

Minimum Aberration and Related Designs in Fractional Factorials

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1 Introduction

In this book, we have so far been concerned with optimal fractional factorial plans under specified linear models. The present chapter differs in spirit from the previous ones in the sense of allowing a greater flexibility in the model. Our objective here is to present a concise introduction to the basics of certain newly emerging areas like minimum aberration, search and supersaturated designs. In the process, we shall come across new kinds of optimality criteria.

Section 2 begins with a discussion on the classical regular fractions and introduces minimum aberration designs. Search and supersaturated designs are then briefly discussed in Sections 3 and 4 respectively.

2 Regular Fractions and Minimum Aberration Designs

Consider the set up of a symmetric m^n factorial where $m(\geq 2)$ is a prime or a prime power. Following Bose (1947), it is then possible to develop an approach to fractional factorial plans providing a significant insight into the underlying issues. This utilizes the properties of a finite field and calls for the use of a notational system different from the one employed in the earlier chapters.

Let $\rho_0, \rho_1, \dots, \rho_{m-1}$ be the elements of $GF(m)$, where $\rho_0 (= 0)$ and $\rho_1 (= 1)$ are the identity elements with respect to the operations of addition and multiplication respectively. A typical treatment combination $j_1 \dots j_n$ of an m^n factorial can be identified with the vector $(\rho_{j_1}, \dots, \rho_{j_n})'$ ($0 \leq j_1, \dots, j_n \leq m-1$). The m^n treatment combinations are thus represented by the m^n vectors of the form $\mathbf{z} = (z_1, \dots, z_n)'$, where $z_i \in GF(m)$ for each i . In other words, the treatment combinations are identified with the points of the n -dimensional finite Euclidean geometry $EG(n, m)$, based on $GF(m)$; see e.g., Raghavarao (1971, pp. 357-359). The effect of a treatment combination represented by $\mathbf{z}$ will be denoted by $\tau(\mathbf{z})$.

For symmetric prime or prime powered factorials, there is a convenient way of representing treatment contrasts belonging to factorial effects. Some preliminaries are needed in this context. Let $\mathbf{a} = (a_1, \dots, a_n)'$ be any fixed nonnull vector over $GF(m)$. Then it is easy to see that each of the sets

$$V_j(\mathbf{a}) = \{\mathbf{z} = (z_1, \dots, z_n)' : \mathbf{a}'\mathbf{z} = \rho_j\}, \quad 0 \leq j \leq m-1, \quad (2.1)$$

has cardinality m^{n-1} , and that these sets provide a disjoint partition of the class of all treatment combinations. The sets in (2.1) are collectively called a parallel pencil of $(n-1)$ -flats of $EG(n, m)$ and hence $\mathbf{a}$ itself is said to represent a pencil. Let

$$S_j(\mathbf{a}) = \sum_{\mathbf{z} \in V_j(\mathbf{a})} \tau(\mathbf{z}), \quad 0 \leq j \leq m-1. \quad (2.2)$$

Then a treatment contrast L is said to belong to the pencil $\mathbf{a}$ if it is of the form

$$L = \sum_{j=0}^{m-1} l_j S_j(\mathbf{a}), \quad (2.3)$$

where $l_0, \dots, l_{m-1}$ are real numbers, not all zeros, satisfying $\sum_{j=0}^{m-1} l_j = 0$. Clearly, there are $m-1$ linearly independent treatment contrasts belonging to the pencil $\mathbf{a}$.

It is easy to see from (2.1) that two pencils, say $\mathbf{a}$ and $\mathbf{a}^*$, satisfying $\mathbf{a}^* = \lambda \mathbf{a}$ for some $\lambda (\neq 0) \in GF(m)$, induce the same disjoint partition of the class of all treatment combinations. As such, hereafter, pencils with proportional entries will be considered as identical. Thus there are $(m^n - 1)/(m - 1)$ distinct pencils no two of which have proportional entries. For any two distinct pencils $\mathbf{a}, \mathbf{b}$ and $0 \leq j, j' \leq m-1$, it follows from (2.1) that the set $V_j(\mathbf{a}) \cap V_{j'}(\mathbf{b})$ has cardinality m^{n-2} . Hence from (2.2) and (2.3), the following lemma can be proved easily.

Lemma 2.1 *Treatment contrasts belonging to distinct pencils are mutually orthogonal. ■*

We are now in a position to connect pencils with factorial effects. A pencil $\mathbf{a} = (a_1, \dots, a_n)'$ is said to belong to a factorial effect $F_{i_1} \dots F_{i_g}$ ($1 \leq i_1 < \dots < i_g \leq n; 1 \leq g \leq n$) provided

$$\begin{aligned} a_i &\neq 0, \quad \text{if } i = i_1, \dots, i_g, \\ &= 0, \quad \text{otherwise.} \end{aligned} \quad (2.4)$$

Comparing (2.1)–(2.4) with the contents of Section 1.2 (see the discussion following (1.2.1)), it is not hard to see that if a pencil $\mathbf{a}$ belongs to a factorial effect $F_{i_1} \dots F_{i_g}$ then every treatment contrast belonging to $\mathbf{a}$ also belongs to $F_{i_1} \dots F_{i_g}$. Now, pencils with proportional entries are identical and hence, by (2.4), there are $(m-1)^{g-1}$ distinct pencils belonging to $F_{i_1} \dots F_{i_g}$. Each of them carries $m-1$ linearly independent treatment contrasts. Thus, in view of Lemma 2.1, these pencils provide a representation for the treatment contrasts belonging to $F_{i_1} \dots F_{i_g}$ in terms of $(m-1)^{g-1}$ mutually orthogonal sets of contrasts with $m-1$ linearly independent contrasts in each set. This accounts for a complete collection of $(m-1)^g$ linearly independent treatment contrasts belonging to the factorial effect $F_{i_1} \dots F_{i_g}$.

For example, let $m = 3$ and $n = 2$. Then the pencils $(1, 0)'$ and $(0, 1)'$ belong to main effects F_1 and F_2 respectively while interaction $F_1 F_2$ is represented by the two distinct pencils $(1, 1)'$ and $(1, 2)'$, each carrying two linearly independent treatment contrasts, the contrasts belonging to distinct pencils being mutually orthogonal.

A regular fraction of an m^n factorial, where $m (\geq 2)$ is a prime or a prime power, is specified by any k ($1 \leq k < n$) linearly independent pencils, say $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)}$, and consists of treatment combinations $\mathbf{z}$ satisfying

$$A\mathbf{z} = \mathbf{c},$$

where A is a $k \times n$ matrix with rows $\mathbf{a}^{(i)'}$, $1 \leq i \leq k$, and $\mathbf{c}$ is a fixed $k \times 1$ vector over $GF(m)$. In what follows, the specific choice of $\mathbf{c}$ is inconsequential and hence for notational simplicity we assume, without loss of generality, that $\mathbf{c} = \mathbf{0}$, the null vector over $GF(m)$. Then a regular fractional factorial plan is given by, say,

$$d(A) = \{\mathbf{z} : A\mathbf{z} = \mathbf{0}\}. \quad (2.5)$$

Since the rows of the $k \times n$ matrix A are given by linearly independent pencils, it is clear that $d(A)$ contains $N = m^{n-k}$ distinct treatment combinations. By (2.5), these m^{n-k} treatment combinations, or equivalently points of $EG(n, m)$, form a subgroup of $EG(n, m)$, the group operation being componentwise addition.

We now proceed to explore the properties of $d(A)$ assuming that the underlying linear model is given by (2.3.3) (with d there replaced by $d(A)$), where the only cognizable effects are those due to treatment combinations. With reference to $d(A)$, a pencil $\mathbf{a}$ is called a defining pencil if

$$\mathbf{a} \in \mathcal{M}(A'), \quad (2.6)$$

where $\mathcal{M}(\cdot)$ denotes the column space of a matrix. Since pencils with proportional entries are identical, and A has full row rank, there are $(m^k - 1)/(m - 1)$ distinct defining pencils. For every defining pencil $\mathbf{a}$, by (2.5) and (2.6), $\mathbf{a}'\mathbf{z} = 0$ whenever $\mathbf{z} \in d(A)$ which, in view of (2.1), yields $d(A) \subset V_0(\mathbf{a})$. Therefore, from (2.2) and (2.3), the following result is evident.

Lemma 2.2 *No treatment contrast belonging to any defining pencil is estimable in $d(A)$.*

■

Lemmas 2.3–2.5 below help in understanding the status of $d(A)$ relative to pencils other than the defining ones. We need to introduce the important notion of alias sets for presenting these lemmas. Let $\mathcal{B}(\equiv \mathcal{B}(A))$ be the set of pencils which are not defining pencils. There are

$$\frac{m^n - 1}{m - 1} - \frac{m^k - 1}{m - 1} = \frac{m^k(m^{n-k} - 1)}{m - 1}$$

distinct pencils in $\mathcal{B}$. Any two members of $\mathcal{B}$, say $\mathbf{b}$ and $\mathbf{b}^*$, are said to be aliases of each other if

$$\mathbf{b} - \mathbf{b}^* \in \mathcal{M}(A'). \quad (2.7)$$

Since $A = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)})'$ has full row rank, it is easily seen that any $\mathbf{b}(\in \mathcal{B})$ has m^k distinct aliases, namely,

$$\mathbf{b}(\lambda_1, \dots, \lambda_k) = \mathbf{b} + \sum_{i=1}^k \lambda_i \mathbf{a}^{(i)} \quad (\lambda_1, \dots, \lambda_k \in GF(m)), \quad (2.8)$$

including itself. In fact, by (2.7), the relationship of being aliased is an equivalence relation that partitions $\mathcal{B}$ into $(m^{n-k} - 1)/(m - 1)$ equivalence classes each of cardinality m^k . Any such equivalence class is called an alias set with reference to the fraction $d(A)$. Thus the pencils $\mathbf{b}(\lambda_1, \dots, \lambda_k)$, defined in (2.8), constitute the alias set containing $\mathbf{b}$.

Lemma 2.3 *Let $\mathbf{b}, \mathbf{b}^*(\in \mathcal{B})$ be aliases of each other and*

$$L = \sum_{j=0}^{m-1} l_j S_j(\mathbf{b}) \quad \text{and} \quad L^* = \sum_{j=0}^{m-1} l_j S_j(\mathbf{b}^*) \quad (2.9)$$

be treatment contrasts belonging to $\mathbf{b}$ and $\mathbf{b}^*$ respectively. Then the parts of L and L^* , that involve only the treatment combinations included in $d(A)$, are identical.

Proof. By (2.1), (2.2), (2.5) and (2.9), the part of L , that involves only the treatment combinations included in $d(A)$, is given by, say,

$$L(A) = \sum_{j=0}^{m-1} l_j S_j(\mathbf{b}, A), \quad (2.10)$$

where

$$S_j(\mathbf{b}, A) = \sum_{\mathbf{z} \in V_j(\mathbf{b}, A)} \tau(\mathbf{z}), \quad (2.11)$$

and

$$V_j(\mathbf{b}, A) = V_j(\mathbf{b}) \cap d(A) = \{\mathbf{z} : \mathbf{b}'\mathbf{z} = \rho_j, A\mathbf{z} = \mathbf{0}\}. \quad (2.12)$$

Similarly, the part of L^* , that involves only the treatment combinations included in $d(A)$, equals $L^*(A)$ which is obtained upon replacement of $\mathbf{b}$ by $\mathbf{b}^*$ everywhere in (2.10)–(2.12). Since $\mathbf{b}$ and $\mathbf{b}^*$ are aliases of each other, by (2.7) and (2.12), $V_j(\mathbf{b}, A) = V_j(\mathbf{b}^*, A)$ for every j , whence the result follows. ■

Remark 2.1 Contrasts with matching coefficients, such as L and L^* of (2.9), will be called corresponding contrasts. Lemma 2.3 shows that corresponding contrasts belonging to pencils, that are members of the same alias set, cannot be distinguished on the basis of the fraction $d(A)$. In this sense, pencils belonging to the same alias set are said to be confounded with one another. Interestingly, as $\mathbf{b} \notin \mathcal{M}(A')$ in the set up of Lemma 2.3, the set $V_j(\mathbf{b}, A)$ considered in (2.12) has cardinality m^{n-k-1} for each j , so that by (2.10), (2.11), $L(A)$ itself is a contrast involving the treatment combinations in $d(A)$. ■

Lemma 2.4 Let $\mathcal{B}^*$ be an alias set and consider corresponding treatment contrasts $\sum_{j=0}^{m-1} l_j S_j(\mathbf{b}^*)$ for $\mathbf{b}^* \in \mathcal{B}^*$. Then

$$\sum^* \sum_{j=0}^{m-1} l_j S_j(\mathbf{b}^*) = m^k \sum_{j=0}^{m-1} l_j S_j(\mathbf{b}, A), \quad (2.13)$$

where $\sum^*$ denotes sum over $\mathbf{b}^* \in \mathcal{B}^*$ and $\mathbf{b}$ is any particular member of $\mathcal{B}^*$.

Proof. Let Z be the set of all the m^n treatment combinations. For $0 \leq j \leq m-1$, $\mathbf{b}^* \in \mathcal{B}^*$ and $\mathbf{z} \in Z$, define the indicator $\phi_j(\mathbf{b}^*, \mathbf{z})$ which assumes the value 1 if $\mathbf{z} \in V_j(\mathbf{b}^*)$ and the value 0 otherwise, i.e., by (2.1),

$$\begin{aligned} \phi_j(\mathbf{b}^*, \mathbf{z}) &= 1, \text{ if } (\mathbf{b}^*)'\mathbf{z} = \rho_j, \\ &= 0, \text{ otherwise.} \end{aligned} \quad (2.14)$$

From (2.2), (2.3) and (2.14)

$$\sum^* \sum_{j=0}^{m-1} l_j S_j(\mathbf{b}^*) = \sum_{j=0}^{m-1} l_j \sum_{\mathbf{z} \in Z} \left\{ \sum^* \phi_j(\mathbf{b}^*, \mathbf{z}) \right\} \tau(\mathbf{z}). \quad (2.15)$$

But $\mathbf{b} \in \mathcal{B}^*$ and hence the pencils given by (2.8) constitute the set $\mathcal{B}^*$. Therefore, by (2.14), for any fixed j and $\mathbf{z}$ the quantity $\sum^* \phi_j(\mathbf{b}^*, \mathbf{z})$ equals the cardinality of the set

$$\{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)' : \boldsymbol{\lambda}' A \mathbf{z} = \rho_j - \mathbf{b}' \mathbf{z} \text{ and } \lambda_i \in GF(m) \text{ for each } i\}.$$

Consequently, by (2.5) and (2.12)

$$\begin{aligned} \sum^* \phi_j(\mathbf{b}^*, \mathbf{z}) &= m^k \text{ if } \mathbf{z} \in V_j(\mathbf{b}, A), \\ &= 0 \text{ if } \mathbf{z} \in d(A) - V_j(\mathbf{b}, A), \\ &= m^{k-1} \text{ if } \mathbf{z} \notin d(A). \end{aligned}$$

If one substitutes the above in (2.15), recalls (2.11) and uses the fact that $\sum_{j=0}^{m-1} l_j = 0$, then the result follows. ■

From (2.10) and Remark 2.1, it is clear that the right hand side of (2.13) is a contrast involving only the treatment combinations included in $d(A)$. Therefore, the right hand side and hence the left hand side of (2.13) will be estimable in $d(A)$. In other words, while pencils belonging to the same alias set are confounded with one another (*vide* Remark 2.1), the sum of corresponding contrasts belonging to such pencils is estimable in $d(A)$. Thus any treatment contrast belonging to a pencil $\mathbf{b} \in \mathcal{B}$ is estimable in $d(A)$ if and only if corresponding contrasts belonging to all other pencils that are aliased with $\mathbf{b}$ are ignorable.

We shall say that a pencil is estimable in $d(A)$ if so is every treatment contrast belonging to it. Similarly, if every treatment contrast belonging to a pencil is ignorable then the pencil itself will be called ignorable. Then from Lemmas 2.3, 2.4 and the discussion in the last paragraph, the following is evident.

Lemma 2.5 *A pencil $\mathbf{b} \in \mathcal{B}$ is estimable in $d(A)$ if and only if all other pencils that are aliased with $\mathbf{b}$ are ignorable.* ■

Example 2.1 With reference to a 3^5 factorial, consider a regular fraction $d(A)$ which is given by (2.5) with

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 & 2 \end{pmatrix},$$

and consists of $N = 27$ distinct treatment combinations. Here $m = 3, n = 5, k = 2$ and, by (2.6), the distinct defining pencils are

$$(1, 1, 0, 2, 0)', (1, 2, 1, 0, 2)', (1, 0, 2, 1, 1)' \text{ and } (0, 1, 1, 1, 2)'.$$

No main effect pencil (i.e., one that corresponds to a main effect and has exactly one nonzero entry) is a defining pencil. By (2.8), the alias set containing the main effect pencil $(1, 0, 0, 0, 0)'$ is given by

$$\begin{aligned} &\{(1, 0, 0, 0, 0)', (1, 1, 2, 0, 1)', (0, 1, 2, 0, 1)', \\ &(1, 2, 0, 1, 0)', (0, 0, 1, 2, 2)', (1, 2, 2, 2, 1)', \\ &(0, 1, 0, 2, 0)', (1, 1, 1, 1, 2)', (1, 0, 1, 2, 2)'\}. \end{aligned}$$

Similarly, one can find the alias sets containing all other main effect pencils and check that no two distinct main effect pencils have been aliased with each other. Hence if all factorial effects involving two or more factors, i.e., all interactions, are absent then the pencils belonging to these effects are ignorable and, therefore, by Lemma 2.5, all main effect pencils are estimable in the regular fraction $d(A)$ under consideration. ■

Note that in Example 2.1, there are three or more nonzero entries in each defining pencil. Hence the conclusion in this example regarding the estimability of all main effect pencils under the absence of all interactions is, in fact, a consequence of the following general result.

Theorem 2.1 *In a regular fraction, all contrasts belonging to factorial effects involving f or less factors are estimable under the absence of all factorial effects involving $t + 1$ or more factors ($1 \leq f \leq t \leq n - 1$) if and only if each defining pencil has at least $f + t + 1$ nonzero entries.*

Proof. Let $\mathbf{b}$ be any pencil involving f or less nonzero entries. By (2.4) and Lemmas 2.2 and 2.5, $\mathbf{b}$ is estimable in a regular fraction under the absence of all factorial effects involving $t + 1$ or more factors if and only if

- (i) $\mathbf{b}$ is not a defining pencil, and
- (ii) each other pencil aliased with $\mathbf{b}$ has at least $t + 1$ nonzero entries.

By (2.8), the above conditions hold for every pencil $\mathbf{b}$ having f or less nonzero entries if and only if each defining pencil has at least $f + t + 1$ nonzero entries. Hence the result follows. ■

In view of Theorem 2.1, the minimum number of nonzero entries in a defining pencil plays an important role in determining the behaviour of a regular fraction. Following Box and Hunter (1961a,b), this minimum number is called the resolution of such a fraction. In Remark 2.2 below, we shall indicate the link between this definition of resolution and what was introduced in Section 2.4 and followed in the previous chapters. Observe that each defining pencil in Example 2.1 has three or four nonzero entries. Hence the plan considered there has resolution three according to the present definition. In view of Theorem 2.1, a regular fraction of resolution one or two cannot ensure the estimability of all main effect contrasts even under the absence of all interactions. As such, hereafter, only regular fractions of resolution three or more will be considered. Then the following corollary is evident from Theorem 2.1.

Corollary 2.1 *A regular fraction of resolution $R(\geq 3)$ keeps all contrasts belonging to factorial effects involving f or less factors estimable under the absence of all factorial effects involving $R - f$ or more factors, whenever the integer f satisfies $1 \leq f \leq \frac{1}{2}(R - 1)$.* ■

The next result connects regular fractions with orthogonal arrays.

Theorem 2.2 *The treatment combinations included in a regular fraction of resolution R , when written as rows, represent a symmetric orthogonal array of strength $R - 1$.*

Proof. Consider a regular fraction $d(A)$ as given by (2.5). Since the $k \times n$ matrix A has full row rank, there exists an $(n - k) \times n$ matrix C , with elements from $GF(m)$ and also of full row rank, such that the row spaces of A and C are orthocomplements of each other, i.e.,

$$CA' = O. \quad (2.16)$$

By (2.5), then the m^{n-k} distinct vectors in the row space of C represent the treatment combinations in $d(A)$, written as rows. Furthermore, if $d(A)$ has resolution R then every $(n - k) \times (R - 1)$ submatrix of C has rank $R - 1$ for otherwise, by (2.16), there exists

a nonnull vector in $\mathcal{M}(A')$, i.e., a defining pencil (cf. (2.6)), with less than R nonzero entries. This setting is precisely the same as that in Lemma 3.4.1 and the rest of the proof follows using the same arguments as there. ■

Remark 2.2 From Theorems 2.6.1 and 2.2, it is clear that a regular fraction of resolution $R(\geq 3)$ represents a universally optimal Resolution (f, t) plan, in the sense of Section 2.4 and within the class of all plans having the same number of runs, for every choice of integers f, t such that $f + t = R - 1$ and $1 \leq f \leq t \leq n - 1$. The reader may refer back to Remark 2.6.2 as well in this connection. ■

Remark 2.3 Theorem 2.2 shows that the necessary conditions for the existence of symmetric orthogonal arrays, notably Rao's bound, are also applicable to regular fractions. Further details on the issue of existence of regular fractions in the case of two-level factorials are available in Draper and Lin (1990). Interestingly, if $R \geq 3$ in Theorem 2.2 then no two distinct columns of the matrix C , of full row rank, are proportional, so that the n columns of C can be interpreted as distinct points of the finite projective geometry $PG(n - k - 1, m)$ (cf. Remark 3.4.3). Because of the duality between the matrices C and A in the sense of having row spaces that are orthocomplements of each other, it follows that a regular fraction of resolution three or more is equivalent to a set of n distinct points of a finite projective geometry such that the matrix, given by these points as columns, has full row rank. Since $PG(n - k - 1, m)$ contains $(m^{n-k} - 1)/(m - 1)$ distinct points, given m, n and k , then a regular fraction of resolution three or more exists if and only if

$$n \leq (m^{n-k} - 1)/(m - 1). \quad (2.17)$$

■

Corollary 2.1 suggests that given m, n and k , one should look for a regular fraction having resolution as high as possible. However, there may be more than one regular fraction with the maximum possible resolution and, although they all share the same universal optimality property as indicated in Remark 2.2 under a specified model, one may discriminate among them on the basis of their defining pencils and aliasing patterns when interest lies in studying the consequences of possible model variation. To illustrate this point, we revisit Example 2.1. There $m = 3, n = 5, k = 2$, and no regular fraction can have resolution greater than three. This follows noting that if a fraction of resolution four is available then, by Theorem 2.2, one gets a symmetric $OA(27, 5, 3, 3)$ which is nonexistent by Remark 3.4.1. The fraction discussed in Example 2.1 is, indeed, of the maximum possible resolution, namely three. However, in the same set up, the fraction $d(\tilde{A})$, where

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 \end{pmatrix},$$

has defining pencils

$$(1, 1, 0, 2, 0)', (1, 0, 1, 0, 2)', (1, 2, 2, 1, 1)', (0, 1, 2, 2, 1)',$$

and is also of resolution three. Observe that $d(A)$ has only one defining pencil with exactly three nonzero entries while there are two such pencils in $d(\tilde{A})$. Hence, by explicitly writing the alias sets or directly from (2.8), it can be seen that in $d(A)$ there are three distinct two-factor interaction pencils (i.e., pencils with exactly two nonzero entries) that are aliased

with the main effect pencils while in $d(\tilde{A})$ the number of such two-factor interaction pencils equals six. Therefore, if one is not fully confident about the absence of all two-factor interactions, then $d(A)$ seems to be preferable to $d(\tilde{A})$ although both of them have the highest possible resolution. More generally, if $W_3(\geq 0)$ be the number of pencils with exactly three nonzero entries in any regular fraction of resolution three or more, then from (2.8) it may be verified that $3W_3$ distinct two-factor interaction pencils get aliased in such a fraction with the main effect pencils. Thus, given any two resolution three fractions, the one with a smaller value of W_3 will be preferred under possible uncertainty about the model.

The considerations mentioned in the last paragraph lead to the criterion of minimum aberration introduced by Fries and Hunter (1980) for regular fractions. Under possible model uncertainty, this criterion is particularly meaningful when all pencils with the same number of nonzero entries are considered equally important and, as usual, the lower order factorial effects are supposed to be more important than the higher order ones. With reference to a regular fraction $d(A)$, as defined in (2.5), let $W_i(A)$ be the number of defining pencils with exactly i nonzero entries ($1 \leq i \leq n$). Then the sequence

$$\{W_1(A), W_2(A), W_3(A), \dots, W_n(A)\}$$

is called the wordlength pattern of $d(A)$ following a coding theoretic terminology. Given m, n and k , consider two regular fractions $d(A)$ and $d(\tilde{A})$ and suppose u is the smallest integer such that $W_u(A) \neq W_u(\tilde{A})$. Then $d(A)$ is said to have less aberration than $d(\tilde{A})$ if $W_u(A) < W_u(\tilde{A})$. A regular fraction $d(A)$ is said to represent a minimum aberration (MA) design if there is no other regular fraction having less aberration than $d(A)$. Obviously, an MA design has the highest possible resolution as well in a given context.

In the set up of Example 2.1, the wordlength patterns of $d(A)$ and $d(\tilde{A})$ are $\{0, 0, 1, 3, 0\}$ and $\{0, 0, 2, 1, 1\}$ respectively, so that $d(A)$ has less aberration than $d(\tilde{A})$. In fact, Chen, Sun and Wu (1993) noted that $d(A)$ is the MA design here.

Franklin (1984) gave some tables of MA designs. Theoretical characterizations for such designs were obtained by Chen and Wu (1991) for $k = 3, 4$, and by Chen (1992) for $m = 2, k = 5$. Using an efficient computational algorithm, Chen, Sun and Wu (1993) compiled a catalogue of regular fractions which are good under the criterion of aberration. This catalogue incorporates, in particular, MA designs for $m = 2$ and 3 over a practically useful range. Laycock and Rowley (1995) suggested a method of generating all regular fractions for given m, n and k . Their results can help in finding MA designs.

The combinatorics underlying MA designs is deep and much new ground has recently been broken via consideration of complementary sets. From Remark 2.3, recall that a regular fraction of resolution three or more is equivalent to a set of n distinct points of the finite projective geometry $PG(n - k - 1, m)$. When n is close to the upper bound in (2.17), the complementary set in $PG(n - k - 1, m)$ has a small cardinality. In this and several other situations, the theoretical study of MA designs can be greatly facilitated by expressing the wordlength pattern of a regular fraction in terms of the complementary set. The reader is referred to Chen and Hedayat (1996), Tang and Wu (1996) and Suen *et al.* (1997) for these interesting developments.

Wu and Zhang (1993) extended the idea of MA designs to the case of $2^{n_1} \times 4^{n_2}$ factorials constructed by the method of grouping discussed in Section 4.6. This work was followed up by Mukerjee and Wu (1997a).

In an attempt to provide more explicit statistical justification for MA designs, Cheng, Steinberg and Sun (1998) and Cheng and Mukerjee (1998) considered the criterion of estimation capacity. According to this criterion, a regular fraction $d(A)$ of resolution

three or more is evaluated on the basis of the largeness of the quantities $E_u(A)$, $u \geq 1$, where $E_u(A)$ is the number of possible models, involving all the main effect pencils and exactly u two-factor interaction pencils, that $d(A)$ can estimate when all interactions involving three or more factors are absent. The sequence $\{E_1(A), E_2(A), \dots\}$ is called the estimation capacity sequence of $d(A)$. The computational as well as theoretical results reported by these authors indicate that the criteria of minimum aberration and estimation capacity are quite in agreement.

Observe that estimation capacity is essentially a criterion of model robustness. We refer to Paik and Federer (1973), Hedayat *et al.* (1974), Raktue (1976), Monette (1983), Hedayat and Pesotan (1990), Meyer *et al.* (1996) and Filliben and Li (1997), among others, for results having an impact on the study of model robustness from other considerations.

The issue of blocking for regular fractions has also received attention in the literature. In the context of an m^n factorial, a blocked regular fraction $d(A, B)$ is specified by $k + s$ linearly independent pencils $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)}, \mathbf{b}^{(1)}, \dots, \mathbf{b}^{(s)}$, where $k, s \geq 1$ and $k + s < n$. It consists of m^s blocks, indexed by the $s \times 1$ vectors $\boldsymbol{\xi}$ over $GF(m)$, such that a typical block contains m^{n-k-s} distinct treatment combinations $\mathbf{z}$ satisfying

$$A\mathbf{z} = \mathbf{0}, \quad B\mathbf{z} = \boldsymbol{\xi}, \quad (2.18)$$

where

$$A = (\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(k)})', \quad B = (\mathbf{b}^{(1)}, \dots, \mathbf{b}^{(s)})'.$$

A treatment contrast is called a between block contrast in $d(A, B)$ if it involves only the treatment combinations included in $d(A, B)$ and, in addition, all treatment combinations belonging to the same block appear in it with the same coefficient. The defining pencils and alias sets are again dictated by (2.6) and (2.7). Then it is easy to see that given any alias set $\mathcal{B}^*$, either

$$\mathbf{b} \in \mathcal{M}(A', B') - \mathcal{M}(A') \text{ for every } \mathbf{b} \in \mathcal{B}^*, \quad (2.19)$$

or,

$$\mathbf{b} \notin \mathcal{M}(A', B') \text{ for every } \mathbf{b} \in \mathcal{B}^*. \quad (2.20)$$

Consider now any treatment contrast L belonging to a pencil $\mathbf{b}$ in $\mathcal{B}^*$. As before, the part of L , that involves only the treatment combinations included in $d(A, B)$, is given by $L(A)$ as specified in (2.10)–(2.12). If (2.19) holds then by (2.12) and (2.18), the set $V_j(\mathbf{b}, A)$ is the union of m^{s-1} distinct blocks for each j . Hence then by (2.10)–(2.12), $L(A)$ is a between block contrast in $d(A, B)$. On the other hand, if (2.20) holds then by (2.12) and (2.18), each set $V_j(\mathbf{b}, A)$ has $m^{n-k-s-1}$ treatment combinations in common with every block, so that by (2.10)–(2.12), $L(A)$ is orthogonal to every between block contrast. From the points just noted, the following can be deduced:

- (i) If (2.19) holds for any alias set $\mathcal{B}^*$, then no treatment contrast belonging to any pencil in $\mathcal{B}^*$ remains estimable in $d(A, B)$ under the presence of block effects. In this sense, then the alias set $\mathcal{B}^*$ is said to be confounded with blocks.
- (ii) If (2.20) holds for any alias set $\mathcal{B}^*$, then Lemma 2.5 is applicable to $\mathcal{B}^*$ even in the presence of block effects, i.e., any pencil $\mathbf{b}$ belonging to such an alias set is estimable in $d(A, B)$ if and only if all other pencils that are aliased with $\mathbf{b}$ are ignorable.

By (2.8) and (2.19), the number of alias sets that are confounded with blocks equals $(m^s - 1)/(m - 1)$. Recalling that there are $(m^{n-k} - 1)/(m - 1)$ alias sets altogether, this leaves $(m^{n-k} - m^s)/(m - 1)$ alias sets which satisfy (2.20) and are not confounded with blocks.

Example 2.2 This is in continuation of Example 2.1. With reference to a 3^5 factorial, consider a blocked regular fraction $d(A, B)$, where A is as shown earlier in Example 2.1 and B consists of a single row $(0, 0, 1, 0, 1)$. Here $m = 3, n = 5, k = 2, s = 1$, and $d(A, B)$ can be obtained explicitly from (2.18). Note that $d(A, B)$ consists of 27 distinct treatment combinations which are the same as those included in $d(A)$ of Example 2.1 but now arranged in three blocks of size 9 each. The defining pencils are as shown in Example 2.1 and none of these is a main effect pencil. Since $s = 1$, only one alias set gets confounded with blocks. By (2.19), this confounded alias set is given by

$$\begin{aligned} &\{(0, 0, 1, 0, 1)', (1, 1, 1, 2, 1)', (1, 1, 2, 2, 2)', \\ &(1, 2, 2, 0, 0)', (1, 2, 0, 0, 1)', (1, 0, 0, 1, 2)', \\ &(1, 0, 1, 1, 0)', (0, 1, 2, 1, 0)', (0, 1, 0, 1, 1)'\}, \end{aligned}$$

and it does not contain any main effect pencil. Thus the alias sets containing main effect pencils are not confounded with blocks and the same conclusion as in Example 2.1 holds regarding the estimability of the main effect pencils under the absence of all interactions. ■

The reader may verify that the plan shown in Example 7.4.1 is also a blocked regular fraction with $m = 2, n = 4, k = 1, s = 2$, $\mathbf{a}^{(1)} = (1, 1, 1, 1)'$, $\mathbf{b}^{(1)} = (1, 1, 0, 0)'$, $\mathbf{b}^{(2)} = (1, 0, 1, 0)'$. We refer to Box, Hunter and Hunter (1978) for more details on the principles underlying the issue of blocking. Very recently, several extensions of the notion of resolution and aberration to blocked regular fractions have been proposed and studied in the literature from various viewpoints. The reader is referred to Bisgaard (1994), Sun *et al.* (1997), Sitter *et al.* (1997), Mukerjee and Wu (1997b) and Chen and Cheng (1997) for these developments.

Regular fractions and extensions thereof have been studied from many other perspectives. Srivastava *et al.* (1984) explored parallel flats fractions which consist of treatment combinations $\mathbf{z}$ satisfying

$$A\mathbf{z} \in \{\mathbf{c}_1, \dots, \mathbf{c}_q\},$$

where A is as in (2.5) and $\mathbf{c}_1, \dots, \mathbf{c}_q$ are $q(> 1)$ distinct $k \times 1$ vectors over $GF(m)$. This work was followed up by Srivastava (1987), Buhadra and Anderson (1995), Srivastava and Li (1996) and Liao *et al.* (1996) among others, and we refer to Ghosh (1996, Section 7) for a review. Cheng and Li (1993) investigated the construction of regular fractions of two-level factorials when certain treatment combinations are considered infeasible and hence debarred. Several authors considered the problem of finding regular fractions when interest lies in specified factorial effects, e.g., the main effects and some specified two-factor interactions. In this connection, we refer to Franklin (1985), Wu and Chen (1992) and Sun and Wu (1994) where more references can be found. Accounts of further group theoretic developments related to regular fractions, including those associated with a complex linear model, are available in Kobilinsky (1985), Bailey (1985), Kobilinsky and Monod (1995) and the references therein.

3 Search Designs

The theory of search linear models, due to Srivastava (1975), plays a major role in finding appropriate fractional factorial plans under model uncertainty. We begin by presenting Theorem 3.1 below which is central to this theory and then adopt it to our context.

Consider the linear model

$$\mathbb{E}(\mathbf{Y}) = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2, \quad \mathbb{D}(\mathbf{Y}) = \sigma^2 I_N, \quad (3.1)$$

where $\mathbf{Y}$ is the $N \times 1$ observational vector, $X = [X_1 \dot{\vdots} X_2]$ is the known design matrix, $\boldsymbol{\beta} = (\boldsymbol{\beta}_1', \boldsymbol{\beta}_2')'$ is the parametric vector, σ^2 is the common variance of the observations, and I_N is the identity matrix of order N . It is known that $\boldsymbol{\beta}_2$ can be partitioned as

$$\boldsymbol{\beta}_2 = (\boldsymbol{\beta}_{21}', \dots, \boldsymbol{\beta}_{2a}')', \quad (3.2)$$

and that among $\boldsymbol{\beta}_{21}, \dots, \boldsymbol{\beta}_{2a}$, at most k are nonnull, where k is a known positive integer which is quite small relative to a . There is, however, no knowledge about which of $\boldsymbol{\beta}_{21}, \dots, \boldsymbol{\beta}_{2a}$ are possibly nonzero and what their values are. The objectives are to estimate $\boldsymbol{\beta}_1$ and search and estimate the possibly nonzero entities among $\boldsymbol{\beta}_{21}, \dots, \boldsymbol{\beta}_{2a}$. If these objectives are met, then following Srivastava (1975), the design d associated with the observational vector $\mathbf{Y}$ in (3.1) is said to have resolving power $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k)$. Let

$$X_2 = [X_{21} \dot{\vdots} \dots \dot{\vdots} X_{2a}]$$

be the partitioned form of X_2 corresponding to (3.2). Then the following result, due to Srivastava (1975), holds.

Theorem 3.1 (a) For the design d to have resolving power $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k)$ in the set up described above, it is necessary that the matrix

$$X(i_1, \dots, i_{2k}) = [X_{i_1} \dot{\vdots} X_{i_{2k}}] \quad (3.3)$$

has full column rank for every $i_1, \dots, i_{2k}$ ($1 \leq i_1 < \dots < i_{2k} \leq a$).

(b) If in addition, $\sigma^2 = 0$, then the necessary condition stated above is also sufficient for d to have resolving power $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k)$.

Proof. (a) Let d have resolving power $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k)$. If possible, suppose the stated rank condition does not hold. Then the matrix $X(i_1, \dots, i_{2k})$ does not have full column rank for some $i_1, \dots, i_{2k}$, say, without loss of generality, for $i_j = j$ ($1 \leq j \leq 2k$). Then by (3.3), there exist vectors $\boldsymbol{\xi}, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{2k}$, of appropriate orders and not all null, such that

$$X_1\boldsymbol{\xi} + \sum_{j=1}^{2k} X_{2j}\boldsymbol{\xi}_j = \mathbf{0}. \quad (3.4)$$

If $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{2k}$ are all null then $\boldsymbol{\xi} \neq \mathbf{0}$, so that X_1 is not of full column rank. But then $\boldsymbol{\beta}_1$ is not estimable and this contradicts the fact that d has resolving power $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, k)$. Hence $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{2k}$ are not all null. Without loss of generality, suppose $\boldsymbol{\xi}_1 \neq \mathbf{0}$.

Consider now two models M_1 and M_2 such that according to M_1

$$\mathbb{E}(\mathbf{Y}) = X_1\boldsymbol{\xi}^* + \sum_{j=1}^k X_{2j}\boldsymbol{\xi}_j, \quad \mathbb{D}(\mathbf{Y}) = \sigma^2 I_N, \quad (3.5)$$

and under M_2 ,

$$\mathbb{E}(\mathbf{Y}) = X_1(\boldsymbol{\xi}^* - \boldsymbol{\xi}) + \sum_{j=k+1}^{2k} X_{2j}(-\boldsymbol{\xi}_j), \quad \mathbb{D}(\mathbf{Y}) = \sigma^2 I_N, \quad (3.6)$$

where ξ^* is any vector of appropriate order. By (3.5) and (3.6), the models M_1 and M_2 correspond to (3.1) with

$$\beta_1 = \xi^*, \beta_{2j} = \xi_j \ (1 \leq j \leq k), \beta_{2j} = \mathbf{0} \ (k+1 \leq j \leq a),$$

and

$$\beta_1 = \xi^* - \xi, \beta_{2j} = \mathbf{0} \ (1 \leq j \leq k \text{ and } 2k+1 \leq j \leq a), \beta_{2j} = -\xi_j \ (k+1 \leq j \leq 2k),$$

respectively. Thus under both M_1 and M_2 , at most k of $\beta_{21}, \dots, \beta_{2a}$ are nonnull. However, the set of possibly nonnull vectors among $\beta_{21}, \dots, \beta_{2a}$ under M_1 is different from that under M_2 . For instance, $\beta_{21}(=\xi_1)$ is nonnull under M_1 but null under M_2 . On the other hand, by (3.4)–(3.6), the expression for $E(\mathbf{Y})$ under (3.5) is the same as that under (3.6). Therefore, on the basis of $\mathbf{Y}$, the models M_1 and M_2 are not distinguishable from each other although the sets of possibly nonnull vectors among $\beta_{21}, \dots, \beta_{2a}$ are different under these two models. This again contradicts the fact that d has resolving power (β_1, β_2, k) . Hence part (a) of the theorem follows.

(b) let $\mathcal{C}$ be the class of subsets of $\{1, 2, \dots, a\}$, with cardinality k or less. The empty set is also a member of $\mathcal{C}$. If $\sigma^2 = 0$ then, in view of (3.1) and the fact that among $\beta_{21}, \dots, \beta_{2a}$ at most k are nonnull, there exists a set C in $\mathcal{C}$ and vectors $\beta_1(C), \beta_{2i}(C)$ ($i \in C$), such that

$$\mathbf{Y} = X_1\beta_1(C) + \sum_{i \in C} X_{2i}\beta_{2i}(C), \quad (3.7)$$

with probability unity, and

$$\beta_{2i}(C) \neq \mathbf{0} \text{ for each } i \in C. \quad (3.8)$$

Assume that the rank condition stated in part (a) holds. In order to prove that then d has resolving power (β_1, β_2, k) , we first show that $\mathbf{Y}$ cannot be of the form (3.7), subject to (3.8), simultaneously for two distinct sets in $\mathcal{C}$. If possible, suppose there are two such distinct sets, say C_1 and C_2 , in $\mathcal{C}$. Then by (3.7) and (3.8),

$$X_1\beta_1(C_1) + \sum_{i \in C_1} X_{2i}\beta_{2i}(C_1) = X_1\beta_1(C_2) + \sum_{i \in C_2} X_{2i}\beta_{2i}(C_2), \quad (3.9)$$

where

$$\beta_{2i}(C_1) \neq \mathbf{0} \text{ for each } i \in C_1, \quad (3.10)$$

and

$$\beta_{2i}(C_2) \neq \mathbf{0} \text{ for each } i \in C_2. \quad (3.11)$$

Since the sets C_1 and C_2 are not identical, it follows from (3.9)–(3.11) that the matrix $[X_1 : X_2^*]$, where

$$X_2^* = [\dots, X_{2i}, \dots]_{i \in C_1 \cup C_2},$$

does not have full column rank. But then the assumed rank condition is violated since $C_1 \cup C_2$ has cardinality at most $2k$. Thus we arrive at a contradiction and it follows that $\mathbf{Y}$ is of the form (3.7), subject to (3.8), for a unique set C in $\mathcal{C}$. Furthermore, invoking the rank condition again, the vectors $\beta_1(C)$ and $\beta_{2i}(C)$ ($i \in C$) in (3.7) are unique. Hence, given $\mathbf{Y}$, there is no ambiguity about the true model so that d has resolving power (β_1, β_2, k) . ■

Remark 3.1 If $\sigma^2 = 0$ and the rank condition stated in part (a) of Theorem 3.1 holds then the true model can be identified with probability unity. The following considerations may then help in actually finding the unique set C and hence the true model (3.7). For any H in $\mathcal{C}$, let $X(H) = [X_1 : X_2(H)]$ where

$$X_2(H) = [\dots, X_{2i}, \dots]_{i \in H}.$$

Under the rank condition, the columns of $X_2(H)$ are linearly independent. Write

$$Q(H) = X(H)\{X(H)'X(H)\}^{-1}X(H)'$$

for the orthogonal projector on the column space of $X(H)$. Also, let

$$\{X(H)'X(H)\}^{-1}X(H)'Y = \begin{bmatrix} \gamma_1 \\ \gamma_2(H) \end{bmatrix},$$

where

$$\gamma_2(H) = (\dots, \gamma_{2i}(H)', \dots)'_{i \in H},$$

and the orders of γ_1 and $\gamma_{2i}(H)$ are the same as the number of columns of X_1 and X_{2i} respectively. Then it is not hard to see that a member H of $\mathcal{C}$ equals the unique set C , as envisaged in (3.7), if and only if

$$Q(H)Y = Y, \text{ and } \gamma_{2i}(H) \neq \mathbf{0} \text{ for each } i \in H,$$

in which case $\gamma_{2i}(H)$ equals $\beta_{2i}(C)$ for every $i \in H$. Hence one may proceed to devise an algorithm for the actual determination of the true model (3.7). ■

Remark 3.2 For $\sigma^2 > 0$, even under the rank condition, identification of the true model is not possible with probability unity. As suggested in Srivastava (1975, 1996), the following search procedure is then helpful. Let $\mathcal{C}_0$ be a subclass of $\mathcal{C}$ consisting of those subsets of $\{1, 2, \dots, a\}$ which have cardinality exactly k . For any $H \in \mathcal{C}_0$, let $M(H)$ be the submodel of (3.1) that incorporates β_1 and β_{2i} ($i \in H$) and takes $\beta_{2i} = \mathbf{0}$ for $i \notin H$. For each $H \in \mathcal{C}_0$, compute

$$\text{RSS}(H) = Y'[I_N - Q(H)]Y,$$

which is the residual sum of squares on the basis of $M(H)$. Conclude that $M(H_0)$ is the true model if $H_0 \in \mathcal{C}_0$ is such that $\text{RSS}(H) \geq \text{RSS}(H_0)$ for every $H \in \mathcal{C}_0$. ■

We now discuss how Theorem 3.1 can be applied to study fractional factorial plans under model uncertainty. We consider a general $m_1 \times \dots \times m_n$ factorial set up, resume the use of the Kronecker notation and return to the model (2.3.3) given by

$$\mathbb{E}(Y) = X_d\tau, \quad \mathbb{D}(Y) = \sigma^2 I_N, \quad (3.12)$$

where, as before, Y is the $N \times 1$ observational vector under a plan d involving N runs, τ is the vector of effects of the treatment combinations arranged in lexicographic order, and

$$X_d = ((\chi_d(u; j_1 \dots j_n))),$$

with the indicator $\chi_d(u; j_1 \dots j_n)$ assuming the value 1 if the u th observation according to d corresponds to the treatment combination $j_1 \dots j_n$ and the value zero otherwise.

Consider a scenario where one wishes to use a model that incorporates the general mean and the lower order factorial effects and possibly excludes the higher order factorial effects but is uncertain about the status of the remaining factorial effects. To be more specific, suppose the following information is available about the underlying model for some integers f, t ($0 \leq f < t \leq n$) and $k(> 0)$:

- (a) all factorial effects involving $t + 1$ or more factors are absent;
- (b) among all factorial effects involving more than f but not more than t factors, at most k are present where k is small compared to the total number of such effects and it is unknown which of these effects are possibly present.

Because of (a) and (b), the factorial effects can be classified into three categories:

- (i) those involving more than t factors: by (a), these need not be included in the model;
- (ii) those involving more than f but not more than t factors: by (b), these have to be included in the model with the knowledge that at most k of them are actually present;
- (iii) those involving f or less factors: these, as well as the general mean, are to be included in the model.

Consideration of the intermediate category (ii), arising from (b) above, induces flexibility in the model and helps in avoiding rigid assumptions regarding the absence of specific effects in this category. The condition (b) implies that among the effects in category (ii) only a few are possibly present without spelling out what these possibly present effects are. Thus this condition signifies both effect sparsity and model uncertainty. Note that if, in particular, $t = n$ then (a) and (i) above do not arise, i.e., none of the factorial effects is straightforwardly assumed to be absent. On the other hand, if $f = 0$ then by (b) even some main effects are potentially absent.

For ease in reference, we recall some notation from Chapter 2. Let Ω_ν be the set of binary n -tuples with at most ν components unity and the matrix $P^{\mathbf{x}}$ be as defined in (2.2.4) and (2.2.5). Also, let $\beta_{\mathbf{x}}, \beta^{(1)}$ and $P^{(1)}$ be as in (2.3.7)–(2.3.9). Then, in consideration of (i)–(iii) above and analogously to (2.3.12), under (a) and (b), the model (2.3.3), or equivalently (3.12), can be expressed as

$$E(\mathbf{Y}) = X_d(P^{(1)})'\beta^{(1)} + \sum_{f,t} X_d(P^{\mathbf{x}})'\beta_{\mathbf{x}}, \quad D(\mathbf{Y}) = \sigma^2 I_N, \quad (3.13)$$

where $\sum_{f,t}$ denotes sum over $\mathbf{x} \in \Omega_t - \Omega_f$ and among the $\beta_{\mathbf{x}}$, $\mathbf{x} \in \Omega_t - \Omega_f$, at most k are nonnull. Recall that $\beta^{(1)}$ represents the general mean and complete sets of orthonormal contrasts belonging to factorial effects involving at most f factors. Also, for each $\mathbf{x} = x_1 \dots x_n \in \Omega_t - \Omega_f$, by (2.3.7), $\beta_{\mathbf{x}}$ represents a complete set of orthonormal contrasts belonging to factorial effect $F^{\mathbf{x}} (\equiv F_1^{x_1} \dots F_n^{x_n})$, which obviously involves more than f but not more than t factors. The objectives are to estimate $\beta^{(1)}$ and search and estimate the possibly nonnull $\beta_{\mathbf{x}}$, $\mathbf{x} \in \Omega_t - \Omega_f$. In the spirit of Srivastava (1975), a plan d meeting these objectives will be called a search design with resolving power (f, t, k) .

Comparing (3.1) and (3.13), Theorem 3.1 can now be paraphrased in the context of fractional factorial plans as follows.

Theorem 3.2 (a) For a plan d to be a search design with resolving power (f, t, k) , it is necessary that the matrix

$$X_d[(P^{(1)})' : (P^{\mathbf{x}(1)})' : \dots : (P^{\mathbf{x}(2k)})']$$

has full column rank for every choice of $2k$ distinct members $\mathbf{x}(1), \dots, \mathbf{x}(2k)$ of $\Omega_t - \Omega_f$.
 (b) If $\sigma^2 = 0$ then the necessary condition stated above is also sufficient for d to be a search design with resolving power (f, t, k) . ■

The points noted in Remarks 3.1 and 3.2 are relevant also in connection with Theorem 3.2.

Example 3.1 Consider the set up of a 2^n factorial. For $0 \leq i \leq n$, let T_i be the set of treatment combinations $j_1 \dots j_n$ such that among $j_1, \dots, j_n$, exactly i equal 1 and the rest equal zero. Clearly, T_i has cardinality $\binom{n}{i}$. Following Srivastava and Ghosh (1976) and Srivastava and Arora (1991), consider the plans d_1 and d_2 consisting of the treatment combinations in

$$T_1 \cup T_{n-2} \cup T_{n-1} \cup T_n \text{ and } T_0 \cup T_1 \cup T_2 \cup T_n$$

respectively. As noted by these authors, the plans d_1 and d_2 satisfy the rank condition of Theorem 3.2 for (i) $f = 2, t = n, k = 1$, and (ii) $f = 1, t = n, k = 2$ respectively. Also, when $n \geq 7$, Shirakura and Tazawa (1992) observed that the plan d_3 given by the treatment combinations in $T_2 \cup T_{n-2} \cup T_n$ satisfies the same rank condition for $f = 2, t = 3$ and $k = 2$. We refer to the original papers for the proofs and other related results. ■

The issue of construction of plans satisfying the rank condition of Theorem 3.2 has been studied extensively in the literature. While much of this work relates to 2^n factorials, some progress has also been made with other kinds of factorials including asymmetric factorials; see e.g., Anderson and Thomas (1980), Chatterjee and Mukerjee (1986), Ghosh and Zhang (1987) and Chatterjee (1990, 1991). We refer to Gupta (1990) and Ghosh (1996) for informative reviews and further references. A comprehensive discussion, linking search linear models with other related topics, is available in Srivastava (1996).

Several authors investigated the search design problem for $\sigma^2 > 0$ and studied the probability of identifying the true model. The reader is referred to Srivastava and Mallenby (1985), Mukerjee and Chatterjee (1994) and Shirakura *et al.* (1997) for such results.

4 Supersaturated Designs

In an $m_1 \times \dots \times m_n$ factorial, suppose it is considered reasonable to assume the absence of all interactions and further suppose interest lies in the general mean and the main effects. Then the model corresponds to (2.3.12) with $f = t = 1$. From Theorem 2.3.2, it is known that at least

$$N_0 = 1 + \sum_{i=1}^n (m_i - 1) \quad (4.1)$$

runs are needed to keep the general mean and all the main effect contrasts estimable. A saturated plan meets this objective in exactly N_0 runs. However, particularly in exploratory studies, the number of factors, n , can be quite large and, from practical considerations, even N_0 runs may not be affordable.

One can still make some headway if there is reason to believe that among the n factors only a few are active in the sense of having nonnegligible main effects. As with search linear models, the few active factors are, however, typically not known *a priori*. In such situations, plans involving even less than N_0 runs may be considered with a view to estimating the general mean and identifying the nonnegligible main effects possibly via data analytic techniques like stepwise selection or ridge regression (*cf.* Lin (1993, 1995)). Plans of this kind are known as supersaturated designs.

The literature on supersaturated designs relates mostly to 2^n factorials. Hence, we take $m_1 = \dots = m_n = 2$, so that (4.1) reduces to $N_0 = n + 1$. Let d be a supersaturated design involving $N (< n + 1)$ runs. Then, as in (6.3.4) and (6.3.5), the information matrix

under d for the parametric vector $\beta^{(1)}$, representing the general mean and the main effects, is

$$\mathcal{I}_d = v^{-1} K_d K_d' \quad (4.2)$$

where $v = 2^n$ and the elements of the $(n+1) \times N$ matrix K_d are ± 1 . In fact, from (6.3.1), (6.3.5) and the definition of the matrix G_d involved in (6.3.4), it is not hard to see that

$$K_d = \begin{bmatrix} 1 & 1 & \dots & 1 \\ p_d(1,1) & p_d(1,2) & \dots & p_d(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ p_d(n,1) & p_d(n,2) & \dots & p_d(n,N) \end{bmatrix}, \quad (4.3)$$

where, for $1 \leq i \leq n$ and $1 \leq u \leq N$, the quantity $p_d(i, u)$ equals $+1$ if the i th factor appears at level 1 in the u th run of d and -1 otherwise.

Suppose the levels 0 and 1 of each factor appear equally often in d . Then d is called balanced and by (4.2), (4.3)

$$\mathcal{I}_d = v^{-1} \begin{bmatrix} N & \mathbf{0}' \\ \mathbf{0} & S_d \end{bmatrix}, \quad (4.4)$$

where $S_d = ((s_d(i, j)))$ is an $n \times n$ matrix given by

$$\begin{aligned} s_d(i, i) &= N, \quad 1 \leq i \leq n, \\ s_d(i, j) &= \sum_{u=1}^N p_d(i, u) p_d(j, u), \quad 1 \leq i \neq j \leq n. \end{aligned}$$

Had $s_d(i, j)$ been zero for every $i \neq j$, then by (4.4), $\mathcal{I}_d$ would have equalled $(N/v)I_{n+1}$ and a universal optimality result on d could have been claimed as in Lemma 2.5.2. Although the fact that $N < n+1$ rules out such a possibility (*cf.* (4.2)), the plan d may be anticipated to perform well in detecting the possibly present main effects provided the quantities $s_d(i, j)$, $i \neq j$, are small in magnitude. From this consideration, Booth and Cox (1962) proposed the criterion

$$E_d(s^2) = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} \{s_d(i, j)\}^2$$

as a performance characteristic of a balanced supersaturated design. Several other criteria for choosing supersaturated designs have been discussed in Li and Wu (1997) where a brief review of the literature is also available.

Algebraic as well as algorithmic methods for constructing supersaturated designs, that minimize $E_d(s^2)$ or behave well with respect to other criteria, have been investigated by several authors. The reader is referred to Lin (1993, 1995), Wu (1993), Nguyen (1996), Li and Wu (1997), Tang and Wu (1997) and Cheng (1997) for the details of such constructions. More references are available in Cheng (1998). We also refer to Gupta and Chatterjee (1998) for a review of supersaturated designs highlighting their relationship with search designs, and to Chatterjee and Gupta (1997) for some results on supersaturated designs for general m^n symmetric factorials.

The idea of supersaturated designs depends critically on effect sparsity in the form of the supposition that only a few of the n factors are actually active. The same consideration has led to the study of another phenomenon, namely projectivity. Following Box and Tyssedal (1996), a fractional factorial plan involving n factors has projectivity $n^*(< n)$ if

for every choice of n^* factors it produces a complete factorial. As before, suppose only a few factors are active though it is not known *a priori* what these active factors are. One may then wish to identify the active factors via an initial analysis, based on a model incorporating only the general mean and the main effects, and then proceed to explore a model that involves only the active factors so identified and interactions among them. Since the active factors are not known *a priori*, the feasibility of such an analysis, which possibly exploits data analytic techniques, depends on the projectivity of the plan used. Several authors, like Lin and Draper (1992), Cheng (1995), Wang and Wu (1995) and Box and Tyssedal (1996), examined fractional factorial plans based on orthogonal arrays with regard to projectivity and related issues. We refer to Cheng (1998) for a review of the work in this area together with a discussion on the connection with search designs.

While concluding this chapter, we emphasize again that our aim here was to familiarize the reader with only the fundamentals of minimum aberration, search and supersaturated designs. Nevertheless, it is hoped that this introduction, together with the references cited, will help in pursuing further studies on these emerging areas.

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