. (2002).
The purpose of this talk is to explain the connection of 'Search for Optimal Covariates' Model Designs' to the MOAs and discuss some of the available results in this fascinating area of research.

We utilize Hadamard matrices and $M O L S$ to deduce that the absolute maximum number
of $W$-matrices, can be constructed in some cases. This work is in continuation to the research findings contained in Das et. al. (2002).
Without loss of generality, we assume that Hadamard matrices considered here have all the elements of last row and last column equal to 1 .
Theorem Let $H_{m}$ and $H_{n}$ exist for some $m$ and $n$. Then a $W_{m n}$-set of cardinality $(m-1)(n-1)$ exists.
Proof : Let $h_{i}$ denote the $i^{\text {th }}$ column of $H_{m}$ for $i=1,2, \cdots, m-1$ and $x_{j}^{\prime}$ denote the $j^{\text {th }}$ row of $H_{n}$ for $j=1,2, \cdots, n-1$. Consider the set $S=\left\{W^{(i j)}, i=1,2, \cdots, m-1\right.$ and $j=1,2, \cdots, n-1$. $\}$ where $W^{(i j)}$ is defined as

$$
W^{(i j)}=h_{i} x_{j}^{\prime}
$$

Observe that $e_{m}^{\prime} W^{(i j)}=e_{m}^{\prime} h_{i} x_{j}^{\prime}=0 . x_{j}^{\prime}=\phi^{\prime}$ and $W^{(i j)} e_{n}=h_{i} x_{j}^{\prime} e_{n}=h_{i} .0=\phi$. Further $e_{m}^{\prime}\left(h_{i} * h_{p}\right)=0$ for $i \neq p$ and $\left(x_{j}^{\prime} * x_{q}^{\prime}\right) e_{n}=0$ for $j \neq q$.
Also

$$
\begin{aligned}
e_{m}^{\prime}\left(W^{(i j)} * W^{(p q)}\right) e_{n} & =e_{m}^{\prime}\left[\left(h_{i} x_{j}^{\prime}\right) *\left(h_{p} x_{q}\right)\right] e_{n} \\
& =e_{m}^{\prime}\left(h_{i} * h_{p}\right) \cdot\left(x_{j}^{\prime} * x_{q}^{\prime}\right) e_{m} \\
& =0 \text { as } i \neq p \text { or } j \neq q .
\end{aligned}
$$

Thus the above defined set $S$ is $W_{m n}$-set of cardinality $(m-1)(n-1)$.
Consider the case where $m=8$ and $n=4$.

$$
\begin{gathered}
H_{m}=\left[\begin{array}{rrrrrrrr}
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \\
H_{n}=\left[\begin{array}{rrrr}
1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right]
\end{gathered}
$$

Then $W^{(32)}$ constructed using third column of $H_{m}$ and second row of $H_{n}$ is

$$
\begin{aligned}
W^{(32)} & =\left(\begin{array}{r}
1 \\
-1 \\
1 \\
-1 \\
-1 \\
1 \\
-1 \\
1
\end{array}\right)\left(\begin{array}{rrrr} 
\\
-1 & -1 & 1 & 1
\end{array}\right) \\
= & {\left[\begin{array}{rrrr}
-1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1
\end{array}\right] }
\end{aligned}
$$

In a similar way all the $(m-1)(n-1)$ matrices $W^{(i j)}$ of $W_{m \times n}$-set can be constructed for $i=1,2, \cdots, m-1$ and $j=1,2, \cdots n-1$.

Corollary 1 Let $b=p m$ and $v=q n$ when $p$ and $q$ are integers and $H_{m}$ and $H_{n}$ exist. Then a $W_{b v}$-set of cardinality $(m-1)(n-1)$ can be constructed.
Proof: Construct the set $T$ as $\left\{V^{(i j)}, i=1,2, \cdots, m-1, j=1,2, \cdots, n-1\right\}$ where $V^{(i j)}$ is the matrix of $p \times q$ blocks with each block same as $W^{(i j)}$ in $S$ of the above theorem. $T$ satisfies required properties.
Corollary 2 When $H_{n}$ exists and $m \equiv 2(\bmod 4)$ a $W_{m \times n}$-set of cardinality at least $(n-1)$ can be constructed.

Remark 1 This result strengthens the corresponding result in Liski-Mandal-Shah-Sinha (2001) considerably in terms of increasing the number of covariates to be included in the model.
Now we discuss about connection with MOAs. An $n \times k$ matrix, with entries of the $i^{t h}$ column being from a set of $s_{i}(>1)$ levels of the $i^{\text {th }}$ factor $F_{i}, i=1,2, \cdots, k$, such that in every $n \times d$ submatrix $(1 \leq d \leq k)$, all possible $1 \times d$ vectors appear equally often, is called an Orthogonal Array of strength $d$, having $n$ runs and $k$ factors and is denoted by $\mathrm{OA}\left(n, k, s_{1} \times s_{2} \times \cdots \times s_{k}, d\right)$.
When $s_{1}=s_{2}=\cdots=s_{k}$, the OA is called a Fixed or Symmetrical Array; otherwise, it is referred to as a Mixed or an Asymmetrical Orthogonal Array.
In this talk the following results from Rao et. al. (2002) will also be discussed.

1. A set of $m W$-matrices each of order $r \times v$ under the CRD set-up coexists with an $\operatorname{MOA}\left(r v, 1+m, v \times 2^{m}, 2\right)$.
2. A set of $m W$-matrices each of order $b \times v$ under the RBD set-up coexists with an $\operatorname{MOA}\left(b v, 2+m, v \times b \times 2^{m}, 2\right)$.
3. (a) If there exist Hadamard matrices of orders $r$ and $v$, then the $\operatorname{MOA}(r v, 1+m, v \times$ $\left.2^{m}, 2\right)$ exists where $m=v(r-1)$.
(b) If there exist Hadamard matrices of orders $b$ and $v$, then the $\operatorname{MOA}(v b, 2+m, v \times$ $\left.b \times 2^{m}, 2\right)$ exists where $m=(b-1)(v-1)$.
4. (a) If $r$ is congruent of $2 \bmod v$, and if $(r-1)$ is a prime or a prime power and, further, if a Hadamard matrix of order $v$ exists, then the $\operatorname{MOA}(r v, 1+m, v \times$ $\left.2^{m}, 2\right)$ exists where $m=v(r-1)$.
(b) If $b$ is congruent of $2 \bmod v$, and if $(b-1)$ is a prime or a prime power and, further, if a Hadamard matrix of order $v$ exists, then the $\mathrm{MOA}(b v, 2+m, b \times$ $\left.v \times 2^{m}, 2\right)$ exists where $m=(b-1)(v-1)-(b-2)$.
5. Maximum value of $m$ for which there exists an $\operatorname{MOA}\left(v^{2}, 2+m, v \times v \times 2^{m}, 2\right)$ tends to infinity as $v$ tends to infinity, $v$ being even.
6. Analytical proof of existence of $\operatorname{MOA}\left(12,5,3 \times 2^{4}, 2\right)$ is provided.

Some of the above results have been substantially strengthened in Rao et. al. (2002).

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