



Canonical reduced words and signed descent length enumeration in Coxeter groups

Umesh Shankar¹ · Sivaramakrishnan Sivasubramanian¹

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Abstract

Reifegerste and independently, Petersen and Tenner studied a statistic drops on permutations in $\mathfrak{S}_n$. Two other studied statistics on $\mathfrak{S}_n$ are depth and exc. Using descents in *canonical reduced words* of elements in $\mathfrak{S}_n$, we give an involution $f_A : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ that leads to a neat formula for the signed trivariate enumerator of drops, depth, exc in $\mathfrak{S}_n$. This gives a simple formula for the signed univariate drops enumerator in $\mathfrak{S}_n$. For the type-B Coxeter group $\mathfrak{B}_n$, using similar techniques, we show analogous univariate results. For the type-D Coxeter group, we get analogous but inductive univariate results. Under the famous Foata-Zeilberger bijection ϕ_{FZ} which takes permutations to restricted Laguerre histories, we show that permutations π and $f_A(\pi)$ map to the same Motzkin path, but have different history components. Using the Foata-Zeilberger bijection, we also get a continued fraction for the generating function enumerating the pair of statistics drops and MAD. Graham and Diaconis determined the mean and the variance of the Spearman metric of disarray $D(\pi)$ when one samples π from $\mathfrak{S}_n$ at random. As an application of our results, we get the mean and variance of the statistic drops(π) when we sample π from $\mathcal{A}_n$ at random.

Keywords Permutation statistics · Signed enumeration · Descent length · Coxeter group

Mathematics Subject Classification 05A05 · 05A15 · 05A19

1 Introduction

Let $\mathfrak{S}_n$ be the set of permutations on $[n] = \{1, 2, \dots, n\}$. While studying an optimization problem motivated by a sorting algorithm called *straight selection sort*,

✉ Umesh Shankar
204093001@iitb.ac.in; umeshshankar@outlook.com

Sivaramakrishnan Sivasubramanian
krishnan@math.iitb.ac.in

¹ Department of Mathematics, Indian Institute of Technology, Bombay, Mumbai 400076, India

Petersen and Tenner in [22] came up with the following statistic they call *depth* for elements of $\mathfrak{S}_n$ and $\mathfrak{B}_n$, the hyperoctahedral group. For the permutation group $\mathfrak{S}_n$, they gave a simple alternate expression for depth. Define the *excedance set* of $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{S}_n$ as $\text{ExcSet}(\pi) = \{i \in [n-1] : \pi_i > i\}$. Define $\text{depth}(\pi) = \sum_{i \in \text{ExcSet}(\pi)} (\pi_i - i)$.

Bagno, Biagioli, Novick and Woo in [1] extended the notion of depth to all Coxeter groups. A closely related statistic on $\mathfrak{S}_n$ was defined by Graham and Diaconis in [8]. For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{S}_n$, Graham and Diaconis defined $s(\pi) = \sum_{i=1}^n |\pi_i - i|$ to be its Spearman disarray measure. Knuth in [17, Problem 5.1.1.28] calls $s(\pi)$ as the “total displacement” of π . It is clear that for all $\pi \in \mathfrak{S}_n$, $s(\pi) = 2 \cdot \text{depth}(\pi)$. Though a generating function for depth does not seem to have a simple form, Petersen and Guay-Paquet [21] expressed the generating function for depth as a continued fraction.

Define a very similar statistic *descent drops*, denoted $\text{drops}(\pi)$ as follows. Firstly, for $\pi = \pi_1, \pi_2, \dots, \pi_n$, define $\text{DescSet}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$. Define

$$\text{drops}(\pi) = \sum_{i \in \text{DescSet}(\pi)} (\pi_i - \pi_{i+1}). \quad (1)$$

Both these statistics have been studied by Reifegerste in [23]. For $\pi \in \mathfrak{S}_n$, define $\text{exc}(\pi) = |\text{ExcSet}(\pi)|$ and $\text{des}(\pi) = |\text{DescSet}(\pi)|$. Reifegerste in [23] and Petersen and Tenner independently in [22], showed that the distribution of the ordered pairs $(\text{depth}(\pi), \text{exc}(\pi))$ and $(\text{drops}(\pi), \text{des}(\pi))$ coincides when summed over $\mathfrak{S}_n$. Their result [23, Proposition 1.1] and [22, Theorem 1.3] is as follows.

Theorem 1 (Reifegerste, Petersen and Tenner) *When $n \geq 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{depth}(\pi)} t^{\text{exc}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{drops}(\pi)} t^{\text{des}(\pi)}$$

This result in particular shows that when $n \geq 1$, $\sum_{\pi \in \mathfrak{S}_n} q^{\text{drops}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} q^{\text{depth}(\pi)}$. “Mantaci [16, Proposition 1] showed the following identity for signed enumeration of exc over $\mathfrak{S}_n$. [Mantaci] When $n \geq 1$, we have $\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{exc}(\pi)} = (1-t)^{n-1}$. Sivasubramanian in [24] showed that enumerating $\text{depth}(\pi)$ along with its sign over $\mathfrak{S}_n$ gave interesting results. In [24], $\text{depth}(\pi)$ was termed *excedance length* of π and denoted $\text{ExcLen}(\pi)$. For $\pi \in \mathfrak{S}_n$, define $\text{inv}(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}|$. He proved the following in [24, Theorem 9].

Theorem 2 (Sivasubramanian) *When $n \geq 1$, we have*

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} q^{\text{depth}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} q^{\text{ExcLen}(\pi)} = (1-q)^{n-1}.$$

Thus, a natural question is the signed enumeration of the drops in $\mathfrak{S}_n$. Here, we show that enumerating signed drops in $\mathfrak{S}_n$, gives the same result. Even more surprising is that the signed joint distribution of the three statistics, exc , depth and drops , factors nicely. For $n \geq 1$, define $\text{SgnExcDepDrop}_n(t, p, q) =$

$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{exc}(\pi)} p^{\text{depth}(\pi)} q^{\text{drops}(\pi)}$. Our first main result of this paper is the following.

Theorem 3 For $n \geq 1$, we have $\text{SgnExcDepDrop}_n(t, p, q) = (1 - tpq)^{n-1}$.

This result generalises the results of Mantaci [16, Proposition 1] and Theorem 2 while giving us the following new corollary. For $n \geq 1$, define $\text{SgnDrop}_n(q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} q^{\text{drops}(\pi)}$.

Corollary 4 When $n \geq 1$, we have $\text{SgnDrop}_n(q) = (1 - q)^{n-1}$.

Our proof of Theorem 3 uses a weight preserving, sign reversing involution f_A based on the idea of *canonical reduced words*. In Sect. 2, we cover the relevant material on reduced words in type-A and type-B Coxeter groups. In Sect. 3, we define our involution and in Sect. 4, we show that our involution preserves the drops statistic. The famous Foata-Zeilberger bijection from [10] uses the idea of restricted Laguerre history and maps permutations in $\mathfrak{S}_n$ bijectively to weighted Motzkin paths of length n . For an introduction to Motzkin paths, we suggest the paper [19] by Oste and Van der Jeugt and for heights in Motzkin paths, we refer the reader to [3] by Brennan and Knopfmacher. In Sect. 4.1, we show that our involution f_A preserves the shape of the Motzkin path under the Foata-Zeilberger bijection. Using this observation, we finish the proof of Theorem 3 and also show that under the Foata-Zeilberger bijection, the fixed points of our involution f_A map to Motzkin paths whose height is at most one.

The connection to Motzkin paths helps us get a continued fraction expression for the pair of statistics (depth, inv). This form for the continued fraction was observed and conjectured by Petersen in his email to Zeilberger [20]. In Theorem 29 we give our proof of this conjecture. Using a bijection of Clarke, Steingrimsen and Zeng [5], we get a continued fraction expression for the pair of statistics (drops, MAD) over $\mathfrak{S}_n$ (see Sect. 5 for details).

In $\mathfrak{S}_n$, as the definition of drops depends on *descents*, we consider a straightforward generalization to type-B and type-D Coxeter groups. In Sect. 6, we prove a type-B counterpart (see Theorem 31) of Corollary 4, where we enumerate type-B drops with signs, in $\mathfrak{B}_n$ using very similar ideas on reduced words. Our type-D counterpart is Theorem 34 and is given in Sect. 7. Our type-D result is not proved using canonical reduced words but inductively.

Lastly, we give two application of our enumeration results. Among other results, Graham and Diaconis in [8, Theorem 1] determined the mean and variance of the **Spearman-disarray metric** statistic when permutations are sampled randomly from $\mathfrak{S}_n$. In Subsect. 8.1, we determine the mean and the variance of the drops statistic when permutations are randomly sampled from $\mathcal{A}_n$. In Subsect. 8.2, we show that our reduced word-based involution gives a matching in the Bruhat order on the type-A and type-B Weyl groups. Our matching is different from the one given by Jones [15] for the type-A case.

2 Canonical reduced words

We briefly cover the relevant material on canonical reduced words in type-A and type-B Coxeter groups needed for our proof. We point the reader to Humphreys [14], Stembridge [26] and Garsia [12] as good background and references for this topic.

Let (W, S) be a Coxeter system. As each element $w \in W$ can be written as a product of generators from S , its *length* is the minimal number of generators required in any such product. The type-A Coxeter group $\mathfrak{S}_n$ is generated by $S = \{s_1, s_2, \dots, s_{n-1}\}$ where s_i is the transposition $(i, i + 1)$. The hyperoctahedral group $\mathfrak{B}_n$ is the type-B Coxeter group consisting of permutations of the set $[\pm n] = \{-n, -(n - 1), \dots, -1, 1, 2, \dots, n\}$ satisfying $\pi(-i) = -\pi(i)$ for $1 \leq i \leq n$. It is known that $\mathfrak{B}_n$ is generated by $S = \{s_0, s_1, \dots, s_{n-1}\}$ where s_0 only switches the sign of the first element. When $1 \leq i < n$, s_i is an identical transposition as in the $\mathfrak{S}_n$ case. Descents in Coxeter groups are related to reduced words as follows. Let $w \in W$ have length $\ell(w)$. Then, $s \in \text{DescSet}(w)$ iff $\ell(ws) < \ell(w)$ (see Björner and Brenti [2]). We will write “ w has a descent at position i ” to mean $s_i \in \text{DescSet}(w)$.

For both type-A and type-B Coxeter groups, recall that multiplying an element $w = w_1, w_2, \dots, w_n \in W$ (given in one-line notation), by s_i for $i = 1, \dots, n - 1$ on the right swaps w_i and w_{i+1} . For type-B Coxeter groups, multiplying w by s_0 on the right flips the sign of w_1 . In both $\mathfrak{S}_n$ and $\mathfrak{B}_n$, we denote the empty word by 1. To get the reduced word for π , we start from the identity permutation and apply s_i on positions get to π .

2.1 Type-A Coxeter groups

Given $\pi \in \mathfrak{S}_n$, we start from the identity permutation and mimic an iterative “reverse sorting” process to get to π and use this to define the canonical reduced word $\text{canon_reduc}(\pi)$. We have $(n - 1)$ stages which we number from 1 to $n - 1$ in increasing order. For $1 \leq i \leq n - 1$, at the i -th stage, we ensure that the i -th element from the right agrees with π . At subsequent stages, these elements which are set in their correct place do not move. It is clear that after performing this for $(n - 1)$ stages, we will get π . Further, it is clear that the factorization obtained for π has length $\text{inv}(\pi)$ and so we term the concatenated string of factors of this algorithm as $\text{canon_reduc}(\pi)$. Note that we have started correction from the rightmost element. Alternatively, one could have started correction from the leftmost element (this is done by Garsia, see [12, Theorem 1.1.1]).

Example 5 In one-line notation, let $\pi = 4, 1, 5, 2, 3 \in \mathfrak{S}_5$. We start with $\pi_0 = \text{id} = 1, 2, 3, 4, 5$. Since 3 is the last letter of π , we move the 3 to the last position by applying $r_4 = s_3s_4$ (from left to right) to π_0 . This gives us $\pi_1 = 1, 2, 4, 5, 3$. Since 2 appears in the penultimate position of π , we apply $r_3 = s_2s_3$ to π_1 to get $\pi_2 = 1, 4, 5, 2, 3$. Since 5 is in its correct position, we apply $r_2 = 1$ to π_2 to get π_3 (in this case $\pi_3 = \pi_2$). Finally, we apply $r_1 = s_1$ to π_3 to get $\pi_4 = \pi$. Thus, we get the canonical factorisation of π as $\text{canon_reduc}(\pi) = [r_4][r_3][r_2][r_1]$, that is $\text{canon_reduc}(\pi) = [s_3s_4][s_2s_3][1][s_1]$. Square brackets are only given above for added clarity.

To see the set in which r_i belongs, let $W^{(i)} = \{1, s_i, s_{i-1}s_i, \dots, s_1s_2 \cdots s_i\}$. By this reverse sorting process, it is easy to see that for all $\pi \in \mathfrak{S}_n$, we will get $\text{canon_reduc}(\pi) = [r_{n-1}][r_{n-1}] \cdots [r_1]$ where $r_i \in W^{(i)}$.

Remark 6 In $\mathfrak{S}_n$, it is easy to see that $W^{(1)}$ has only elements of length 0 or 1 in it.

2.2 Type-B Coxeter groups

The basic idea is the same in this case as well. The only difference is that now, to get elements with a negative sign, we need to move it to the first place, apply s_0 and move it back to its correct position. As opposed to the $\mathfrak{S}_n$ case, we will have n stages as at the last stage, the first element could possibly get negated. Thus if we use the same notation as in Subsect. 2.1, we will get factors $[r_n][r_{n-1}] \cdots [r_1]$. To see where the factors r_i belong, for $1 \leq i \leq n$, define

$$W^{(i)} = \{1, s_{i-1}, s_{i-2}s_{i-1}, \dots, s_0s_1 \cdots s_{i-1}, s_1s_0s_1 \cdots s_{i-1}, \dots, s_{i-1} \cdots s_1s_0s_1 \cdots s_{i-1}\}.$$

Note that $W^{(1)} = \{1, s_0\}$.

Remark 7 As in the type-A case, note that $W^{(1)}$ has only elements of length 0 or 1 in it.

We start from the identity permutation $\text{id} \in \mathfrak{B}_n$ and analogously perform operations to move elements of $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$ to the right such that for $1 \leq i \leq n$, after i iterations, the i rightmost elements are exactly as they appear in π . We will get $\text{canon_reduc}(\pi) = [r_n][r_{n-1}] \cdots [r_2][r_1]$, where $r_i \in W^{(i)}$ and as before, square brackets are only given for clarity. For example, if in one-line notation, we have $\sigma = 4, 1, \bar{5}, 2, \bar{3}$, then one can check that applying $r_5 = s_2s_1s_0s_1s_2s_3s_4$ to id results in $1, 2, 4, 5, \bar{3}$. Thus, the rightmost element is in its correct place. Proceeding in a similar manner, one can check that $\text{canon_reduc}(\sigma) = [s_2s_1s_0s_1s_2s_3s_4][s_2s_3][s_2s_1s_0s_1s_2][s_1][1]$.

3 Two sign reversing involutions and some properties

Let W be either $\mathfrak{S}_n$ or $\mathfrak{B}_n$. We look at the indices of the canonical reduced word for each $w \in W$. This is a sequence of integers of length $\ell(w)$. For a finite sequence $x = x_1, x_2, \dots, x_t$ of integers, define its ascent set to be $\text{AscSet}(x) = \{i \in [t-1] : x_i < x_{i+1}\}$. For $w \in W$, recall $\text{canon_reduc}(w)$ is the canonical reduced word of w . As the generators are denoted by s_r where r is an integer, let $\text{indx_canon_reduc}(w)$ be the sequence of these indices of the generators that appear in $\text{canon_reduc}(w)$. Let $\text{AscSet}(\text{indx_canon_reduc}(w))$ be the ascent set of the word $\text{indx_canon_reduc}(w)$.

Remark 8 In both the type-A and the type-B case, the following is easy to see. Since all elements of W_i always end with s_i and start with some s_j with $j < i$, there cannot

be an ascent in $\text{indx_canon_reduc}(w)$ between the last letter of r_i and the first letter of r_j when $i > j$.

By Remark 8, in $\text{indx_canon_reduc}(w)$, ascents can appear only between two letters, where both letters are elements of $W^{(i)}$. Further, all ascents will be between consecutive numbers. The word *consecutive* is used with respect to the total order on the generators. That is, all ascents in $\text{indx_canon_reduc}(w) = r_1, r_2, \dots, r_\ell$ will be at indices j for some $j < \ell$, where $r_j = a$ and $r_{j+1} = a + 1$. Consider Example 5 and recall $\pi = 4, 1, 5, 2, 3 \in \mathfrak{S}_5$. We clearly have its indices $\text{indx_canon_reduc}(\pi) = 3, 4, 2, 3, 1$ and $\text{AscSet}(\text{indx_canon_reduc}(\pi)) = \{1, 3\}$.

3.1 Type-A Coxeter groups

We define the involution f_A for the type-A case first. For $w \in \mathfrak{S}_n$, if $\text{indx_canon_reduc}(w)$ has ascents, then let $i \in \text{AscSet}(\text{indx_canon_reduc}(w))$ be its largest element. That is, consider the last occurrence of an ascent. From the discussion in the first paragraph of Sect. 3, such an ascent will be of the form $a, a + 1$ (as it arises from $s_a s_{a+1}$). In w , change $s_a s_{a+1}$ to $s_a s_{a+1} s_a$. As this modified expression is a product of generators, we get an element $w' \in W$ and define $f_A(w) = w'$. If $\text{indx_canon_reduc}(w)$ does not have descents, then define $f_A(w) = w$.

Alternatively, we could have defined the map f_A as follows. Let $w \in \mathfrak{S}_n$ and let its canonical reduced word be $\text{canon_reduc}(w) = [r_{n-1}][r_{n-2}] \cdots [r_2][r_1]$ with $r_i \in W^{(i)}$. Let $\ell(r_i)$ be the number of s_j 's in r_i . Consider the *smallest* i such that $\ell(r_i) \geq 2$. If $\ell(r_i) \geq 2$, by Remark 6, $i \geq 2$ and the string r_i will end with $s_{i-1} s_i$. The map f_A converts r_i to $r_i s_{i-1}$ and does not change any other r_k 's. As $f_A(w)$ is a product of generator elements, after a possible reduction (using $s_i^2 = 1$), we will get an element $w' \in W$ and define $f_A(w) = w'$. If $\pi \in \mathfrak{S}_n$ is such that $\ell(r_i) \leq 1$ for all i , then define $f_A(\sigma) = \sigma$. We give an example of the bijection f_A below.

Example 9 From Example 5, when $\pi = 4, 1, 5, 2, 3$, $\text{canon_reduc}(\pi) = [s_3 s_4][s_2 s_3][1][s_1]$. Thus, $f_A(\pi) = [s_3 s_4][s_2 s_3 s_2][1][s_1] = [s_3 s_4][s_2 s_3][s_2][s_1]$ and so $f_A(\pi) = 5, 1, 4, 2, 3$.

3.2 Type-B Coxeter groups

For $\sigma \in \mathfrak{B}_n$, let $\text{canon_reduc}(\sigma) = [r_n][r_{n-1}] \cdots [r_1]$. We differentiate between the r_i 's using the following definition. For $1 < i \leq n$, define $r_i \in W^{(i)}$ as being nml (short for *near-maximal-length*) if $r_i = u_i$ or if $r_i = v_i$ where $u_i = s_{i-1} s_{i-2} \cdots s_0 s_1 \cdots s_{i-1} \in W^{(i)}$ and $v_i = s_{i-2} \cdots s_0 s_1 \cdots s_{i-1} \in W^{(i)}$.

We define an involution g_B on $\mathfrak{B}_n$ in two parts as follows. First, consider the case when $\sigma \in \mathfrak{B}_n$ is such that $\text{canon_reduc}(\sigma)$ has a factor r_i which is nml. In this case, let i be the largest index with r_i being nml (that is, r_i is the leftmost nml factor). We change u_i to v_i and vice versa. That is, if $r_i = u_i$, then $\text{canon_reduc}(\sigma) = [r_n] \cdots [u_i] \cdots [r_1]$. In this case, define $g_B(\sigma) = [r_n] \cdots [v_i] \cdots [r_1]$. If $r_i = v_i$, then, $\text{canon_reduc}(\sigma) = [r_n] \cdots [v_i] \cdots [r_1]$. In this case, define $g_B(\sigma) = [r_n] \cdots [u_i] \cdots [r_1]$.

If $\sigma \in \mathfrak{B}_n$ does not have nml factors in $\text{canon_reduc}(\sigma)$, we proceed as done in the type-A involution f_A . In this case, let t be the smallest index with $\ell(r_t) \geq 2$. As seen before, we can show $t \geq 2$. Further, r_t will end with s_{t-1} . Define $g_B(\sigma) = [r_n] \cdots [r_{t+1}][r_t s_{t-2}][r_{t-1}] \cdots [r_1]$. Lastly, if $\sigma \in \mathfrak{B}_n$ is such that $\text{canon_reduc}(\sigma)$ has neither a nml factor, nor a factor r_i with $\ell(r_i) \geq 2$, then define $g_B(\sigma) = \sigma$. We give two examples of our bijection g_B below.

Example 10 Let $\sigma = 3, 1, \bar{5}, 2, \bar{4} \in \mathfrak{B}_5$. We have $\text{canon_reduc}(\sigma) = [r_5][r_4][r_3][r_2][r_1]$ where $r_5 = s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4$, $r_4 = s_2 s_3$, $r_3 = s_2 s_1 s_0 s_1 s_2$, $r_2 = s_1$ and $r_1 = 1$. It is easy to see that $\text{canon_reduc}(\sigma)$ has two nml factors viz $r_5 = u_5$ and $r_3 = v_3$. Among them, the leftmost nml factor is r_5 . We replace $r_5 = u_5$ by $r_5 = v_5$. Thus, we get $g_B(\sigma) = [s_4 s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4][s_2 s_3][s_2 s_1 s_0 s_1 s_2][s_1][1]$, that is $g_B(\sigma) = 3, 1, \bar{4}, 2, \bar{5}$.

Example 11 Let $\sigma = \bar{6}, 1, 5, 3, 4, \bar{2} \in \mathfrak{B}_6$. Then, $\text{canon_reduc}(\sigma) = [r_6][r_5][r_4][r_3][r_2][r_1]$ where $r_6 = s_2 s_1 s_0 s_1 s_2 s_3 s_4 s_5$, $r_5 = s_3 s_4$, $r_4 = s_2 s_3$, $r_3 = 1$, $r_2 = s_1$ and $r_1 = s_0$. Since there is no nml factor, we use the type-A style involution and change $r_4 = s_2 s_3$ to $s_2 s_3 s_2$ to get $g_B(\sigma) = [r_6][r_5][s_2 s_3 s_2][r_3][r_2][r_1]$. Hence, we get $g_B(\sigma) = \bar{5}, 1, 6, 3, 4, \bar{2}$.

3.3 An intermediate sequence of elements of W

In this subsection, for W being both the type-A and type-B Coxeter groups, given $w \in W$, we define a sequence of elements starting with the respective identity elements id and ending at w . This sequence will be used in the proof of Lemmas 19 and 33.

Let W have n generators and let $w \in W$ have $\text{canon_reduc}(w) = [r_n][r_{n-1}] \cdots [r_1]$. Define $w_{n+1} = \text{id}$ and for $1 \leq i \leq n$, define $w_i = \prod_{j=n}^i [r_j]$. Since r_j is a product of generators, $w_i \in W$. Clearly, $w_1 = w$ and we think of the sequence w_j as j decreases from $j = n+1$ to $j = 1$ as a sequence of elements starting from id and getting closer to w . It is not necessary for elements w_j of this sequence to be distinct (see Example 12). We call elements w_j as **the sequence of intermediate elements** of w . It is clear from the definition of w_i that $\text{canon_reduc}(w_i) = \prod_{j=n}^i [r_j] \prod_{j=i-1}^1 [1] = \prod_{j=n}^i [r_j]$.

Example 12 Let $w = 3, 1, \bar{5}, 2, \bar{4} \in \mathfrak{B}_5$. Then, as seen in Example 10, $\text{canon_reduc}(\sigma) = [r_5][r_4][r_3][r_2][r_1]$ where $r_5 = s_3 s_2 s_1 s_0 s_1 s_2 s_3 s_4$, $r_4 = s_2 s_3$, $r_3 = s_2 s_1 s_0 s_1 s_2$, $r_2 = s_1$ and $r_1 = 1$. In this case, we have $w_6 = \text{id}$, $w_5 = [r_5] = 1, 2, 3, 5, \bar{4}$, $w_4 = [r_5][r_4] = 1, 3, 5, 2, \bar{4}$, $w_3 = 1, 3, \bar{5}, 2, 4$ and $w_2 = w_1 = 3, 1, \bar{5}, 2, \bar{4} = w$.

3.4 A few lemmas

We prove a few lemmas about the map f_A defined above and give without proof similar lemmas about the map g_B .

Lemma 13 If $w \in \mathfrak{S}_n$, then $f_A(w)$ perhaps after using the relation $s_i^2 = 1$ is the canonical reduced word of some $w' \in \mathfrak{S}_n$.

Proof Let $T \subseteq \mathfrak{S}_n$ be those elements $w \in W$ such that $\text{indx_canon_reduc}(w)$ has no ascents. As T is the set of fixed points of f_A , if $w \in T$, then $f_A(w) = w$ and so the lemma trivially follows. Thus, assume that $w \in W - T$. Recall that if $\text{canon_reduc}(w) = [r_n] \cdot [r_{n-1}] \cdots [r_2] \cdot [r_1]$, then we modify the last index j such that $\ell(r_j) \geq 2$. As given in the alternate definition of the map f_A , $j \geq 2$ and so $j - 1 \geq 1$. Further, $\ell(r_{j-1})$ takes only two values 0 or 1 and both values give rise to canonical reduced words of some elements $u, w \in W$.

We claim that f_A swaps the two possibilities for $\ell(r_{j-1})$. To see this, suppose $\ell(r_{j-1}) = 1$. Then $r_{j-1} = s_{i-1}$ and since f_A changes r_j to $r_j s_{i-1}$, after using the relation $s_{i-1}^2 = 1$, we get that $r_{j-1} = 1$, i.e. $\ell(r_{j-1}) = 0$. Similarly, if $\ell(r_{j-1}) = 0$, then as f_A changes r_j to $r_j s_{j-1}$, we will have $r_{j-1} = s_{j-1}$. Hence $\ell(r_{j-1}) = 1$. Thus, f_A switches these two possibilities for r_{j-1} and does not change other r_j 's. Hence $f_A(w)$ is also a canonically reduced word, completing the proof. $\square$

In the next two lemmas, we claim that f_A is an involution and that it reverses the sign on non-fixed points.

Lemma 14 *The map f_A defined above is an involution. i.e. $f_A(f_A(w)) = w$ for all $w \in W$.*

Proof Let T be the set defined in the proof of Lemma 13. The lemma is trivial if $w \in T$. If $w \in W - T$, then, from the proof of Lemma 13 it follows that if j is the smallest index such that $\ell(r_j) \geq 2$, then f_A flips the two possible values 0 and 1 of $\ell(r_{j-1})$. Hence $f_A(f_A(w)) = w$, completing the proof. $\square$

Lemma 15 *Let $W = \mathfrak{S}_n$ and T be the fixed point set of f_A defined in Lemma 13. For $w \in W - T$, $\text{inv}(w) \not\equiv \text{inv}(f_A(w)) \pmod{2}$.*

Proof If $\text{canon_reduc}(w)$ is a product of k generators, then by definition, $\text{inv}(w) \equiv k \pmod{2}$. By construction, $f_A(w)$ is a product of $k + 1$ or $k - 1$ transpositions (as $\ell(r_{j-1})$ changes parity). By the Deletion property (see [2, Pg 17]), $f_A(w)$ has length p where $p \not\equiv k \pmod{2}$. Thus the signs of w and $f_A(w)$ are different. $\square$

An analogous result is true for the involution g_B and since the proof is not very different, we merely state the next result and omit its proof.

Lemma 16 *Let W be the type-B Coxeter group and let $w \in W$. Then, $g_B(w)$ perhaps after using the relation $s_i^2 = 1$ will give a reduced decomposition of some $w' \in W$. Further, the map g_B is an involution and reverses sign on the set of its non-fixed points.*

The following lemma will be needed for our results in Sect. 4.1. Since it involves the map f_A , we present it here.

Lemma 17 *If $f_A(\pi) \neq \pi$, then there exists a transposition (a, b) such that $(a, b)f_A(\pi) = \pi$. Further, if i is the index where the canonical reduced word decompositions of π and $f_A(\pi)$ differ and if w_{i+1} is the $(i + 1)$ -st intermediate permutation of π , then a, b are the letters at the positions $i, i + 1$ in w_{i+1} , respectively.*

Proof Let $\pi = [r_{n-1}] \dots [r_1]$ and $f_A(\pi) = [t_{n-1}] \dots [t_1]$ be the canonical reduced word decompositions of π and $f_A(\pi)$, respectively. Let i be the only index where they differ. Then, we have $r_i = s_i t_i$. Furthermore, by our involution, $l(r_j) \leq 1$ for all $j \leq i$. Let $2 \leq i' < i$ be the largest index such that $r_{i'} = 1$. Then, for each $i' < j < i$, $r_j = t_j = s_j$. Let w_j, z_j , for $1 \leq j \leq n$, be the intermediate permutations of $\pi, f_A(\pi)$, respectively. Since $r_j = t_j$ for all $j \geq i + 1$, we have $w_j = z_j$ for $j \geq i + 1$.

Without loss of generality, let $r_i = 1$ and $t_i = s_i$. Since $w_{i+1} = z_{i+1}$, suppose that $w_{i+1} = x_1 \dots x_{i-1} b a x_{i+2} \dots x_n$. This implies $w_{i+1} \cdot r_i = w_i = x_1 \dots x_{i-1} b a x_{i+2} \dots x_n$. Since $r_j = s_j$ for $i' < j < i$, $w_{i'} = w_i \prod_{i' < j < i} s_j$ and

we get $w_{i'} = x_1 \dots x_{i'} b x_{i'+1} \dots x_{i-1} a x_{i+2} \dots x_n$.

Similarly, since $t_i = s_i$, we have $z_i = x_1 \dots x_{i-1} a b x_{i+2} \dots x_n$. $z_{i'} = z_i \prod_{i' < j < i} s_j$

and so $z_{i'} = x_1 \dots x_{i'} a x_{i'+1} \dots x_{i-1} b x_{i+2} \dots x_n$. By our choice of i' , $r_{i'} = t_{i'} = 1$ and for $1 \leq j < i'$, the transpositions r_j, t_j act only on the subword $x_1 \dots x_{i'}$ for $z_{i'}$ and $w_{i'}$, respectively. Therefore, π and $f_A(\pi)$ differ only by the transposition (a, b) and a, b are the letters at positions $i, i + 1$, respectively in w_{i+1} . $\square$

4 Proof of Theorem 3

Recalling the definition of intermediate permutations from Subsection 3.3, we first prove the following lemma.

Lemma 18 *Let $W = \mathfrak{S}_n$ and let $w \in W$ have intermediate elements w_i for $i \in [n]$. For $i \in [n]$, let the one-line notation of w_i be $w_i = a_1, a_2, \dots, a_n$. Then, $a_1 < a_2 < \dots < a_i$.*

Proof The proof is by reverse induction on i with the base case being when $i = n$. As $\pi_n = \text{id}$, the statement of the lemma is satisfied. Let the result be true when $i = k$ and let $\text{canon_reduc}(w) = [r_{n-1}][r_{n-2}][r_2][r_1]$. We will prove that the claim holds when $i = k - 1$. When $i = k$, if $w_k = a_1, a_2, \dots, a_n$, then by induction, $a_1 < a_2 < \dots < a_k$. As $w_{k-1} = w_k[r_{k-1}]$ and since $r_{k-1} \in W^{k-1}$, we see that some element a_j where $j \in [k]$ comes to position k after moving the block of elements $[a_{j+1}, \dots, a_k]$ one position to the left. Thus, if the one-line notation of w_{k-1} is $w_{k-1} = a'_1, a'_2, \dots, a'_n$, then this movement clearly does not alter the relative order among the elements $a'_1, a'_2, \dots, a'_{k-1}$. Thus $a'_1 < a'_2 < \dots < a'_{k-1}$, completing the proof. $\square$

Lemma 19 *Let $W = \mathfrak{S}_n$. For $w \in W$, $\text{drops}(w) = \text{drops}(f_A(w))$.*

Proof If $f_A(w) = w$, the lemma is trivial. Thus, assume $f_A(w) = z$, with $z \neq w$. Let $\text{canon_reduc}(w) = [r_{n-1}][r_{n-2}][r_2][r_1]$ and $\text{canon_reduc}(z) = [q_{n-1}][q_{n-2}][q_2][q_1]$. Let t be the smallest index such that $\ell(r_t) \geq 2$. We have seen that $t \geq 2$ and that $r_i = q_i$ for all indices $i \in [n - 1]$, $i \neq t - 1$. For $1 \leq i \leq n$, recall the intermediate permutations w_i of w and z_i of z . We have seen that either $\ell(r_{t-1}) = 1, \ell(q_{t-1}) = 0$ or vice versa. Without loss of generality, assume that $\ell(r_{t-1}) = 1$. Since $r_{t-1} \in W^{t-1}$, $r_{t-1} = s_{t-1}$ and $q_{t-1} = 1$. Thus, we have $w_{t-1} = w_t \cdot s_{t-1}$ and $z_{t-1} = z_t$. This is depicted pictorially in Fig. 1.

$$\begin{array}{c}
w_n \xrightarrow{r_{n-1}} w_{n-1} \longrightarrow \cdots w_{t+1} \xrightarrow{r_t} w_t \xrightarrow{s_{t-1}} w_{t-1} \xrightarrow{r_{t-2}} w_{t-2} \longrightarrow \cdots w_1 = w \\
z_n \xrightarrow{r_{n-1}} z_{n-1} \longrightarrow \cdots z_{t+1} \xrightarrow{r_t} z_t \xrightarrow{1} z_{t-1} \xrightarrow{q_{t-2}} z_{t-2} \longrightarrow \cdots z_1 = z
\end{array}$$

Fig. 1 Intermediate permutations

We prove the stronger assertion that $\text{drops}(w_i) = \text{drops}(z_i)$ for all $i \in [n]$ by reverse induction on i . The claim is true when $i = n$ as $w_n = z_n = \text{id}$. Further, as $r_i = q_i$ when $n - 1 \geq i \geq t$, we clearly have $w_i = z_i$ when $n \geq i \geq t$. Hence $\text{drops}(w_i) = \text{drops}(z_i)$ when $t \leq i \leq n$.

Since the canonical reduced word for w_t (and z_t) ends in s_t , position $t \in \text{DescSet}(w_t)$ and $t \in \text{DescSet}(z_t)$. In one-line notation, let $w_t = z_t = z_{t-1} = x_1, x_2, \dots, x_n$ and let $w_{t-1} = y_1, y_2, \dots, y_n$. Since $w_{t-1} = w_t \cdot s_{t-1}$, we have

$$x_t = y_{t-1}, x_{t-1} = y_t \text{ and } x_{t+1} = y_{t+1}. \quad (2)$$

By Lemma 18 applied to w_t , we get $x_1 < x_2 < \cdots < x_t$. Since a reduced word for w_{t-1} ends in s_{t-1} , position $t - 1$ is a descent for w_{t-1} and thus we have $y_{t-1} > y_t$. Using (2) we rephrase this as $x_t > x_{t-1}$. Further, as $\ell(r_t) \geq 2$, and as $w_{t+1} \cdot r_t = w_t$, the element x_{t+1} occurred at an index smaller than $t - 1$ in w_{t+1} and moved to the t -th position in w_t . During this move, clearly, a block of $\ell(r_t)$ elements got shifted by one position to the left and x_{t+1} took the position $t + 1$. Applying Lemma 18 to w_{t+1} , we get

$$x_{t+1} < x_{t-1} < x_t. \quad (3)$$

Note that (3) is about the relative order of elements of w_t . Further, as a reduced word for w_t ends in s_t , position t will be a descent of w_t . Thus, $\text{drops}(w_t) = (x_t - x_{t+1}) + D$ where $D = \sum_{c \in \text{DescSet}(w_t) - \{t\}} (x_c - x_{c+1})$. Since $w_t \cdot s_{t-1} = w_{t-1}$, by Lemma 13, $\text{canon_reduc}(w_{t-1}) = [r_{n-1}] \cdot [r_{n-2}] \cdots [r_t][r_{t-1}]$. Since $\ell(r_t) \geq 2$, r_t ends with $s_{t-1}s_t$. Since $r_{t-1} = s_{t-1}$, a reduced word for w_{t-1} ends with $s_{t-1}s_t s_{t-1}$. Thus, some reduced word for w_{t-1} has the form $f s_{t-1}s_t s_{t-1}$ for some f written as a product of generators. Using the Coxeter relations, this reduced word can be changed to $f s_t s_{t-1} s_t$. Thus, there are two reduced words for w_{t-1} , one which ends with s_t and another which ends with s_{t-1} . Thus $t - 1, t \in \text{DescSet}(w_{t-1})$. Clearly, the other descents of w_{t-1} are identical to that of w_t . Thus, w_{t-1} has descents at positions $t - 1, t$ and at $k \in \text{DescSet}(w_t) - \{t\}$. Thus

$$\begin{aligned}
\text{drops}(w_{t-1}) &= (y_{t-1} - y_t) + (y_t - y_{t+1}) + D = (y_{t-1} - y_{t+1}) + D \\
&= (x_t - x_{t+1}) + D = \text{drops}(w_t) = \text{drops}(z_{t-1})
\end{aligned}$$

where we have used (2) in the last line.

We next show that $\text{drops}(w_{t-2}) = \text{drops}(z_{t-2})$. When $k > 2$, the same argument will show that $\text{drops}(w_{t-k}) = \text{drops}(z_{t-k})$. Since t is the smallest index with $\ell(r_t) \geq 2$,

whenever $k \geq 2$, we will have $\ell(r_{t-k}) = 0/1$. If $\ell(r_{t-2}) = 0$, then $r_{t-2} = 1$ and there is nothing to prove. Thus, we assume that $r_{t-2} = q_{t-2} = s_{t-2}$.

Lemma 18 applied to z_t gives $x_{t-2} < x_{t-1}$. As $z_{t-2} = z_t \cdot s_{t-2}$, these are precisely the elements swapped in z_t to get z_{t-2} . Thus, $\text{DescSet}(z_{t-2}) = \text{DescSet}(z_t) \cup \{t-2\}$. Similarly, we claim that $\text{DescSet}(w_{t-2}) = \text{DescSet}(w_t) \cup \{t-2\}$. To see this, recall that as $w_t = x_1, x_2, \dots, x_n$, by (3), we have $x_{t+1} < x_{t-1} < x_t$. If $w_{t-2} = y'_1, y'_2, \dots, y'_n$, as $w_{t-2} = w_t \cdot s_{t-1} \cdot s_{t-2}$, we get

$$y'_{t+1} = x_{t+1}, y'_{t-2} = x_t, y'_{t-1} = x_{t-2}, \text{ and } y'_t = x_{t-1}. \quad (4)$$

Further, if $z_{t-2} = x'_1, x'_2, \dots, x'_n$, as $z_{t-2} = z_t \cdot s_{t-2}$, we have $x'_{t-2} = x_{t-1}$, $x'_{t-1} = x_{t-2}$ and $x'_t = x_t$. In w_t , since position t is a descent, we have $x_t > x_{t+1}$ and let $D = \sum_{q \in \text{DescSet}(w_t) - \{t\}} (x_q - x_{q+1})$. Recall that $\text{DescSet}(z_{t-2}) = \text{DescSet}(z_t) \cup \{t-2\}$ and that $\text{DescSet}(w_{t-2}) = \text{DescSet}(w_t) \cup \{t-2\}$. Thus,

$$\begin{aligned} \text{drops}(z_{t-2}) &= \text{drops}(z_t) + (x'_{t-2} - x'_{t-1}) = \text{drops}(z_t) + (x_{t-1} - x_{t-2}) \\ &= D + (x'_t - x'_{t+1}) + (x_{t-1} - x_{t-2}) = D + (x_t - x_{t+1}) + (x_{t-1} - x_{t-2}) \\ &= D + (x_t - x_{t-2}) + (x_{t-1} - x_{t+1}) = D + (y'_{t-2} - y'_{t-1}) + (y'_t - y'_{t+1}) \\ &= \text{drops}(w_{t-2}) \end{aligned}$$

where the last line follows from (4). An equivalent argument which we omit shows that $\text{drops}(w_{t-k}) = \text{drops}(z_{t-k})$ for $k > 2$, completing the proof. $\square$

4.1 The FZ map and height in Motzkin paths

Before we state our main result of this section, we need some definitions. For a permutation $\pi \in \mathfrak{S}_n$, we call an index $1 \leq i \leq n$ to be

- a *cyclic peak* denoted as $i \in \text{CPk}(\pi)$ if $\pi^{-1}(i) < i > \pi(i)$,
- a *cyclic valley* denoted as $i \in \text{CVal}(\pi)$ if $\pi^{-1}(i) > i < \pi(i)$,
- a *cyclic double ascent* denoted as $i \in \text{Cda}(\pi)$ if $\pi^{-1}(i) < i < \pi(i)$,
- a *cyclic double descent* denoted as $i \in \text{Cdd}(\pi)$ if $\pi^{-1}(i) > i > \pi(i)$,
- a *fixed point* denoted as $i \in \text{Fix}(\pi)$ if $\pi(i) = i$.

Definition 20 A 2-Motzkin path of length n is a word $w = s_1 s_2 \dots s_n$ on the alphabet $\{\text{N}, \text{S}, \text{E}, \text{dE}\}$ (abbreviations for North, South, East and dotted East, respectively) such that for each i , the height or level $h_i(w)$ of the i^{th} step of w , defined by

$$h_i(w) = \#\{j | j < i, s_j = \text{N}\} - \#\{j | j < i, s_j = \text{S}\}$$

is non-negative and is equal to zero when $i = n$.

Definition 21 A restricted Laguerre history of length n is an ordered pair $(\mathbf{s}, \mathbf{p})$, where $\mathbf{s} = (s_1, s_2, \dots, s_n)$ is a 2-Motzkin path of length n and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{N}^n$ satisfies $0 \leq p_i \leq h_i(\mathbf{s})$ when $s_i = \text{N}, \text{E}$ and $0 \leq p_i \leq h_i(\mathbf{s}) - 1$ when $s_i = \text{S}, \text{dE}$. We denote the set of restricted Laguerre histories of length n by $\mathcal{LH}_n^*$.

We refer to the 2-Motzkin path $\mathbf{s}$ of a restricted Laguerre history $(\mathbf{s}, \mathbf{p})$, as its shape, denoted by $\text{sh}(\mathbf{s}, \mathbf{p})$, and the vector $\mathbf{p} = (p_1, \dots, p_n)$ as the labels of the restricted Laguerre history, denoted by $\mathbf{h}(\mathbf{s}, \mathbf{p})$.

Next, we recall the famous Foata-Zeilberger bijection [10] between $\mathfrak{S}_n$ and $\mathcal{LH}_n^*$ using cyclic statistics. Given $\sigma \in \mathfrak{S}_n$, define $\phi_{FZ}(\sigma) = (\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n^*$ where for $i = 1, 2, \dots, n$,

$$s_i = \begin{cases} N & \text{if } i \in \text{Cval}(\sigma), \\ S & \text{if } i \in \text{Cpk}(\sigma), \\ E & \text{if } i \in \text{Cda}(\sigma) \cup \text{Fix}(\sigma), \\ \text{dE} & \text{if } i \in \text{Cdd}(\sigma), \end{cases}$$

with $p_i = \text{nest}_i(\sigma)$ where $\text{nest}_i(\sigma) = |\{j | j < i < \sigma(i) < \sigma(j) \text{ or } \sigma(j) < \sigma(i) \leq i < j\}|$. Define $\text{nest}(\sigma) = \sum_{i=1}^n \text{nest}_i(\sigma)$. To determine the cyclic statistics of $f_A(\pi) \in \mathfrak{S}_n$, we need a few lemmas.

Lemma 22 *For $1 \leq i \leq n$, if w_i are the intermediate permutations of $\pi \in \mathfrak{S}_n$, then $w_i(j) \geq j$ for $1 \leq j \leq i$.*

Proof This follows from the proof of Lemma 18. $\square$

We use Lemma 22 to bound the a, b that appear in Lemma 17.

Lemma 23 *Let i be the index where $\pi, f_A(\pi)$ differ in their canonical reduced word decomposition and let a, b be as in Lemma 17, then $a \geq i + 1$ and $b \geq i + 2$.*

Proof By Lemma 17, we know that a, b are in positions $i, i + 1$, respectively, of w_{i+1} . Without loss of generality, assume that a is in position i and b is in position $i + 1$. We know that $w_{i+1} = w_{i+2} \cdot r_{i+1}$ and $l(r_{i+1}) \geq 2$ (by the definition of f_A). Since $l(r_{i+1}) \geq 2$, a had to have been in the position $i + 1$ and b had to be in position $i + 2$ of w_{i+2} because a is moved to position i and b moved to position $i + 1$ in w_{i+1} . Therefore, by Lemma 22 for w_{i+2} , we get $a \geq i + 1$ and $b \geq i + 2$. This completes the proof. $\square$

With these lemmas in hand, we now prove our main result of this section. If $i \in \text{CPk}(\pi)$ (respectively, $\text{CVal}(\pi)$, $\text{Cda}(\pi) \cup \text{Fix}(\pi)$, $\text{Cdd}(\pi)$), then we will show that $i \in \text{CPk}(f_A(\pi))$ (respectively, $\text{CVal}(f_A(\pi))$, $\text{Cda}(f_A(\pi)) \cup \text{Fix}(f_A(\pi))$, $\text{Cdd}(f_A(\pi))$). This will prove that the shape of the associated restricted Laguerre histories is the same.

Proposition 24 *Let $\pi \in \mathfrak{S}_n$, and f_A denote our involution defined in Subsect. 3.1. Then,*

$$\text{sh}(\phi_{FZ}(\pi)) = \text{sh}(\phi_{FZ}(f_A(\pi))).$$

Proof We show that if $i \in \text{CPk}(\pi)$ (respectively, $\text{CVal}(\pi)$, $\text{Cda}(\pi) \cup \text{Fix}(\pi)$, $\text{Cdd}(\pi)$), then $i \in \text{CPk}(f_A(\pi))$ (respectively, $\text{CVal}(f_A(\pi))$, $\text{Cda}(f_A(\pi)) \cup \text{Fix}(f_A(\pi))$, $\text{Cdd}(f_A(\pi))$). Let a, b be such that $(a, b)f_A(\pi) = \pi$. Without loss of

generality, assume that a appears to the left of b in the one-line notation of π . Thus, we have

$$\pi = \pi(1), \dots, a, \dots, \pi(i), b, \pi(i+2), \dots, \pi(n) \text{ and} \\ f_A(\pi) = \pi(1), \dots, b, \dots, \pi(i), a, \pi(i+2), \dots, \pi(n).$$

If $\pi^{-1}(a) = i'$, then π and $f_A(\pi)$ are different only at the indices $i', i+1$. Therefore, we have the following simple relations. If $j \notin \{i', i+1\}$, then $\pi(j) = f_A(\pi)(j)$ and if $j \notin \{a, b\}$, then $\pi^{-1}(j) = f_A(\pi)^{-1}(j)$. To prove the claim, we only need to check for the following three cases.

- Case 1, when $j = i+1$ or $j = i'$:** Let $j = i'$, then $\pi(j) = a$. Since $a \geq i+1$, by Lemma 22 and $j = i' \leq i$, we have $j < \pi(j) = a$. If $\pi(j) = a$, then $f_A(\pi)(j) = b$. We have $j < b = f_A(\pi)(j)$. Since $i' < a$ and $i' < b$, $j = i'$ is not equal to either a or b and therefore, $\pi^{-1}(j) = f_A(\pi)^{-1}(j)$. These produce the same letters under ϕ_{FZ} .

Let $j = i+1$, then $\pi(j) = b$. Since $j < b$, it can be a . If not, we have $\pi^{-1}(j) = f_A(\pi)^{-1}(j)$. If $a = i+1 = j$, then $f_A(\pi)^{-1}(a) = a = f_A(\pi)(i+1) = a$ and $i' = \pi^{-1}(a) < i+1 < \pi(i+1) = b$, then they both produce the same letters under ϕ_{FZ} as cyclic double ascents and fixed points both produce the same letter.

- Case 2, when $j = a$ or $j = b$:** If $j = a$, then $j \geq i+1$ and we further split into two cases $a = i+1$ and $a \geq i+2$. The former case was covered earlier in Case 1. For the latter case, clearly, the letters $\pi(j)$ will coincide for π and $f_A(\pi)$ when $j \geq i+2$. Therefore, we have $\pi(a) = f_A(\pi)(a)$. Also, $\pi^{-1}(a) = i' < i+1 \leq a$ and $f_A(\pi)^{-1}(a) = i+1 < i+2 \leq a$. Thus, they produce the same letters under the map ϕ_{FZ} .

Suppose $j = b$, then $j \geq i+2$ and the letters $\pi(j)$ coincide for π and $f_A(\pi)$ for $j \geq i+2$. Further, $\pi^{-1}(b) = i+1 < b$ and $f_A(\pi)^{-1}(b) = i' < i+2 \leq b$. Thus, they produce the same letters under the map ϕ_{FZ} .

- Case 3, when $j = \pi(a)$ or $j = \pi(b)$ (i.e. $j = f_A(\pi)(a)$ or $j = f_A(\pi)(b)$):** Since $a \geq i+1$, $\pi(a) = f_A(\pi)(a)$ and thus we have $\pi^{-1}(j) = f_A(\pi)^{-1}(j) = a$. If j is neither $i+1$ nor i' , then, $\pi(j) = f_A(\pi)(j)$. These j produce the same letters under ϕ_{FZ} . If we have $j = i'$, we then have $a > i' < a$ and $f_A(\pi)^{-1}(i') = a > i' < f_A(\pi)(i') = b$. Therefore, they produce the same letters.

If we assume $j = i+1$, either $a = i+1$ or $a > i+1 < a$. The case when $a = i+1$ has been covered earlier in Case 1. If $a > i+1$, then $a > i+1 < f_A(\pi)(i+1) = a$. They both have cyclic valleys at $i+1$ and, therefore, produce the same letters under ϕ_{FZ} .

Since $b \geq i+2$, $\pi(b) = f_A(\pi)(b)$ we have $\pi^{-1}(j) = f_A(\pi)^{-1}(j) = b$. If j is not $i+1$ or i' , then $\pi(j) = f_A(\pi)(j)$. These j produce the same letters under ϕ_{FZ} . If we have $j = i'$, we have $\pi^{-1}(i') = b > i' < \pi(i') = a$ and $f_A(\pi)^{-1}(i') = b > i' < f_A(\pi)(i') = b$. Therefore, they produce the same letters. If we have $j = i+1$, then $b = \pi^{-1}(i+1) > i+1 < \pi(i+1) = b$ $a \geq i+1 < f_A(\pi)(i+1) = a$. However, the case when $j = a = i+1$ has been covered in Case 1. They both have cyclic valleys at $i+1$ and, therefore, produce the same letters under ϕ_{FZ} .

The rest of the indices have the same values and positions in π , $f_A(\pi)$, by Lemma 17. This completes the proof. $\square$

Petersen and Guay-Paquet in [21] defined a map $\phi_{\text{Pet-GP}}$ from $\mathfrak{S}_n$ to the set Motz_n of Motzkin paths of length n that is closely related to the Foata-Zeilberger bijection. In a Motzkin path we do not differentiate between the E and dE steps unlike a 2-Motzkin path. Let $\phi_{\text{Pet-GP}}$ be the map described by Petersen and Guay-Paquet. Petersen and Guay-Paquet show that $\phi_{\text{Pet-GP}}(\pi) = \text{sh}(\phi_{FZ}(\pi))$. For $\pi \in \mathfrak{S}_n$; they show that $\phi_{\text{Pet-GP}}(\pi)$ is not injective but is surjective onto Motz_n . They used it to get a continued fraction expression for the generating function of the depth statistic using the shape of the Motzkin path corresponding to each permutation.

To determine the number of pre-images a Motzkin path has, Petersen and Guay-Paquet defined the *weight* of step p_i as

$$\omega_i = \begin{cases} h_i & \text{if } p_i = N \text{ or } p_i = S, \\ 2h_i + 1 & \text{if } p_i = E, \end{cases}$$

and the weight of a path $p = p_1 \cdots p_n$ to be the product $\omega(p) = \omega_1 \cdots \omega_n$. They showed (see [21, Proposition 4]) that the weight of a Motzkin path p equals the number of permutations in its preimage $\phi^{-1}(p)$.

Proposition 25 (Petersen and Guay-Paquet) *Let $p \in \text{Motz}_n$. Then*

$$\omega(p) = |\{\pi \in \mathfrak{S}_n : \phi_{\text{Pet-GP}}(\pi) = p\}|$$

The following is an easy corollary of the previous proposition.

Corollary 26 *For $p \in \text{Motz}_n$, if p has maximum height strictly more than 1, then, under the map $\phi_{\text{Pet-GP}}$, there are even number of permutations $\pi \in \mathfrak{S}_n$ with $\phi_{\text{Pet-GP}}(\pi) = p$. If p has maximum height at most 1, there are odd number of permutations $\pi \in \mathfrak{S}_n$ with $\phi_{\text{Pet-GP}}(\pi) = p$.*

Next, we give a bijection between the set of Motzkin paths of length n with maximum height at most 1 and subsets of $[n]$ with even cardinality. This will imply that the set of Motzkin paths of length n with maximum height at most 1 has cardinality 2^{n-1} .

Proposition 27 *There is a bijection between the number of Motzkin paths of length n whose height is at most 1 and the number of subsets of $[n]$ whose cardinality is even.*

Proof Let $S \subseteq [n]$ be such that $|S|$ is even and let $S = \{s_1 \dots s_{|S|}\}_{<}$ be S with its elements written in ascending order. For $1 \leq i \leq \frac{|S|}{2}$, declare the s_{2i-1} -th step of the Motzkin path to be ‘N’, the s_{2i} -th step of the Motzkin path to be ‘S’ and the rest of steps to be ‘E’. This process is clearly reversible and gives us a bijection. $\square$

Combining all the results above, we conclude that there is a bijection between the fixed points of f_A and Motzkin paths whose maximum height is at most 1.

Proposition 28 *The fixed points of the involution f_A get mapped bijectively, under $\phi_{\text{Pet-GP}}$, to Motzkin paths whose maximum height is at most 1.*

Proof We know that the fixed points of the involution f_A are permutations whose reduced words r_i satisfy $\ell(r_i) \leq 1$. Clearly, there are 2^{n-1} such permutations. By Proposition 24, if the cardinality of the set of permutations that map to a particular Motzkin path is odd, then it contains at least one permutation that is fixed by f_A . There are exactly 2^{n-1} such paths and exactly 2^{n-1} fixed points. Therefore, every fixed point must map to a different Motzkin path whose maximum height is at most 1. $\square$

We, now, finish the proof of Theorem 3.

Proof of Theorem 3 Since the involution f_A preserves the shape of the 2-Motzkin path under the Foata-Zeilberger bijection, we have $\text{area}(\phi_{FZ}(\pi)) = \text{area}(\phi_{FZ}(f_A(\pi)))$. Thus $\text{depth}(\pi) = \text{depth}(f_A(\pi))$ because the area under the Motzkin path is the depth of the permutation. Since the shape is preserved, so is the number of N, dE steps. The sum of the number of N, dE is exactly the number of iexc (exc of the inverse of the permutation) of the permutation, and this is also preserved under the involution. In Lemma 19, we showed that the map preserves drops as well. Denote by $\text{FIX}(f_A)$ the permutations fixed under the involution f_A . From these three observations, we have

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{iexc}(\pi)} p^{\text{depth}(\pi)} q^{\text{drops}(\pi)} = \sum_{\pi \in \text{FIX}(f_A)} (-1)^{\text{inv}(\pi)} t^{\text{iexc}(\pi)} p^{\text{depth}(\pi)} q^{\text{drops}(\pi)}$$

The permutations in $\text{FIX}(f_A)$ are the ones whose canonical reduced word is of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ where $n > i_1 > i_2 > \cdots > i_k \geq 1$. Clearly, such permutations are in bijection with subsets L of $[n-1]$ and thus $|\text{FIX}(f_A)| = 2^{n-1}$. For each s_{i_j} , the iexc increases by 1. Since these are adjacent transpositions, the depth and drops increases by 1. Therefore, if $\pi \in \text{FIX}(f_A)$, then $\text{depth}(\pi) = \text{drops}(\pi) = \text{iexc}(\pi) = \text{inv}(\pi)$. Therefore,

$$\sum_{\pi \in \text{FIX}(f_A)} (-1)^{\text{inv}(\pi)} t^{\text{iexc}(\pi)} p^{\text{depth}(\pi)} q^{\text{drops}(\pi)} = (1 - tpq)^{n-1}$$

Let RC be the reverse-complement bijection. It is easy to check that RC is sign preserving and

$$(\text{iexc}(\pi), \text{depth}(\pi), \text{drops}(\pi)) = (\text{exc}(\text{RC}(\pi)), \text{depth}(\text{RC}(\pi)), \text{drops}(\text{RC}(\pi))).$$

Therefore,

$$\begin{aligned} & \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{iexc}(\pi)} p^{\text{depth}(\pi)} q^{\text{drops}(\pi)} \\ &= \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\text{RC}(\pi))} t^{\text{iexc}(\text{RC}(\pi))} p^{\text{depth}(\text{RC}(\pi))} q^{\text{drops}(\text{RC}(\pi))} \\ &= \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{exc}(\pi)} p^{\text{depth}(\pi)} q^{\text{drops}(\pi)} \end{aligned}$$

This finishes the proof. $\square$

5 Continued fractions for (depth, inv) and (drops, MAD)

In this section, we calculate a continued fraction expansion for the pair (drops, MAD). The MAD statistic was defined and shown to be a Mahonian statistic by Clarke, Steingrímsson and Zeng [5]. Any $\pi \in \mathfrak{S}_n$ can be uniquely decomposed into *descent blocks* (maximal descending subwords). Denote the first and last letter of each block B of length at least two by $c(B)$, $o(B)$, respectively. The *right embracing number* of $\pi(i)$ is the number of descent blocks strictly to the right of the block containing $\pi(i)$ and for which $c(B) > \pi(i) > o(B)$. The sum of all right embracing numbers is denoted by $\text{Res}(\pi)$. Then, $\text{MAD}(\pi) := \text{drops}(\pi) + \text{Res}(\pi)$.

We will get a continued fraction expansion for the pair (depth, inv) via the Foata-Zeilberger bijection and the theory of continued fractions due to Flajolet [9]. Clarke, Steingrímsson and Zeng showed bijectively that the pairs (drops, MAD) and (depth, inv) are equidistributed over $\mathfrak{S}_n$. Thus, we can recast our results using this pair. We get a J-type continued fraction expansion where the denominators have linear and quadratic terms. For some background on continued fractions, we refer the reader to the book by Goulden and Jackson [13]. Define the (depth, inv)-enumerating polynomial over $\mathfrak{S}_n$ as

$$F_n(x, q) = \sum_{w \in \mathfrak{S}_n} x^{\text{depth}(w)} q^{\text{inv}(w)}.$$

Our result of this section is the following. It was conjectured by Petersen in his email to Zeilberger (see [20]).

Theorem 29 *The ordinary generating function of $F_n(x, q)$ has the following Jacobi continued fraction expansion:*

$$\sum_{n \geq 0} F_n(x, q) t^n = \frac{1}{1 - c_0 t - \frac{b_1 t^2}{1 - c_1 t - \frac{b_2 t^2}{\ddots \frac{b_{n+1} t^2}{1 - c_n t - \ddots}}}}$$

where $c_n = x^n q^n ([n]_q + [n+1]_q)$ and $b_{n+1} = x^{2n+1} q^{2n+1} ([n+1]_p)^2$ for $n \geq 0$.

Proof For $\pi \in \mathfrak{S}_n$, if $\phi_{FZ}(\pi) = (\mathbf{s}, \mathbf{p})$, then we have the following.

$$\text{depth}(\pi) = \text{area}(\phi_{FZ}(\pi)), \quad \text{area}(\phi_{FZ}(\pi)) = \sum_{i=1}^n h_i(\mathbf{s}),$$

$$\text{inv}(\pi) = \text{area}(\phi_{FZ}(\pi)) + \text{nest}(\pi) = \sum_{i=1}^n h_i(\mathbf{s}) + \text{nest}(\pi) = \sum_{i=1}^n (h_i(\mathbf{s}) + p_i).$$

The first two equations are well known in the literature, see [21],[5]. The third equation was noted by Viennot, de Medicis [6].

$$\begin{aligned} F_n(x, q) &= \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n^*} x^{\text{area}(\mathbf{s}, \mathbf{p})} q^{\text{area}(\mathbf{s}, \mathbf{p}) + \sum_{i=1}^n p_i} \\ &= \sum_{(\mathbf{s}, \mathbf{p}) \in \mathcal{LH}_n^*} \prod_{s_i=\text{N}} x^{h_i} q^{h_i+p_i} \prod_{s_i=\text{S}} x^{h_i} q^{h_i+p_i} \prod_{s_i=\text{dE}} x^{h_i} q^{h_i+p_i} \prod_{s_i=\text{E}} x^{h_i} q^{h_i+p_i} \\ &= \sum_{\mathbf{s} \in 2\text{-Motz}_n} \prod_{s_i=\text{N}} x^{h_i} q^{h_i} [h_i + 1]_q \prod_{s_i=\text{S}} x^{h_i} q^{h_i} [h_i]_q \prod_{s_i=\text{dE}} x^{h_i} q^{h_i} [h_i]_q \\ &\quad \prod_{s_i=\text{E}} x^{h_i} q^{h_i} [h_i + 1]_q \end{aligned}$$

where 2-Motz_n is the set of 2-Motzkin paths of length n . Therefore, by Flajolet's theory of continued fractions [9], we have completed our proof. $\square$

By the bijection of Clarke-Steingrímsson-Zeng in [5], we know that $(\text{depth}, \text{inv})$ is equidistributed with $(\text{drops}, \text{MAD})$. Therefore, we have

$$F_n(x, q) = \sum_{w \in \mathfrak{S}_n} x^{\text{depth}(w)} q^{\text{inv}(w)} = \sum_{w \in \mathfrak{S}_n} x^{\text{drops}(w)} y^{\text{MAD}(w)}.$$

As a corollary, we can enumerate the permutations, with respect to depth and inv, in the pre-image of Motzkin paths under the $\phi_{\text{Pet-GP}}$ map.

Corollary 30 *Let $p \in \text{Motz}_n$. Then*

$$\begin{aligned} \sum_{w \in \phi_{\text{Pet-GP}}^{-1}(p)} q^{\text{inv}(w)} x^{\text{depth}(w)} &= \prod_{s_i=\text{N}} x^{h_i} q^{h_i} [h_i + 1]_q \prod_{s_i=\text{S}} x^{h_i} q^{h_i} [h_i]_q \\ &\quad \prod_{s_i=\text{E or dE}} x^{h_i} q^{h_i} ([h_i]_q + [h_i + 1]_q) \end{aligned} \quad (5)$$

6 Type-B Coxeter groups

For $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n \in \mathfrak{B}_n$, let $\text{Negs}(\sigma) = \{i \in [n] : \sigma_i < 0\}$ be the set of indices where σ takes negative values and let $\text{NegSum}(\sigma) = -(\sum_{i \in \text{Negs}(\sigma)} \sigma_i)$ be the absolute value of the sum of the negative components of σ . Recall the definition of the number of type-A inversions of σ as $\text{inv}_A(\sigma) = |\{1 \leq i < j \leq n : \sigma_i > \sigma_j\}|$. Here, comparison is done with respect to the standard order on $\mathbb{Z}$. Define the number of inversions of $\sigma \in \mathfrak{B}_n$ as $\text{inv}_B(\sigma) = \text{NegSum}(\sigma) + \text{inv}_A(\sigma)$. The above definition is

due to Brenti [4] who also shows that $\text{inv}_B(\sigma)$ is the length of σ in the Coxeter-theoretic sense with respect to the generators $s_0, s_1, \dots, s_{n-1}$.

Let $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n \in \mathfrak{B}_n$. Define $\sigma_0 = 0$ and define $\text{DescSet}_B(\sigma) = \{0 \leq i < n : \sigma_i > \sigma_{i+1}\}$. This combinatorial definition of descents can be found in [2]. Recall that $\text{drops}_B(\sigma) = \sum_{i \in \text{DescSet}_B(\sigma)} (\sigma_i - \sigma_{i+1})$. If $\sigma \in \mathfrak{B}_n \cap \mathfrak{S}_n$, then clearly $\text{drops}_B(\sigma) = \text{drops}(\sigma)$. For $n \geq 1$, define $\text{SgnDrop}_{B_n}(q) = \sum_{\sigma \in \mathfrak{B}_n} (-1)^{\text{inv}_B(\sigma)} q^{\text{drops}_B(\sigma)}$. Our main result is the following type-B counterpart of Theorem 4.

Theorem 31 For $n \geq 1$, $\text{SgnDrop}_{B_n}(q) = (1 - q)^n$.

We begin with a few lemmas. The following is a type-B counterpart of Lemma 18. As its proof is similar to the proof of Lemma 18, we omit a proof. The only difference is that we only claim similar results for the intermediate permutations $w_{n+1}, w_n \dots w_2$ and not w_1 as r_1 could be s_0 which flips the sign of the first element.

Lemma 32 Let $W = \mathfrak{B}_n$ and let $w \in W$ have intermediate elements w_i , where $1 \leq i \leq n+1$. For $2 \leq i \leq n+1$, let the one-line notation of w_i be $w_i = a_0 a_1, a_2, \dots, a_n$, where $a_0 = 0$. Then, $a_0 < a_1 < a_2 \dots < a_{i-1}$. Let $\text{canon_reduc}(w) = [r_n][r_{n-1}] \dots [r_1]$ and let $w_1 = b_0 b_1, b_2, \dots, b_n$ where $b_0 = 0$. If $r_1 \neq s_0$, then $b_0 < b_1$.

The following is a type-B counterpart of Lemma 19.

Lemma 33 For $\sigma \in \mathfrak{B}_n$, $\text{drops}_B(\sigma) = \text{drops}_B(g_B(\sigma))$.

Proof Let $T_n = \{\psi \in \mathfrak{B}_n : g_B(\psi) = \psi\}$. The lemma is trivial if $\sigma \in T_n$, and hence, we assume that $\sigma \notin T_n$. We break the proof into two cases depending on whether $\text{canon_reduc}(\sigma)$ has nml factors.

Case 1: (When $\text{canon_reduc}(\sigma)$ has no nml factors.) The argument is very similar to the type-A argument. We start from $w_{n+1} = \text{id}$ and note that at the i -th step, the $(n-i)$ th element from the left gets into its correct place with its correct sign. After we perform $w_{i+1}r_n$, we get w_n and let x_i be the $(n-i)$ th element from the left in w_i (that is the element which came into its correct place in w_i). We might get into trouble only when x_i came from lesser than two positions to the left of $(n-i)$ AND if x_i is negative. Otherwise, it can be seen that (2) holds and we are done.

Thus, for all $i \in [n]$, we have $r_i \neq u_i$ and $r_i \neq v_i$. Let $\psi = g_B(\sigma)$ and let t be the smallest index such that $\ell(r_t) \geq 2$. Recall the intermediate permutations σ_i and ψ_i for $1 \leq i \leq n+1$. Let $\psi_i = \psi_{i+1}q_i$. As in the earlier proof, we will show that $\text{drops}_B(\sigma_i) = \text{drops}_B(\psi_i)$ for $1 \leq i \leq n+1$. For $t \leq i \leq n+1$, we have $\sigma_i = \psi_i$ and hence $\text{drops}_B(\sigma_i) = \text{drops}_B(\psi_i)$. Recall that $\sigma_t = \sigma_{t+1}r_t$, that $\psi_t = \psi_{t+1}r_t s_{t-2}$ and that $\ell(r_j) \leq 1$ for $j < t$. After using the relation $s_i^2 = \text{id}$, we see that either $r_{t-1} = 1$ and $q_{t-1} = s_{t-2}$ or $r_{t-1} = s_{t-2}$ and $q_{t-1} = 1$. Assume that $r_{t-1} = s_{t-2}$ and hence $\sigma_t = \psi_t = \psi_{t-1}$. Further, $r_t \in W^{(t)}$ and $r_t \neq u_t, v_t$. Thus, $\text{drops}_B(\sigma_{t-1}) = \text{drops}_B(\psi_{t-1})$. A similar argument shows that $\text{drops}_B(\sigma_i) = \text{drops}_B(\psi_i)$ for $1 \leq i < t$.

Case 2: (When $\text{canon_reduc}(\sigma)$ has nml factors.) We carry over our notation from Case 1. Let t be the largest index with r_t being nml and without loss of generality let $r_t = u_t$ and $q_t = v_t$. Let $w_{t+1} \cdot u_t = w_t$ and $z_{t+1}v_t = z_t$. Note that $n \geq t \geq 2$.

We first show that $\text{drops}_B(w_t) = \text{drops}_B(z_t)$. We clearly have $w_{t+1} = z_{t+1}$. Let $w_{t+1} = z_{t+1} = x_1, x_2, \dots, x_n$. Since $w_t = w_{t+1}u_t$ and $z_{t+1} = z_t v_t$, we have

$$w_t = x_1, x_2, \dots, x_{t-1}, \overline{x_t}, x_{t+1}, \dots, x_n \text{ and } z_t = x_1, x_2, \dots, x_t, \overline{x_{t-1}}, x_{t+2}, \dots, x_n.$$

We clearly have that the $\text{DescSet}_B(w_t) = \text{DescSet}_B(w_{t+1}) \cup \{t\}$ and $\text{DescSet}_B(z_t) = \text{DescSet}_B(z_{t+1}) \cup \{t\}$. Thus, we get $\text{drops}_B(w_t) = \text{drops}_B(w_{t+1}) + x_{t-1} + x_t$ and $\text{drops}_B(z_t) = \text{drops}_B(z_{t+1}) + x_{t-1} + x_t$. As $w_{t+1} = z_{t+1}$, we get that $\text{drops}_B(w_t) = \text{drops}_B(z_t)$ completing the proof. We next show that $\text{drops}_B(w_{t-1}) = \text{drops}_B(z_{t-1})$. When $k \geq 1$, we have $r_{t-k} = q_{t-k}$. Without loss of generality, assume that $r_{t-1} \neq 1$. Therefore, we have $w_{t-1} = x_1, x_2, \dots, x_{t-1}, \alpha, \overline{x_t}, \dots, x_n$ and $z_{t-1} = x_1, x_2, \dots, x_t, \alpha, \overline{x_{t-1}}, \dots, x_n$, where $\alpha = y_i > 0$ or $\alpha = \overline{y_i}$ for some $1 \leq i < t$.

Subcase (a) (When $\alpha = y_i$) Since $y_i < x_{t-1} < x_t$, we have a descent at position $t-2$. Let $D = \sum_{i \in \text{DescSet}_B(w_t) \cap [t+1, n-1]} (x_i - x_{i+1})$. Then, we have $\text{drops}_B(w_{t-1}) = (x_{t-1} - \alpha) + (\alpha + x_t) + D$ and $\text{drops}_B(z_{t-1}) = (x_{t-1} - \alpha) + (\alpha + x_{t-1}) + D$. Clearly, both are equal.

Subcase (b) (When $\alpha = \overline{y_i}$) We have $\overline{y_i} > \overline{x_{t-1}} > \overline{x_t}$ and therefore $\text{drops}_B(w_{t-1}) = (x_{t-1} + y_i) + (x_t - y_i) + D$ and $\text{drops}_B(z_{t-1}) = (x_t + y_i) + (x_{t-1} - y_i) + D$. Clearly, both are equal. An equivalent argument can be used to show that $\text{drops}_B(w_{t-k}) = \text{drops}_B(z_{t-k})$ when $k \geq 2$ and is, therefore, omitted. The proof is complete. $\square$

With these lemmas in place, we move on to the proof of Theorem 31.

Proof of Theorem 31 Define T_n to be the set of $\sigma \in \mathfrak{B}_n$ such that $\text{indx_canon_reduc}(\sigma)$ has no ascents. These are permutations whose canonical reduced word is of the form $s_{i_1} s_{i_2} \cdots s_{i_k}$ where $n > i_1 > i_2 > \cdots > i_k \geq 0$. As there are n generators, such permutations are clearly in bijection with subsets L of $[n-1]_0$ and thus $|T_n| = 2^n$. Define $K_n = \mathfrak{B}_n - T_n$. Consider the involution f defined in Sect. 3. By Lemmas 16 and 33, $\sum_{\sigma \in K_n} (-1)^{\text{inv}_B(\sigma)} q^{\text{drops}_B(\sigma)} = 0$ and thus $\sum_{\pi \in \mathfrak{B}_n} (-1)^{\text{inv}_B(\sigma)} q^{\text{drops}_B(\sigma)} = \sum_{\pi \in T_n} (-1)^{\text{inv}_B(\sigma)} q^{\text{drops}_B(\sigma)}$.

For $L \subseteq [n-1]_0$, $L = \{i_1, i_2, \dots, i_k\}$, we write $\langle L \rangle$ or $\{i_k, i_{k-1}, \dots, i_1\}_>$ for the sequence of elements of L arranged in decreasing order. We write $v = \prod_{i \in \langle L \rangle} s_i$ to denote the signed permutation $\sigma \in T_n$ whose canonical reduced word is v . Clearly, the signed permutation σ with canonical reduced word $\prod_{i \in \langle L \rangle} s_i$ has $\text{inv}_B(\sigma) = (-1)^{|L|}$. Further, if we cluster the elements of L into consecutive elements, then it is easy to see that $\text{drops}_B(\sigma) = |L|$. Thus,

$$\sum_{\sigma \in T_n} (-1)^{\text{inv}_B(\sigma)} q^{\text{drops}_B(\sigma)} = \sum_{L \subseteq [n]} (-1)^{|L|} q^{|L|} = (1 - q)^n.$$

The proof is complete. $\square$

7 Type-D Coxeter groups

Recall that $\mathfrak{D}_n$ the type-D Weyl group is the subset of signed permutations $\sigma \in \mathfrak{B}_n$ such that $|\text{Negs}(\sigma)| \equiv 0 \pmod{2}$. That is, it is the set of signed permutations

with an even number of negative elements. For $\sigma \in \mathfrak{D}_n$, Brenti [4] showed that its number of inversions is given by $\text{inv}_D(\sigma) = \text{inv}_B(\sigma) - |\text{Negs}(\sigma)|$. When $n \geq 2$, for $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n \in \mathfrak{D}_n$, define $\sigma_0 = -\sigma_2$. Define $\text{DescSet}_D(\sigma) = \{0 \leq i < n : \sigma_i > \sigma_{i+1}\}$. Define $\text{drops}_D(\sigma) = \sum_{i \in \text{DescSet}_D(\sigma)} (\sigma_{i+1} - \sigma_i)$ and define $\text{SgnDrop}_{D_n}(q) = \sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)}$. The main result of this section is the following.

Theorem 34 For $n \geq 2$, $\text{SgnDrop}_{D_n}(q) = (1 - q^3)(1 - q)^{n-1}$.

First, note that the combinatorial definition of inv_D can be extended to any signed permutation in $\mathfrak{B}_n$ without any changes. Define $\text{dropz}(\pi) = \begin{cases} \text{drops}_B(\pi) & \text{if } \pi_1 > 0, \\ \text{drops}_B(\pi) - \pi_1 & \text{if } \pi_1 < 0. \end{cases}$

Lemma 35 When $n \geq 2$, we have $\sum_{\pi \in \mathfrak{B}_n - \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} q^{\text{dropz}(\pi)} = 0$. Similarly, when $n \geq 2$, we have $\sum_{\pi \in \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} q^{\text{dropz}(\pi)} = 0$.

Proof We only prove the first identity and omit a proof of the second as the arguments are very similar. Let A_1 be the elements of $\mathfrak{B}_n - \mathfrak{D}_n$ that have $n-1$ or $\overline{n-1}$ occurring before n or $\overline{n}$ and let A_2 be the elements of $\mathfrak{B}_n - \mathfrak{D}_n$ in which n or $\overline{n}$ occurs before $n-1$ or $\overline{n-1}$. It is clear that $A_1 \sqcup A_2$ is a partition of $\mathfrak{B}_n - \mathfrak{D}_n$ and $|A_1| = |A_2|$. We define a bijection $g : A_1 \mapsto A_2$. We split into cases here to describe our map g : If

1. If $\pi = \sigma_1, n-1, \sigma_2, n, \sigma_3$ where σ_i , for $i = 1, 2, 3$ are contiguous subwords, then set $g(\pi) = \sigma_1, n, \sigma_2, n-1, \sigma_3$. In this case, one can check that $\text{inv}_D(g(\pi)) = \text{inv}_D(\pi) + 1$ and $\text{dropz}(g(\pi)) = \begin{cases} \text{dropz}(\pi) & \text{if } \sigma_3 \neq \phi, \\ \text{dropz}(\pi) + 1 & \text{if } \sigma_3 = \phi. \end{cases}$
2. If $\pi = \sigma_1, \overline{n-1}, \sigma_2, n, \sigma_3$, then set $g(\pi) = \sigma_1, \overline{n}, \sigma_2, n-1, \sigma_3$. In this case, it can be seen that $\text{inv}_D(g(\pi)) = \text{inv}_D(\pi) + 1$ and $\text{dropz}(g(\pi)) = \begin{cases} \text{dropz}(\pi) & \text{if } \sigma_3 \neq \phi, \\ \text{dropz}(\pi) + 1 & \text{if } \sigma_3 = \phi. \end{cases}$
3. If $\pi = \sigma_1, n-1, \sigma_2, \overline{n}, \sigma_3$, then set $g(\pi) = \sigma_1, n, \sigma_2, \overline{n-1}, \sigma_3$. In this case, it can be seen that $\text{inv}_D(g(\pi)) = \text{inv}_D(\pi) - 1$ and $\text{dropz}(g(\pi)) = \text{dropz}(\pi)$.
4. If $\pi = \sigma_1, \overline{n-1}, \sigma_2, \overline{n}, \sigma_3$, then set $g(\pi) = \sigma_1, \overline{n}, \sigma_2, \overline{n-1}, \sigma_3$. In this case, we have $\text{inv}_D(g(\pi)) = \text{inv}_D(\pi) + 1$ and $\text{dropz}(g(\pi)) = \begin{cases} \text{dropz}(\pi) & \text{if either } \sigma_1 \text{ or } \sigma_2 \neq \phi, \\ \text{dropz}(\pi) + 1 & \text{if } \sigma_1, \sigma_2 = \phi. \end{cases}$

Using g , we get

$$\sum_{\pi \in \mathfrak{B}_n - \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} q^{\text{dropz}(\pi)} = \sum_{w \in A_1} \left[(-1)^{\text{inv}_D(w)} q^{\text{dropz}(w)} + (-1)^{\text{inv}_D(g(w))} q^{\text{dropz}(g(w))} \right].$$

On the right-hand side, the only w for which the two terms do not cancel out are when the last letter is n or $n-1$ (by induction, the contribution of

such permutations is clearly $(1 - q) \left(\sum_{\pi \in \mathfrak{B}_{n-1} - \mathfrak{D}_{n-1}} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} \right)$ and when the first two letters are $\overline{n-1}, \bar{n}$ (again, inductively, such permutations contribute $\sum_{\pi \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)}$). Thus, we get

$$\begin{aligned} & \sum_{\pi \in \mathfrak{B}_n - \mathfrak{D}_n} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} \\ &= (1 - q) \sum_{\pi \in \mathfrak{B}_{n-1} - \mathfrak{D}_{n-1}} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} + (q^{2n} - q^{2n-1}) \\ & \quad \sum_{\pi \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)}. \end{aligned}$$

When $n = 2$, it is easy to see that $\sum_{\pi \in \mathfrak{B}_2 - \mathfrak{D}_2} (-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} = 0$. This completes the proof. $\square$

Lemma 36 For natural numbers $n \geq 4$, we have $\text{SgnDrop}_{\mathfrak{D}_n}(q) = (1 - q) \text{SgnDrop}_{\mathfrak{D}_{n-1}}(q)$.

Proof Define A_1 to be the subset of $\mathfrak{D}_n$ having permutations with $n - 1$ or $\overline{n-1}$ occurring before n or $\bar{n}$ and $A_2 = \mathfrak{D}_n - A_1$. It is straightforward to see that these sets have the same cardinality. We describe a bijection $h : A_1 \mapsto A_2$. The map on $\mathfrak{D}_n$ that acts by h on A_1 and h^{-1} on A_2 is an involution. We describe how to map these signed permutations after splitting them into the following cases.

1. If $\pi = \sigma_1, (n-1), \sigma_2, n, \sigma_3$, then set $h(\pi) = \sigma_1, n, \sigma_2, n-1, \sigma_3$. One can check that $\text{inv}_D(h(\pi)) = \text{inv}_D(\pi) + 1$ and $\text{drops}_D(h(\pi)) = \begin{cases} \text{drops}_D(\pi) & \text{if } \sigma_3 \neq \phi, \\ \text{drops}_D(\pi) + 1 & \text{if } \sigma_3 = \phi. \end{cases}$
2. If $\pi = \sigma_1, \overline{n-1}, \sigma_2, n, \sigma_3$, then set $h(\pi) = \sigma_1, \bar{n}, \sigma_2, n-1, \sigma_3$. One can check that $\text{inv}_D(h(\pi)) = \text{inv}_D(\pi) + 1$ and $\text{drops}_D(h(\pi)) = \begin{cases} \text{drops}_D(\pi) & \text{if } \sigma_3 \neq \phi, \\ \text{drops}_D(\pi) + 1 & \text{if } \sigma_3 = \phi. \end{cases}$
3. If $\pi = \sigma_1, n-1, \sigma_2, \bar{n}, \sigma_3$, then set $h(\pi) = \sigma_1, n, \sigma_2, \overline{n-1}, \sigma_3$. We have $\text{inv}_D(h(\pi)) = \text{inv}_D(\pi) - 1$ and $\text{drops}_D(h(\pi)) = \begin{cases} \text{drops}_D(\pi) & \text{if } \sigma_1, \sigma_2 \neq \phi, \\ \text{drops}_D(\pi) + 1 & \text{if } \sigma_1, \sigma_2 = \phi. \end{cases}$
4. If $\pi = \sigma_1, \overline{n-1}, \sigma_2, \bar{n}, \sigma_3$, then set $h(\pi) = \sigma_1, \bar{n}, \sigma_2, \overline{n-1}, \sigma_3$. We have $\text{inv}_D(h(\pi)) = \text{inv}_D(\pi) + 1$ and $\text{drops}_D(h(\pi)) = \begin{cases} \text{drops}_D(\pi) & \text{if } \sigma_1, \sigma_2 \neq \phi, \\ \text{drops}_D(\pi) + 1 & \text{if } \sigma_1, \sigma_2 = \phi. \end{cases}$

Using h , we get

$$\begin{aligned} \text{SgnDrop}_{\mathfrak{D}_n}(q) &= (1 - q) \text{SgnDrop}_{\mathfrak{D}_{n-1}}(q) \\ &+ \sum_{\{\pi \in A_1 : \pi_1 = n-1, \pi_2 = \bar{n}\}} \left((-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} + (-1)^{\text{inv}_D(h(\pi))} q^{\text{drops}_D(h(\pi))} \right) \\ &+ \sum_{\{\pi \in A_1 : \pi_1 = \overline{n-1}, \pi_2 = \bar{n}\}} \left((-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} + (-1)^{\text{inv}_D(h(\pi))} q^{\text{drops}_D(h(\pi))} \right) \end{aligned}$$

We show that both the second and third terms of the sum above are zero. If $\pi_1 = n - 1$, $\pi_2 = \bar{n}$, then clearly, $\pi_1 = n - 1$, $\bar{n}$, σ where $\sigma \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}$. Thus,

$$\begin{aligned} & \sum_{\{\pi \in A_1: \pi_1=n-1, \pi_2=\bar{n}\}} \left((-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} + (-1)^{\text{inv}_D(h(\pi))} q^{\text{drops}_D(h(\pi))} \right) \\ &= \sum_{\sigma \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}} \left((-1)^{\text{inv}_D(\pi)} q^{2n+\text{dropz}(\sigma)} + (-1)^{\text{inv}_D(g(\pi))} q^{2n-1+\text{dropz}(\sigma)} \right) \\ &= ((-1)^{2n-2} q^{2n} + (-1)^{2n-3} q^{2n-1}) \underbrace{\left(\sum_{\sigma \in \mathfrak{B}_{n-2} - \mathfrak{D}_{n-2}} (-1)^{\text{inv}_D(\sigma)} q^{\text{dropz}(\sigma)} \right)}_{=0 \text{ (from (35))}} \end{aligned}$$

Similarly, if $\pi_1 = \overline{n-1}$, $\pi_2 = \bar{n}$, then we have $\pi_1 = \overline{n-1}$, $\bar{n}$, σ where $\sigma \in \mathfrak{D}_{n-2}$. Hence,

$$\begin{aligned} & \sum_{\{\pi \in A_1: \pi_1=\overline{n-1}, \pi_2=\bar{n}\}} \left((-1)^{\text{inv}_D(\pi)} q^{\text{drops}_D(\pi)} + (-1)^{\text{inv}_D(h(\pi))} q^{\text{drops}_D(h(\pi))} \right) \\ &= \sum_{\sigma \in \mathfrak{D}_{n-2}} \left((-1)^{\text{inv}_D(\pi)} q^{2n+\text{dropz}(\sigma)} + (-1)^{\text{inv}_D(h(\pi))} q^{2n-1+\text{dropz}(\sigma)} \right) \\ &= ((-1)^{2n-2} q^{2n} + (-1)^{2n-3} q^{2n-1}) \underbrace{\left(\sum_{\sigma \in \mathfrak{D}_{n-2}} (-1)^{\text{inv}_D(\sigma)} q^{\text{dropz}(\sigma)} \right)}_{=0 \text{ (from (35))}} \end{aligned}$$

Our proof is complete. $\square$

Proof of Theorem (34) It is easy to see that $\text{SgnDrop}_{D_3}(q) = (1 - q^3)(1 - q)^2$. Applying Lemma 36 completes the proof.

8 Two applications of our results

We cover two applications of our results in this section.

8.1 Mean and variance

Graham and Diaconis showed that when one samples permutations from $\mathfrak{S}_n$ at random, then the distribution of its Spearman's measure $D(\pi)$ is asymptotically normal. Their result is the following.

Theorem 37 (Graham and Diaconis) *Let π be a permutation chosen independently and uniformly in $\mathfrak{S}_n$. Then, as $n \rightarrow \infty$,*

$$E(D(\pi)) = \frac{1}{3}n^2 + O(n) \text{ and } \text{var}(D(\pi)) = \frac{2}{45}n^3 + O(n^2) \quad (6)$$

and $D(\pi)$ standardized by (6) is asymptotically normal.

Remark 38 It is easy to note that the expected value of the Spearman disarray when we sample permutations from $\mathfrak{S}_n$ at random can be evaluated by the linearity of expectation and thus we have $E(D(\pi)) = \frac{n^2-1}{3}$. Therefore, the expected value of the statistic drops over $\mathfrak{S}_n$ equals $\frac{n^2-1}{6}$.

We need the following lemma which is implicit in the second proof of [11, Theorem 1.2]. For positive integers n , let $F_n(t) = \sum_{k=0}^n f_{n,k}t^k$ and $G_n(t) = \sum_{k=0}^n g_{n,k}t^k$ be sequences of polynomials with $f_{n,k}, g_{n,k} \geq 0$ for all k . Further, for all positive integers n , let $F_n(1) > 0$ and $G_n(1) > 0$. We will consider the random variable X_f^n (and X_g^n) which takes the value k with probability $\frac{f_{n,k}}{F_n(1)}$ (and $\frac{g_{n,k}}{G_n(1)}$, respectively). The following form of the Lemma appears in [7, Lemma 8] and gives a condition for moments of X_f^n and X_g^n to be identical.

Lemma 39 *Let $F_n(t)$ and $G_n(t)$ be as described above. Suppose we have a sequence of polynomials $H_n(t)$, a nonzero real number λ and a sequence of positive integers ℓ_n such that*

$$F_n(t) = \lambda G_n(t) \pm (1-t)^{\ell_n} H_n(t). \quad (7)$$

Then, for $r < \ell_n$, the r -th moment of X_f^n is identical to the r -th moment of X_g^n .

As a corollary of Theorem 4, Theorem 37 and Lemma 39, we get the following. A scaling is necessary as for all $\pi \in \mathfrak{S}_n$, we have $2 \cdot \text{drops}(\pi) = f(D(\pi))$ where f is the map used by Reifegerste in [23].

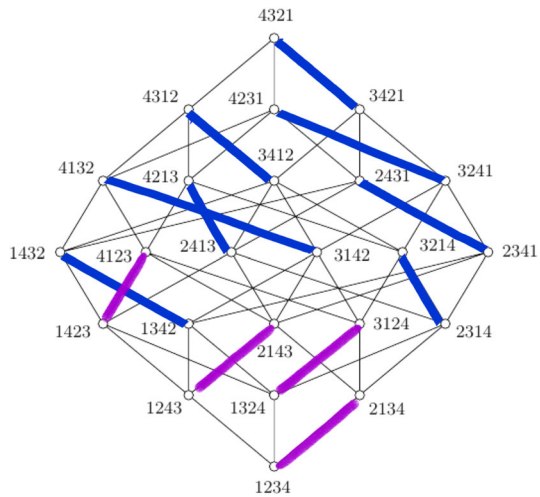
Theorem 40 *Let π be a permutation chosen independently and uniformly in $\mathcal{A}_n$. Then, as $n \rightarrow \infty$,*

$$E(\text{drops}(\pi)) = \frac{1}{2} \cdot \frac{1}{3}n^2 + O(n) \text{ and } \text{var}(\text{drops}(\pi)) = \frac{1}{4} \cdot \frac{2}{45}n^3 + O(n^2). \quad (8)$$

8.2 A complete matching on the Bruhat order

In this section, we show that our sign-reversing involution defined in Sect. 3 gives a matching in the Bruhat order on $\mathfrak{S}_n$ and $\mathfrak{B}_n$. The Bruhat order is a partial order on permutations $\pi \in \mathfrak{S}_n$. There are several definitions of this partial order, and for our purposes, the most suitable is the following. Let $\pi, \psi \in \mathfrak{S}_n$. Define $\psi \leq \pi$ if for some reduced word $s_{i_1}s_{i_2} \cdots s_{i_k}$ of π , there exists a subset $S \subseteq [k]$ such that $\prod_{j \in S} s_{i_j}$ is a reduced word for ψ .

Fig. 2 The Bruhat order on $\mathfrak{S}_4$, taken from page 31 of Björner and Brenti, [2]



Let $\pi \in \mathfrak{S}_n - T_n$ and let $f(\pi)$ be its image. Since π is an involution, assume $\text{inv}(\pi) < \text{inv}(f(\pi))$ (otherwise exchange the roles of π and $f(\pi)$). By Lemma 13, we have $\text{inv}(\pi) = \text{inv}(f(\pi)) - 1$. Further, by Lemma 13 and the definition of the Bruhat order $\pi \leq f(\pi)$ as the canonical reduced word for π is a subword of the canonical reduced word for $f(\pi)$.

Thus, we get a matching in the Bruhat order on elements of $\mathfrak{S}_n - T_n$. We have seen that the elements of T_n are in bijection with subsets of $[n - 1]$ where simple matchings exist (say between subsets *with* and *without* the element s_1 in the reduced word). This gives a matching on the set T_n and combining these two, we get a matching on the Bruhat order of $\mathfrak{S}_n$.

We illustrate the matching for $\mathfrak{S}_4$ in Fig. 2. Both end points of blue edges (seen better on a colour monitor) are elements of $\mathfrak{S}_n - T_n$, while both end points of purple edges are elements in T_n .

Recently Jones [15] gave an explicit matching on arbitrary intervals $[u, v]$ in the Bruhat order where $u \leq v$. His matching depends on a reduced word for the element v and when specialised to the whole Bruhat order on $\mathfrak{S}_n$ (i.e. the interval $[\text{id}, w]$ where w is the unique permutation with largest inv value) is seen to be different from the one just described.

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Data Availability A file with the canonical reduced word for each permutation in $\mathfrak{S}_4$ and the matching on the Bruhat order in Fig. 2 is kept at the url “<http://www.math.iitb.ac.in/~krishnan/4.bijection-output>”.

Declarations

Conflict of interest The authors declare they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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