



Enumerating alternating-runs with sign in Type A and Type B Coxeter groups

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ABSTRACT

We enumerate alternating runs in the alternating group $\mathcal{A}_n \subseteq \mathfrak{S}_n$. This leads us to enumerate the bivariate peak and valley polynomial with sign taken into account. We prove an exact formula for this signed enumerator in $\mathfrak{S}_n$ and show that this polynomial depends on the value of $n \pmod{4}$. If $R_n(t)$ is the polynomial enumerating alternating runs in $\mathfrak{S}_n$, Wilf showed that $(1+t)$ divides $R_n(t)$ and determined the exponent of $(1+t)$ that divides $R_n(t)$. Let $R_n^+(t)$ and $R_n^-(t)$ be the polynomials enumerating alternating runs in $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$ respectively. By finding the exponent of $(1+t)$ that divides $R_n^\pm(t)$, we refine Wilf's result when $n \equiv 2, 3 \pmod{4}$. When $n \equiv 0, 1 \pmod{4}$, we show that the exponent is one short of what Wilf obtains.

As applications of our results, we get moment type identities involving the coefficients of $R_n^\pm(t)$, refinements to enumerating alternating and unimodal permutations in $\mathcal{A}_n$, $\mathfrak{S}_n - \mathcal{A}_n$ and Central Limit Theorems about alternating runs in $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$. Similar results are obtained for type B Coxeter groups.

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1. Introduction

For a positive integer n , let $[n] = \{1, 2, \dots, n\}$. Let $\mathfrak{S}_n$ be the symmetric group on $[n]$. For $\pi \in \mathfrak{S}_n$ with $\pi = \pi_1, \pi_2, \dots, \pi_n$, let $\text{Peak}(\pi) = \{2 \leq i \leq n-1 : \pi_{i-1} < \pi_i > \pi_{i+1}\}$ be its set of peaks and $\text{Valley}(\pi) = \{2 \leq i \leq n-1 : \pi_{i-1} > \pi_i < \pi_{i+1}\}$ be its set of valleys. Let $\text{pk}(\pi) = |\text{Peak}(\pi)|$ and $\text{val}(\pi) = |\text{Valley}(\pi)|$. For an index i with $2 \leq i \leq n-1$, we say that π changes direction at i if $i \in \text{Peak}(\pi) \cup \text{Valley}(\pi)$. Further, we say that π has k alternating runs denoted by $\text{altruns}(\pi) = k$ if π changes directions at $k-1$ indices. Thus, for $\pi \in \mathfrak{S}_n$, we have $\text{altruns}(\pi) = \text{pk}(\pi) + \text{val}(\pi) + 1$. For example, the permutation $\pi = 2, 3, 1, 4, 6, 7, 5$ has 2 peaks, 1 valley and hence 4 alternating runs. Define the alternating runs polynomial in $\mathfrak{S}_n$ as follows:

$$R_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{altruns}(\pi)} = \sum_{k=1}^{n-1} R_{n,k} t^k. \quad (1)$$

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The polynomial $R_n(t)$ is well studied and we refer the reader to Bóna's book [4] for a nice introduction. The earliest reference we know for the polynomial $R_n(t)$ is by André [3]. More recently, $R_n(t)$ has been studied by Canfield and Wilf [8], Chow and Ma [10], Ma [21], Ma et al. [22], and Stanley [29].

A simple recurrence exists for $R_n(t)$ and hence for $R_{n,k}$, the coefficient of t^k in $R_n(t)$. However, a nice formula for $R_n(t)$ does not seem to exist. Several formulae for $R_{n,k}$ exist, but all of them are complicated and involve a non-trivial summation. Stanley in [29], Canfield and Wilf in [8] and later Ma in [21] presented explicit formulae for $R_{n,k}$. Closely related to alternating runs is the number of up-down runs in permutations (see the paper by Gessel and Zhuang [18] for definitions). Gessel and Zhuang in [18] gave plethysm based formulas for the number of up-down runs and other statistics in $\mathfrak{S}_n$.

We are interested in enumerating alternating runs in the alternating group $\mathcal{A}_n \subseteq \mathfrak{S}_n$. Thus, we study the alternating runs polynomial in $\mathfrak{S}_n$ with sign taken into account. For $\pi \in \mathfrak{S}_n$, with $\pi = \pi_1, \pi_2, \dots, \pi_n$, let $\text{inv}(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}|$ denote the number of inversions in π . It is well known that the sign of $\pi \in \mathfrak{S}_n$ equals $(-1)^{\text{inv}(\pi)}$. Signed enumeration of several statistics over various Coxeter groups and some of their subsets is well-known. Désarménien and Foata in [12] enumerated descents with sign in $\mathfrak{S}_n$ and Reiner in [26] enumerated signed descents for types A, B and D Coxeter groups. Mantaci in [23] enumerated excedances with sign in $\mathfrak{S}_n$ and Adin et al. in [1] enumerated signed mahonians in Coxeter groups. Motivated by such results, we define two signed versions of (1):

$$\text{SgnAltrun}_n(t) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} t^{\text{altruns}(\pi)}. \quad (2)$$

Our results actually go through with the more general bivariate version defined below:

$$\text{SgnAltrun}_n(p, q) = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)}. \quad (3)$$

Pan and Zeng in [24] considered a five variable generalization of the Eulerian polynomial with the exponents being the statistics $\text{inv}(\pi)$, $\text{pk}(\pi)$, $\text{val}(\pi)$, double descents $\text{dd}(\pi)$ and double ascents $\text{da}(\pi)$. Enumerating the four variables other than $\text{inv}(\pi)$ gives a generalized Eulerian polynomial. Pan and Zeng gave a q -exponential generating function enumerating this generalized Eulerian polynomial. Though their result involves more variables, it does not seem possible to get our result from theirs. The first main result of this paper is the following neat formula that depends on the value of $n \pmod{4}$.

Theorem 1.1. *For positive integers k , let $n = 4k$ and $n = 4k + 1$. Then,*

$$\text{SgnAltrun}_{4k}(p, q) = \text{SgnAltrun}_{4k+1}(p, q) = 2(1-p)(1-q)(1-pq)^{2(k-1)}.$$

When $n = 4k + 2$ and $n = 4k + 3$, we have

$$\text{SgnAltrun}_{4k+2}(p, q) = \text{SgnAltrun}_{4k+3}(p, q) = 0.$$

We generalize our results to the type B Coxeter Groups. Let $[\pm n] = \{\pm 1, \pm 2, \dots, \pm n\}$ and $\mathfrak{B}_n$ denote the group of signed permutations σ on $[\pm n]$ that satisfy $\sigma(-i) = -\sigma(i)$ for all $i \in [n]$. Chow and Ma in [10] considered the alternating run polynomial over $\mathfrak{B}_n$. The alternating group, which we denote by $\mathcal{A}_n$, is the subgroup consisting of the permutations of $\mathfrak{S}_n$ with an even number of inversions. Like $\mathfrak{S}_n$, $\mathfrak{B}_n$ is a Coxeter group and hence comes with a length function on its elements, which we denote by inv_B . The positive part in $\mathfrak{B}_n$ is defined as its subset containing elements with even length (that is with even inv_B value). In this work, we also enumerate the signed bivariate peak-valley polynomial over type B Coxeter groups. Our main result is Theorem 5.3. Recall that $\mathcal{A}_n \subseteq \mathfrak{S}_n$ is the Alternating group. For ease of writing statements later, we let $\mathfrak{S}_n^+ = \mathcal{A}_n$ and $\mathfrak{S}_n^- = \mathfrak{S}_n - \mathcal{A}_n$. Define

$$R_n^+(t) = \sum_{\pi \in \mathcal{A}_n} t^{\text{altruns}(\pi)} = \sum_{k=1}^{n-1} R_{n,k}^+ t^k \quad \text{and} \quad R_n^-(t) = \sum_{\pi \in \mathfrak{S}_n - \mathcal{A}_n} t^{\text{altruns}(\pi)} = \sum_{k=1}^{n-1} R_{n,k}^- t^k. \quad (4)$$

It is clear that $R_n(t) = R_n^+(t) + R_n^-(t)$. We use the notation $R_n^\pm(t)$ and $R_{n,k}^\pm$ when we refer to both polynomials or their coefficients. Using Theorem 1.1, we trivially get a formula for the univariate version defined in (2). Wilf in [32] showed the following about the exponent of $(1+t)$ that divides $R_n(t)$.

Theorem 1.2 (Wilf). *For positive integers $n \geq 4$, the polynomial $R_n(t)$ is divisible by $(1+t)^m$, where $m = \lfloor (n-2)/2 \rfloor$.*

Wilf's proof was based on the relation between the Eulerian polynomials and $R_n(t)$. Later, Bóna and Ehrenborg in [6] gave an inductive proof of Theorem 1.2. Recently, Bóna in [5] gave a proof of Theorem 1.2 based on group actions. Using Theorem 1.1, we show the following.

Theorem 1.3. For positive integers $n \geq 4$, let $m = \lfloor (n-2)/2 \rfloor$. When $n \equiv 0, 1 \pmod{4}$, the polynomials $R_n^\pm(t)$ are divisible by $(1+t)^{m-1}$. When $n \equiv 2, 3 \pmod{4}$, the polynomials $R_n^\pm(t)$ are divisible by $(1+t)^m$.

Recall that $R_n(t) = R_n^+(t) + R_n^-(t)$. Thus, when $n \equiv 2, 3 \pmod{4}$, Theorem 1.3 clearly refines Theorem 1.2. When $n \equiv 0, 1 \pmod{4}$, Theorem 1.3 falls short of a refinement of Theorem 1.2 as the exponent is lesser by one. However, Example 4.5 shows that our result is the best possible in this case. In Theorem 5.5, we show counterparts of Theorem 1.3 when alternating runs are summed up over the elements with positive sign in type B Coxeter groups. We move on to applications of our signed peak-valley polynomial enumeration.

1.1. Moment-type identities involving $R_{n,k}^\pm$

Moment-like identities involving the $R_{n,k}$'s were given by Comtet [11]. Chow and Ma in [10] deduced their identity from Theorem 1.2. For positive integers n, k , note that $R_{n,k}^\pm = 0$ whenever $k \geq n$. Using Theorem 1.3, we have the following refinement when $n \equiv 2, 3 \pmod{4}$ and near refinement when $n \equiv 0, 1 \pmod{4}$.

Theorem 1.4. For positive integers $n \equiv 0, 1 \pmod{4}$ and positive integers k with $n \geq 2k + 6$, we have

$$\sum_{p \geq 0} (2p+1)^k R_{n,2p+1}^\pm = \sum_{p \geq 0} (2p+2)^k R_{n,2p+2}^\pm. \quad (5)$$

For positive integers $n \equiv 2, 3 \pmod{4}$ and positive integers k with $n \geq 2k + 4$, we have

$$\sum_{p \geq 0} (2p+1)^k R_{n,2p+1}^\pm = \sum_{p \geq 0} (2p+2)^k R_{n,2p+2}^\pm. \quad (6)$$

For positive elements in Type B Weyl groups, we get a similar identity. This is presented as Corollary 5.6.

1.2. Alternating and unimodal permutations in $\mathfrak{S}_n^\pm$

A permutation $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{S}_n$ is said to be alternating if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$, that is, $\pi_i < \pi_{i+1}$ for even indices i and $\pi_i > \pi_{i+1}$ for odd indices i , and is said to be unimodal if there exists an index j for which $\pi_1 < \pi_2 < \dots < \pi_j > \pi_{j+1} > \dots > \pi_n$. We refer the reader to the survey by Stanley [30] for alternating permutations and to the paper by Gannon [17] for unimodal permutations.

Let E_n, E_n^+ , and E_n^- denote the set of alternating permutations in $\mathfrak{S}_n, \mathcal{A}_n$, and $\mathfrak{S}_n - \mathcal{A}_n$ respectively. Similarly, let U_n, U_n^+, U_n^- denote the number of unimodal permutations in $\mathfrak{S}_n, \mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$ respectively. From the univariate version of Theorem 1.1 (Corollary 4.4), we prove the following:

Lemma 1.5. For positive integers $n \geq 2$, we have

$$E_n^+ - E_n^- = \begin{cases} 1 & \text{if } n = 4k, \\ -1 & \text{if } n = 4k + 2, \\ 0 & \text{if } n = 4k + 1 \text{ or } n = 4k + 3. \end{cases} \quad (7)$$

For positive integers $n \geq 2$, we have $U_n^+ = U_n^- = 2^{n-2}$.

André [2] proved that the exponential generating function of E_n is $\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x$. When $n = 0$, we set $E_0^+ = 1$ and $E_0^- = 0$. Using Lemma 1.5, in subsection 4.3, we show the following refinement.

Theorem 1.6. The exponential generating functions of $E_n^\pm$ are as follows:

$$\sum_{n=0}^{\infty} E_n^\pm \frac{x^n}{n!} = \frac{1}{2} \left(\sec x + \tan x \pm \cos x \pm x \right). \quad (8)$$

Steingrímsson in [31] gave results for coloured permutations which includes an exponential generating function (egf henceforth) for the type B alternating permutations. In Theorem 5.9, we give our type B counterpart of Theorem 1.6.

1.3. Central limit theorems about alternating runs in $\mathfrak{S}_n^\pm$

Consider sampling permutations π uniformly at random from a non-empty $S \subseteq \mathfrak{S}_n$. Thus, each $\pi \in S$ is selected with probability $1/|S|$. For any statistic $\text{stat} : \mathfrak{S}_n \rightarrow \mathbb{Z}_{\geq 0}$, consider the random variable X_{stat} that takes the value $\text{stat}(\pi)$ of the sampled π . In this work, we consider $S = \mathfrak{S}_n$, $S = \mathfrak{S}_n^\pm$ and then move to the type B case. Let $A_{\text{stat}, S}(t) = \sum_{\pi \in S} t^{\text{stat}(\pi)}$ denote the ordinary generating function of $\text{stat}(\pi)$ over S . When we write that “the coefficients of $A_{\text{stat}, S}(t)$ are asymptotically normal”, we mean that the random variable X_{stat} has a *Central Limit Theorem* (CLT henceforth) when π is sampled from S . This means that when n is “large enough”, the distribution of X_{stat} is approximately normal.

When $\text{stat} = \text{altruns}$, a CLT for X_{altruns} is known (see Knuth [19, Page 46] and Brockwell and Davis [7, Page 37]). For the statistic $\text{stat}(\pi) = \text{pk}(\pi)$, Ma in [20] showed that the random variable X_{pk} has a CLT. Thus, the coefficients of the peak polynomial are asymptotically normal. Fulman, Kim and Lee in [15] gave similar CLT results for X_{pk} when π is sampled from a conjugacy class of $\mathfrak{S}_n$.

In Subsection 4.4, using the CLT for X_{altruns} over $\mathfrak{S}_n$ and our signed enumeration results, we show that when permutations are sampled from $\mathfrak{S}_n^\pm$, we get similar CLT results. Our type A result is the following.

Theorem 1.7. *The distribution of the coefficients of $R_n^\pm(t)$ is asymptotically normal as $n \rightarrow \infty$. When $n \geq 3$, the random variable X_{altruns} over $\mathfrak{S}_n^\pm$ has mean $(2n - 1)/3$ and when $n \geq 6$, has variance $(16n - 29)/90$.*

Our type B counterpart of Theorem 1.7 is Theorem 5.13 and is given in Subsection 5.4.

2. Preliminaries

We partition $\mathfrak{S}_n$ based on the type of the first and the last pairs. Either pair could be an ascent or a descent. When $n \geq 3$, we get the following four sets:

$$\begin{aligned} \mathfrak{S}_{n,a,a} &= \{\pi \in \mathfrak{S}_n : \pi_1 < \pi_2, \pi_{n-1} < \pi_n\}, & \mathfrak{S}_{n,a,d} &= \{\pi \in \mathfrak{S}_n : \pi_1 < \pi_2, \pi_{n-1} > \pi_n\}, \\ \mathfrak{S}_{n,d,a} &= \{\pi \in \mathfrak{S}_n : \pi_1 > \pi_2, \pi_{n-1} < \pi_n\}, & \mathfrak{S}_{n,d,d} &= \{\pi \in \mathfrak{S}_n : \pi_1 > \pi_2, \pi_{n-1} > \pi_n\}. \end{aligned}$$

We will enumerate the contribution to $\text{SgnAltrun}_n(p, q)$ from each of the four sets. Hence, we define the following alternating runs enumerator polynomial $\text{SgnAltrun}_{n,a,a}(p, q) = \sum_{\pi \in \mathfrak{S}_{n,a,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)}$.

In an identical manner, define $\text{SgnAltrun}_{n,a,d}(p, q)$, $\text{SgnAltrun}_{n,d,a}$ and $\text{SgnAltrun}_{n,d,d}$ over the sets $\mathfrak{S}_{n,a,d}$, $\mathfrak{S}_{n,d,a}$ and $\mathfrak{S}_{n,d,d}$ respectively. To keep track of the case when $\pi \in \mathcal{A}_n$, define

$$\begin{aligned} \mathcal{A}_{n,a,a} &= \{\pi \in \mathcal{A}_n : \pi_1 < \pi_2, \pi_{n-1} < \pi_n\}, & \mathcal{A}_{n,a,d} &= \{\pi \in \mathcal{A}_n : \pi_1 < \pi_2, \pi_{n-1} > \pi_n\}, \\ \mathcal{A}_{n,d,a} &= \{\pi \in \mathcal{A}_n : \pi_1 > \pi_2, \pi_{n-1} < \pi_n\}, & \mathcal{A}_{n,d,d} &= \{\pi \in \mathcal{A}_n : \pi_1 > \pi_2, \pi_{n-1} > \pi_n\}. \end{aligned}$$

It is easy to see when $\pi \in \mathfrak{S}_{n,a,a}$ and when $\pi \in \mathfrak{S}_{n,d,d}$, we have $\text{pk}(\pi) = \text{val}(\pi)$. On the other hand, when $\pi \in \mathfrak{S}_{n,a,d}$, we clearly have $\text{pk}(\pi) = 1 + \text{val}(\pi)$ and when $\pi \in \mathfrak{S}_{n,d,a}$, we have $\text{val}(\pi) = \text{pk}(\pi) + 1$. Thus, for all π , we have $|\text{pk}(\pi) - \text{val}(\pi)| \leq 1$.

For $\pi \in \mathfrak{S}_n$, let π' be obtained from π by deleting the letter n . It is easy to see that there exists $\pi \in \mathcal{A}_n$ such that $\pi' \in \mathcal{A}_{n-1}$ and also there exists $\pi \in \mathcal{A}_n$ such that $\pi' \in \mathfrak{S}_{n-1} - \mathcal{A}_{n-1}$. In particular, if $\pi \in \mathcal{A}_n$ with $\pi_n = n$, then $\pi' \in \mathcal{A}_{n-1}$ and if $\pi \in \mathcal{A}_n$ with $\pi_{n-1} = n$, then $\pi' \in \mathfrak{S}_{n-1} - \mathcal{A}_{n-1}$. An identical statement holds when $\pi \in \mathfrak{S}_n - \mathcal{A}_n$. We will use induction and hence add the letter n to permutations in $\mathfrak{S}_{n-1}$. We need to keep track of the position of the letter n in π as we want to take the sign of the permutations into account. Let $\pi' = \pi_1, \pi_2, \dots, \pi_{n-1} \in \mathfrak{S}_{n-1}$. We term the position before π_1 as the initial or the 0-th gap of π' and the position after π_{n-1} as the final or $(n-1)$ -th gap of π' . For $1 \leq i \leq n-2$, we denote the gap between π_i and π_{i+1} as the i -th gap of π' . For $\pi' \in \mathfrak{S}_{n-1}$, we call the i -th gap of π' to be even if $n-1-i$ is even and the i -th gap of π' to be odd if $n-1-i$ is odd. It is easy to see that if n is added in an *odd gap* of π' , then π and π' have opposite signs and if n is added in an *even gap* of π' , then π and π' have the same sign. The following remark is easy to see. Since the set of gaps is $\{0, 1, \dots, n-1\}$, define $[n-1]_0 = \{0, 1, 2, \dots, n-1\}$.

Remark 2.1. If $\pi' \in \mathfrak{S}_{n-1}$ with $n-1$ being even, the total number of gaps of π' is n which is odd. One can check that in π' , the number of even gaps is $\lfloor n/2 \rfloor + 1$ and the number of odd gaps is $\lfloor n/2 \rfloor$. On the other hand, if $\pi' \in \mathfrak{S}_{n-1}$ with $n-1$ being odd, the total number of gaps is n which is even. Here, in π' , we will have $n/2$ even and $n/2$ odd gaps.

2.1. Two easy bijections and some properties

In this subsection, we mention two easy bijections which we will use several times in this work. Let, $\pi = \pi_1, \dots, \pi_n \in \mathfrak{S}_n$. Define the *complementation bijection* $\text{Compl} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by $\text{Compl}(\pi_1, \pi_2, \dots, \pi_n) = n+1-\pi_1, n+1-\pi_2, \dots, n+1-\pi_n$. Define the *reverse bijection* $\text{Rev} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ by $\text{Rev}(\pi_1, \pi_2, \dots, \pi_n) = \pi_n, \pi_{n-1}, \dots, \pi_1$.

Remark 2.2. The bijection Compl has the following properties:

- P1: For $\pi \in \mathfrak{S}_n$, we have $\text{Valley}(\pi) = \text{Peak}(\text{Compl}(\pi))$ and $\text{Peak}(\pi) = \text{Valley}(\text{Compl}(\pi))$. Thus, $\text{val}(\pi) = \text{pk}(\text{Compl}(\pi))$ and $\text{pk}(\pi) = \text{val}(\text{Compl}(\pi))$.
 P2: Let $\pi \in \mathcal{A}_n$. Then, $\text{Compl}(\pi) \in \mathcal{A}_n$ if and only if $n \equiv 0, 1 \pmod{4}$. Similarly, if $\pi \in \mathfrak{S}_n - \mathcal{A}_n$, then $\text{Compl}(\pi) \in \mathfrak{S}_n - \mathcal{A}_n$ if and only if $n \equiv 0, 1 \pmod{4}$.

Proof. It is easy to see that the peak and the valley sets switch when we complement the permutation π . Thus, P1 is proved. For the second part, we note that $\text{inv}(\pi) + \text{inv}(\text{Compl}(\pi)) = \binom{n}{2}$. As $\binom{n}{2}$ is even if and only if $n \equiv 0, 1 \pmod{4}$, $\text{inv}(\pi)$ and $\text{inv}(\text{Compl}(\pi))$ have the same parity if and only if $n \equiv 0, 1 \pmod{4}$. This completes the proof of P2. $\square$

Lemma 2.3. For positive integers $n \equiv 0, 1 \pmod{4}$, the following hold.

$$\text{SgnAltrun}_{n,a,d}(p, q) = \text{SgnAltrun}_{n,d,a}(q, p), \quad (9)$$

$$\text{SgnAltrun}_{n,a,d}(p, q) = \text{SgnAltrun}_{n,d,d}(p, q). \quad (10)$$

Proof. Consider (9). When $n \equiv 0, 1 \pmod{4}$, by Remark 2.2, Compl is a map from $\mathcal{A}_n$ to $\mathcal{A}_n$. Further, it is clear that Compl is a bijection from $\mathcal{A}_{n,a,d}$ to $\mathcal{A}_{n,d,a}$ that swaps the peak set and the valley set. It is also easy to see that if $\pi \in \mathfrak{S}_{n,a,d}$ then $\text{pk}(\pi) = \text{val}(\pi) + 1$. This proves (9). The proof of (10) is similar. We note that if $\pi \in \mathfrak{S}_{n,a,d} \cup \mathfrak{S}_{n,d,d}$, then, $\text{pk}(\pi) = \text{val}(\pi)$. Thus, there is no need to swap p and q . This completes the proof. $\square$

We next move onto similar properties of the reversing bijection.

Remark 2.4. The bijection Rev has the following properties.

- R1: For $\pi \in \mathfrak{S}_n$, we have $\text{val}(\pi) = \text{val}(\text{Rev}(\pi))$ and $\text{pk}(\pi) = \text{pk}(\text{Rev}(\pi))$.
 R2: Let $\pi \in \mathcal{A}_n$. Then, $\text{Rev}(\pi) \in \mathcal{A}_n$ if and only if $n \equiv 0, 1 \pmod{4}$.

We omit a proof of Remark 2.4 as it follows easily. Using the map Rev , we get the following.

Lemma 2.5. For positive integers $n \equiv 2, 3 \pmod{4}$, the following hold.

$$\text{SgnAltrun}_{n,a,d}(p, q) = 0 \text{ and } \text{SgnAltrun}_{n,d,a}(p, q) = 0, \quad (11)$$

$$\text{SgnAltrun}_{n,a,a}(p, q) = -\text{SgnAltrun}_{n,d,d}(p, q), \quad (12)$$

$$\text{SgnAltrun}_n(p, q) = 0. \quad (13)$$

Proof. We first consider (11). When $n \equiv 2, 3 \pmod{4}$, by Remark 2.4, Rev is a bijection from $\mathcal{A}_n$ to $\mathfrak{S}_n - \mathcal{A}_n$. Further, by Remark 2.4, Rev is a bijection from $\mathcal{A}_{n,a,d}$ to $\mathfrak{S}_{n,a,d} - \mathcal{A}_{n,a,d}$ which preserves the number of peaks and valleys. A similar argument shows (12). Summing (11) and (12) gives (13). $\square$

Remark 2.6. Consider the bijection $\text{Compl} : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$. It is easy to note that π and $\text{Compl}(\pi)$ have the same number of alternating runs. Thus, when $n \geq 2$, each coefficient of $R_n(t)$ is even.

Remark 2.7. Let $\pi' = \pi_1, \pi_2, \dots, \pi_{n-1} \in \mathfrak{S}_{n-1}$ with $\text{Peak}(\pi') = \{k_1, k_2, \dots, k_r\}$. Then, for $1 \leq i \leq r$, both $\psi_1 = \pi_1, \pi_2, \dots, \pi_{k_i-1}, n, \pi_{k_i}, \dots, \pi_{n-1}$ and $\psi_2 = \pi_1, \pi_2, \dots, \pi_{k_i-1}, \pi_{k_i}, n, \dots, \pi_{n-1}$ have the same number of peaks and valleys but have different signs as $\text{inv}(\psi_1) = \text{inv}(\pi') + n - k_i$ and $\text{inv}(\psi_2) = \text{inv}(\pi') + n - k_i - 1$.

3. Recurrences for the signed bivariate polynomials

We want to give recurrences for $\text{SgnAltrun}_{n-1,a,d}(p, q)$, $\text{SgnAltrun}_{n-1,d,a}(p, q)$, $\text{SgnAltrun}_{n-1,a,a}(p, q)$ and lastly $\text{SgnAltrun}_{n-1,d,d}(p, q)$.

We observe that for $n \geq 3$, inserting n at any gap in $\pi' \in \mathfrak{S}_{n-1,d,a}$ cannot give $\pi \in \mathfrak{S}_{n,a,d}$ and likewise, inserting n at any gap in $\pi' \in \mathfrak{S}_{n-1,a,d}$ cannot give $\pi \in \mathfrak{S}_{n,d,a}$. Similarly, inserting n at any position in $\pi' \in \mathfrak{S}_{n-1,a,a}$ cannot give $\pi \in \mathfrak{S}_{n,d,d}$ and inserting n at any position in $\pi' \in \mathfrak{S}_{n-1,d,d}$ cannot give $\pi \in \mathfrak{S}_{n,a,a}$.

We bifurcate the remaining part into two cases based on the parity of n .

3.1. When n is odd

Theorem 3.1. For odd positive integers n , we have

$$q \text{SgnAltrun}_{n-1,a,d}(p, q) = p \text{SgnAltrun}_{n-1,d,a}(p, q).$$

Proof. When n is odd, $n-1 \equiv 0/2 \pmod{4}$. If $n-1 \equiv 0 \pmod{4}$, consider the bijection Compl from $\mathfrak{S}_{n-1,a,d}$ to $\mathfrak{S}_{n-1,d,a}$. We have $\text{pk}(\pi) = \text{val}(\text{Compl}(\pi))$ and $\text{val}(\pi) = \text{pk}(\text{Compl}(\pi))$. Moreover, if $\pi \in \mathfrak{S}_{n-1,a,d}$, then $\text{pk}(\pi) = \text{val}(\pi) + 1 = \text{pk}(\text{Compl}(\pi)) + 1$. Similarly, we get $\text{val}(\pi) = \text{pk}(\pi) - 1 = \text{val}(\text{Compl}(\pi)) - 1$. Thus, if $\pi \in \mathfrak{S}_{n-1,a,d}$, then,

$$q \times p^{\text{pk}(\pi)} q^{\text{val}(\pi)} = p \times p^{\text{pk}(\text{Compl}(\pi))} q^{\text{val}(\text{Compl}(\pi))}. \quad (14)$$

As equation (14) is true for all $\pi \in \mathfrak{S}_{n-1,a,d}$, summing over elements of $\mathfrak{S}_{n-1,a,d}$ completes the proof.

If $n-1 \equiv 2 \pmod{4}$, then both $\text{SgnAltrun}_{n-1,a,d}(p, q) = 0$ and $\text{SgnAltrun}_{n-1,d,a}(p, q) = 0$ by the map Rev . The proof is complete. $\square$

From Lemma 2.3 and Lemma 2.5, we know that instead of getting a recurrence for the four polynomials, it suffices to get a recurrence for just the two polynomials $\text{SgnAltrun}_{n,a,d}(p, q)$ and $\text{SgnAltrun}_{n,a,a}(p, q)$.

Theorem 3.2. For odd positive integers $n \geq 3$, the following recurrence relations hold:

$$\begin{aligned} \text{SgnAltrun}_{n,a,d}(p, q) &= -pq \text{SgnAltrun}_{n-1,a,d}(p, q) - p \text{SgnAltrun}_{n-1,a,a}(p, q) \\ &\quad - p \text{SgnAltrun}_{n-1,d,d}(p, q), \end{aligned} \quad (15)$$

$$\begin{aligned} \text{SgnAltrun}_{n,a,a}(p, q) &= q \text{SgnAltrun}_{n-1,a,d}(p, q) - p \text{SgnAltrun}_{n-1,d,a}(p, q) \\ &\quad + \text{SgnAltrun}_{n-1,a,a}(p, q) \end{aligned} \quad (16)$$

$$= \text{SgnAltrun}_{n-1,a,a}(p, q). \quad (17)$$

Proof. We first prove (15). Note that $\text{SgnAltrun}_{n,a,d}(p, q)$ is the sum of only the following three polynomials:

$$\begin{aligned} \text{SgnAltrun}_{n,a,d}(p, q) &= \sum_{\pi \in \mathfrak{S}_{n,a,d}, \pi' \in \mathfrak{S}_{n-1,a,d}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,a,d}, \pi' \in \mathfrak{S}_{n-1,a,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \\ &\quad + \sum_{\pi \in \mathfrak{S}_{n,a,d}, \pi' \in \mathfrak{S}_{n-1,d,d}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)}. \end{aligned}$$

We consider the contribution from each of the three terms to $\text{SgnAltrun}_{n,a,d}(p, q)$ separately.

1. Suppose $\pi' \in \mathfrak{S}_{n-1,a,d}$ and we want to get all $\pi \in \mathfrak{S}_{n,a,d}$ by inserting n in different gaps of π' . We claim that we cannot insert n in the initial or final gap, as by such an insertion, we will not get $\pi \in \mathfrak{S}_{n,a,d}$. By inserting n in all other gaps of π' , we will get $\pi \in \mathfrak{S}_{n,a,d}$. Let $k_1, k_2, \dots, k_r$ be the peaks of π' . By Remark 2.7, inserting n at the $(k_i - 1)$ -st and k_i -th gaps gives permutations with equal number of peaks and valleys but with different sign. Therefore, inserting n at the gaps $k_1 - 1, k_1, k_2 - 1, k_2, \dots, k_r - 1, k_r$ contributes 0 in total. The remaining gaps are $A = [n-1]_0 - \{0, k_1 - 1, k_1, k_2 - 1, k_2, \dots, k_r - 1, k_r, n-1\}$. It is easy to see that inserting n at any of the gaps from the set A increases both the number of peaks and valleys by 1. Moreover, by Remark 2.1, the total number of even and odd gaps are $\lfloor n/2 \rfloor + 1$ and $\lfloor n/2 \rfloor$ respectively. As $n-1$ is even, both 0 and $n-1$ are even gaps. Hence, the number of even gaps available in the set A is $\lfloor n/2 \rfloor + 1 - (r+2)$. Similarly, the number of odd gaps available in the set A is $\lfloor n/2 \rfloor - r$. Thus, A has one more odd gap than even gaps. Hence, the contribution from $\pi' \in \mathfrak{S}_{n-1,a,d}$ to the sum $\text{SgnAltrun}_{n,a,d}(p, q)$ is $(-pq) \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$.
2. To get $\pi \in \mathfrak{S}_{n,a,d}$ from $\pi' \in \mathfrak{S}_{n-1,a,a}$, there is only one choice: insert n at the $(n-2)$ -nd gap. By inserting n at the $(n-2)$ -nd gap in π' , the number of peaks will increase by 1. Further, the number of inversions will also increase by 1. As $\text{inv}(\pi) = \text{inv}(\pi') + 1$, π and π' are of opposite sign. Thus, the contribution from $\pi' \in \mathfrak{S}_{n-1,a,a}$ to $\text{SgnAltrun}_{n,a,d}(p, q)$ is $(-p) \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$.
3. To get $\pi \in \mathfrak{S}_{n,a,d}$ from $\pi' \in \mathfrak{S}_{n-1,d,d}$, there is again only one choice: insert n at the first gap. By this, the number of peaks will increase by 1. Further, the number of inversions will increase by $n-2$ which is odd. Hence π and π' are of opposite sign. Thus, the contribution from $\pi' \in \mathfrak{S}_{n-1,d,d}$ to $\text{SgnAltrun}_{n,a,d}(p, q)$ is $(-p) \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$.

With these, the proof of (15) is complete. We now consider (16). The polynomial $\text{SgnAltrun}_{n,a,a}(p, q)$ can be written as a sum of the following three polynomials.

$$\begin{aligned} \text{SgnAltrun}_{n,a,a}(p, q) = & \sum_{\pi \in \mathfrak{S}_{n,a,a}, \pi' \in \mathfrak{S}_{n-1,a,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,a,a}, \pi' \in \mathfrak{S}_{n-1,a,d}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \\ & + \sum_{\pi \in \mathfrak{S}_{n,a,a}, \pi' \in \mathfrak{S}_{n-1,d,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \end{aligned}$$

We again consider the contribution to $\text{SgnAltrun}_{n,a,a}(p, q)$ from each of the three terms.

1. Suppose $\pi' \in \mathfrak{S}_{n-1,a,a}$ and we want to get all $\pi \in \mathfrak{S}_{n,a,a}$ by inserting n in different gaps of π' . We clearly cannot insert n in the initial or the penultimate gap, as then the resulting $\pi \notin \mathfrak{S}_{n,a,a}$. Inserting n at the final gap of $\pi' \in \mathfrak{S}_{n-1,a,a}$ does not change the number of peaks and valleys. Let $k_1, k_2, \dots, k_r$ be the peaks of π . By Remark 2.7, the sum of the contribution of the permutations obtained by inserting n at $k_1 - 1, k_1, k_2 - 1, k_2, \dots, k_r - 1, k_r$ equals 0. The remaining gaps are $A = [n-1]_0 - \{0, k_1 - 1, k_1, k_2 - 1, k_2, \dots, k_r - 1, k_r, n-2, n-1\}$. Again, it is easy to see that inserting n at any gap from the set A increases both the number of peaks and valleys by 1. By Remark 2.1, the number of even gaps is $\lfloor n/2 \rfloor + 1$ and the number of odd gaps is $\lfloor n/2 \rfloor$. Moreover, 0 and $n-1$ are even gaps and $n-2$ is an odd gap. Hence, the number of even gaps available in the set A is $\lfloor n/2 \rfloor + 1 - (r+2)$. Similarly, the number of odd gaps available in the set A is $\lfloor n/2 \rfloor - (r+1)$. Thus, the number of odd and even gaps in A are equal. Hence, only the insertion at the final gap contributes to $\text{SgnAltrun}_{n,a,a}(p, q)$. Clearly, the contribution from $\pi' \in \mathfrak{S}_{n-1,a,a}$ to $\text{SgnAltrun}_{n,a,a}(p, q)$ is $(-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$.
2. To get a permutation of $\mathfrak{S}_{n,a,a}$ from a permutation π' of $\mathfrak{S}_{n-1,a,d}$, there is only one choice: insert n at the end of π' . Clearly, doing this increases the number of valleys by 1. Further, the number of inversions remains unchanged and therefore π and π' have same sign. Thus, the contribution from $\pi' \in \mathfrak{S}_{n-1,a,d}$ to $\text{SgnAltrun}_{n,a,a}(p, q)$ is $q \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$.
3. To get a permutation of $\mathfrak{S}_{n,a,a}$ from a permutation π' of $\mathfrak{S}_{n-1,d,a}$, there is again only one choice: insert n at the first gap. Clearly, doing this increases the number of peaks by 1. Clearly, the number of inversions increases by $n-2$ which is odd and hence π and π' are of opposite sign. Thus, the contribution from $\pi' \in \mathfrak{S}_{n-1,d,a}$ to $\text{SgnAltrun}_{n,a,a}(p, q)$ is $(-p) \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$.

The proof of (16) is complete. Theorem 3.1 and (16), give (17). The proof of our theorem is complete. $\square$

3.2. When n is even

When n is even, we have the following counterpart of Theorem 3.2.

Theorem 3.3. For even positive integers $n \geq 2$, we have:

$$\text{SgnAltrun}_{n,a,d}(p, q) = -p \text{SgnAltrun}_{n-1,a,a}(p, q) + p \text{SgnAltrun}_{n-1,d,d}(p, q) \quad (18)$$

$$\begin{aligned} \text{SgnAltrun}_{n,a,a}(p, q) = & (1 + pq) \text{SgnAltrun}_{n-1,a,a}(p, q) + q \text{SgnAltrun}_{n-1,a,d}(p, q) \\ & + p \text{SgnAltrun}_{n-1,d,a}(p, q) \end{aligned} \quad (19)$$

Proof. The moves we make in this proof are similar to those in the proof of Theorem 3.2. Thus, we are a little brief in our proofs of (18) and (19). Clearly, we have

$$\begin{aligned} \text{SgnAltrun}_{n,a,d}(p, q) = & \sum_{\pi \in \mathfrak{S}_{n,a,d}, \pi' \in \mathfrak{S}_{n-1,a,d}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,a,d}, \pi' \in \mathfrak{S}_{n-1,a,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \\ & + \sum_{\pi \in \mathfrak{S}_{n,a,d}, \pi' \in \mathfrak{S}_{n-1,d,d}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \end{aligned}$$

We just state the contribution to $\text{SgnAltrun}_{n,a,d}(p, q)$ from the three terms below and omit its proof as it is very similar to our earlier proof. The contribution from $\pi' \in \mathfrak{S}_{n-1,a,d}$ to $\text{SgnAltrun}_{n,a,d}(p, q)$ is 0. The contribution from $\pi' \in \mathfrak{S}_{n-1,a,a}$ to $\text{SgnAltrun}_{n,a,d}(p, q)$ is $(-p) \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$. The contribution from $\pi' \in \mathfrak{S}_{n-1,d,d}$ to $\text{SgnAltrun}_{n,a,d}(p, q)$ is $p \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$. This completes the proof of (18). We now prove (19). Note that $\text{SgnAltrun}_{n,a,a}(p, q)$ is the sum of the following three polynomials:

$$\begin{aligned} \text{SgnAltrun}_{n,a,a}(p, q) = & \sum_{\pi \in \mathfrak{S}_{n,a,a}, \pi' \in \mathfrak{S}_{n-1,a,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,a,a}, \pi' \in \mathfrak{S}_{n-1,a,d}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \\ & + \sum_{\pi \in \mathfrak{S}_{n,a,a}, \pi' \in \mathfrak{S}_{n-1,d,a}} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} q^{\text{val}(\pi)} \end{aligned}$$

We record the contribution to $\text{SgnAltrun}_{n,a,a}(p,q)$ from these three terms without proof below. The contribution from $\pi' \in \mathfrak{S}_{n-1,a,a}$ to $\text{SgnAltrun}_{n,a,a}(p,q)$ is $(1+pq) \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$. The contribution from $\pi' \in \mathfrak{S}_{n-1,a,d}$ to $\text{SgnAltrun}_{n,a,a}(p,q)$ is $q \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$. The contribution from $\pi' \in \mathfrak{S}_{n-1,d,a}$ to $\text{SgnAltrun}_{n,a,a}(p,q)$ is $p \times (-1)^{\text{inv}(\pi')} p^{\text{pk}(\pi')} q^{\text{val}(\pi')}$. Combining these, completes the proof of (19). $\square$

4. Proof of Theorem 1.1

Towards proving Theorem 1.1, we prove the following.

Theorem 4.1. For positive integers $n = 4k$, the following hold

$$\begin{aligned}\text{SgnAltrun}_{4k,a,a}(p,q) &= \text{SgnAltrun}_{4k,d,d}(p,q) = (1+pq)(1-pq)^{2(k-1)}, \\ \text{SgnAltrun}_{4k,a,d}(p,q) &= -2p(1-pq)^{2(k-1)}, \quad \text{SgnAltrun}_{4k,d,a}(p,q) = -2q(1-pq)^{2(k-1)}, \\ \text{SgnAltrun}_{4k}(p,q) &= 2(1-p)(1-q)(1-pq)^{2(k-1)}.\end{aligned}$$

Proof. We use induction on k . When $k = 1$, the four statements can easily be verified. We assume all the four statements to be true for $k = m$ and prove them for $k = m + 1$. Let $n = 4m + 4$. We prove a seemingly digressive result first. We claim that

$$\text{SgnAltrun}_{4m+1,a,a}(p,q) = \text{SgnAltrun}_{4m,a,a}(p,q), \quad \text{SgnAltrun}_{4m+1,d,d}(p,q) = \text{SgnAltrun}_{4m,d,d}(p,q), \quad (20)$$

$$\text{SgnAltrun}_{4m+1,a,d}(p,q) = \text{SgnAltrun}_{4m,a,d}(p,q), \quad \text{SgnAltrun}_{4m+1,d,a}(p,q) = \text{SgnAltrun}_{4m,d,a}(p,q). \quad (21)$$

We note that both relations in (20) follow from Theorem 3.2. By Theorem 3.2, we have

$$\begin{aligned}\text{SgnAltrun}_{4m+1,a,d}(p,q) &= (-pq) \times \text{SgnAltrun}_{4m,a,d}(p,q) \\ &\quad - p \times \text{SgnAltrun}_{4m,a,a}(p,q) - p \times \text{SgnAltrun}_{4m,d,d}(p,q) \\ &= (-pq)(-2p)(1-pq)^{2(m-1)} - 2p(1+pq)(1-pq)^{2(m-1)} \\ &= (-2p)(1-pq)^{2(m-1)} = \text{SgnAltrun}_{4m,a,d}(p,q).\end{aligned} \quad (22)$$

The second equality above follows by induction. Thus, the first part of (21) holds. As the proof of the second part of (21) is identical, we omit it. We now consider $n = 4m + 4$. We have

$$\begin{aligned}\text{SgnAltrun}_{4m+4,a,a}(p,q) &= (1+pq)\text{SgnAltrun}_{4m+3,a,a}(p,q) = (1+pq)\text{SgnAltrun}_{4m+2,a,a}(p,q) \\ &= (1+pq)[(1+pq)\text{SgnAltrun}_{4m+1,a,a}(p,q) \\ &\quad + q\text{SgnAltrun}_{4m+1,a,d}(p,q) + p\text{SgnAltrun}_{4m+1,d,a}(p,q)] \\ &= (1+pq)[(1+pq)\text{SgnAltrun}_{4m,a,a}(p,q) \\ &\quad + q\text{SgnAltrun}_{4m,a,d}(p,q) + p\text{SgnAltrun}_{4m,d,a}(p,q)] \\ &= (1+pq)[(1+pq)(1+pq)(1-pq)^{2(m-1)} \\ &\quad + q(-2p)(1-pq)^{2(m-1)} + p(-2q)(1-pq)^{2(m-1)}] = (1+pq)(1-pq)^{2m}.\end{aligned} \quad (23)$$

The first equality follows by Theorem 3.3 and Lemma 2.5. The second equality uses Theorem 3.2. In the third equality we again use Theorem 3.3. The fourth equality follows from (20) and (21). The fifth equality uses the induction hypothesis with $n = 4m$. Similarly, we get

$$\begin{aligned}\text{SgnAltrun}_{4m+4,a,d}(p,q) &= (-p)\text{SgnAltrun}_{4m+3,a,a}(p,q) + p\text{SgnAltrun}_{4m+3,d,d}(p,q) \\ &= -2p\text{SgnAltrun}_{4m+3,a,a}(p,q) = -2p\text{SgnAltrun}_{4m+2,a,a}(p,q) \\ &= -2p(1-pq)^{2m}.\end{aligned} \quad (24)$$

Here too, the first equation follows from Theorem 3.3. The second line uses Lemma 2.5 and Theorem 3.2. The last line uses the fact that $\text{SgnAltrun}_{4m+2,a,a}(p,q) = (1-pq)^{2m}$ which we already computed while proving (23). In a similar way one can show that

$$\text{SgnAltrun}_{4m+4,d,a}(p,q) = -2q(1-pq)^{2m}. \quad (25)$$

Summing up (23), (24) and (25) we get

$$\begin{aligned} \text{SgnAltrun}_{4m+4}(p, q) &= 2\text{SgnAltrun}_{4m+4,a,a}(p, q) + \text{SgnAltrun}_{4m+4,a,d}(p, q) + \text{SgnAltrun}_{4m+4,d,a}(p, q) \\ &= 2(1-p)(1-q)(1-pq)^{2m}. \end{aligned} \quad (26)$$

The proof is complete. $\square$

From the proof of Theorem 4.1, we get the following corollary.

Corollary 4.2. For positive integers k , when $n = 4k + 1$, we have

$$\begin{aligned} \text{SgnAltrun}_{4k+1,a,a}(p, q) &= \text{SgnAltrun}_{4k,d,d}(p, q) = (1 + pq)(1 - pq)^{2(k-1)}, \\ \text{SgnAltrun}_{4k+1,a,d}(p, q) &= -2p(1 - pq)^{2(k-1)}, \quad \text{SgnAltrun}_{4k+1,d,a}(p, q) = -2q(1 - pq)^{2(k-1)}, \\ \text{SgnAltrun}_{4k+1}(p, q) &= 2(1 - p)(1 - q)(1 - pq)^{2(k-1)}. \end{aligned}$$

Proof of Theorem 1.1. The proof follows directly from Lemma 2.5, Theorem 4.1 and Corollary 4.2. $\square$

From Theorem 1.1, we immediately have the following corollary.

Corollary 4.3. For positive integers n , we have

$$\sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} p^{\text{pk}(\pi)} = 0 = \sum_{\pi \in \mathfrak{S}_n} (-1)^{\text{inv}(\pi)} q^{\text{val}(\pi)}. \quad (27)$$

It is easy to see that peaks and valleys are equidistributed over $\mathfrak{S}_n$. By Corollary 4.3, we get that they are equidistributed over $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$.

4.1. Multiplicity of $(1 + t)$ as a factor of $R_n^\pm(t)$

Before we get the multiplicity of $(1 + t)$ as a factor of the univariate polynomial $R_n^\pm(t)$, we get the signed univariate polynomial. The following corollary has five results. The one on $\text{SgnAltrun}_n(t)$ has appeared as [14, Theorem 5.5], but the others are new.

Corollary 4.4. For positive integers k , the following hold. When $n = 4k$ and $n = 4k + 1$, we have:

$$\begin{aligned} \text{SgnAltrun}_{n,a,a}(t) &= \text{SgnAltrun}_{n,d,d}(t) = t(1 + t^2)(1 - t^2)^{2k-2}, \\ \text{SgnAltrun}_{n,a,d}(t) &= \text{SgnAltrun}_{n,d,a}(t) = -2t^2(1 - t^2)^{2k-2}, \quad \text{SgnAltrun}_n(t) = 2t(1 - t)^{2k}(1 + t)^{2k-2}. \end{aligned}$$

When $n = 4k + 2$ and $n = 4k + 3$, we have $\text{SgnAltrun}_{n,a,d}(t) = 0$, $\text{SgnAltrun}_{n,d,a}(t) = 0$,

$$\text{SgnAltrun}_{n,a,a}(t) = -\text{SgnAltrun}_{n,d,d}(t) = t(1 - t^2)^{2k}, \quad \text{SgnAltrun}_n(t) = 0.$$

Proof. As $\text{SgnAltrun}_n(t) = t\text{SgnAltrun}_n(p, q)|_{p=q=t}$, the proof follows from Theorem 1.1. $\square$

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We claim that $R_n^\pm(t) = \begin{cases} \frac{R_n(t) \pm 2t(1 - t)^{2k}(1 + t)^{2k-2}}{2} & \text{when } n = 4k, 4k + 1. \\ \frac{R_n(t)}{2} & \text{when } n = 4k + 2, 4k + 3. \end{cases}$ For getting the

above, we have used Corollary 4.4 above. Recall that $m = \lfloor (n - 2)/2 \rfloor$. Hence, when $n = 4k, 4k + 1$, we have $m = 2k - 1$. By Theorem 1.2, the polynomial $R_n(t)$ is divisible by $(1 + t)^{m-1}$ and by Remark 2.6, the polynomial $R_n(t)$ is divisible by 2. Thus, the polynomial $2(1 + t)^m$ divides the polynomial $R_n(t)$ but $2(1 + t)^{m-1}$ divides $\text{SgnAltrun}_n(t)$. Therefore, the polynomial $2(1 + t)^{m-1}$ divides $R_n^\pm(t)$ when $n \equiv 0, 1 \pmod{4}$. Similarly, when $n \equiv 2, 3 \pmod{4}$, the polynomial $2(1 + t)^m$ divides the polynomials $R_n(t)$. This completes the proof. $\square$

Example 4.5. When $n = 4$ and $n = 5$, one can check that $(1 + t)$ divides $R_4(t)$ and $R_5(t)$, but it does not divide the polynomials $R_4^+(t)$, $R_4^-(t)$, $R_5^+(t)$ and $R_5^-(t)$.

$$\begin{aligned}
R_4(t) &= 2t + 12t^2 + 10t^3, & R_5(t) &= 2 + 28t^2 + 58t^3 + 32t^4, \\
R_4^+(t) &= 2t + 4t^2 + 6t^3, & R_5^+(t) &= 2 + 12t^2 + 30t^3 + 16t^4, \\
R_4^-(t) &= 8t^2 + 4t^3, & R_5^-(t) &= 16t^2 + 28t^3 + 16t^4.
\end{aligned}$$

4.2. Moment-type identities

In this subsection, we prove Theorem 1.4. This refines a result from Comtet [11, page 261]. For our proof, we use the proof of the following result of Chow and Ma [10, Corollary 2]. We have paraphrased their result, but from their proof, this change of form will be clear.

Lemma 4.6 (Chow and Ma). *If $(1+t)^m$ divides $f(t) = \sum_{i=0}^n f_i t^i$ (with the convention that $f_p = 0$ when $p > n$), then for positive integers $k \leq m-1$, we have $\sum_{p=0}^n (2p+1)^k f_{2p+1} = \sum_{p=0}^n (2p+2)^k f_{2p+2}$.*

Proof of Theorem 1.4. The proof now follows from Theorem 1.3 and Lemma 4.6. $\square$

4.3. Alternating permutations in $\mathfrak{S}_n^\pm$

Let $[t^k]f(t)$ denote the coefficient of t^k in the polynomial $f(t)$. Moreover, let $[1]f(p, q)$ denote the constant term in the polynomial $f(p, q)$ and $[p]f(p, q)$ denote the coefficient of p in the polynomial $f(p, q)$.

Proof of Lemma 1.5. We at first consider the alternating permutations. Alternating permutations in $\mathfrak{S}_n$ clearly have $n-1$ alternating runs. For even positive integers n , any $\pi \in \text{Alt}_n$ must be in $\mathfrak{S}_{n,d,d}$. In this case, we have $E_n^+ - E_n^- = [t^{n-1}] \text{SgnAltrun}_{n,d,d}(t)$. Similarly, for odd positive integers n , any $\pi \in \text{Alt}_n$ must be in $\mathfrak{S}_{n,d,a}$. In this case, we have $E_n^+ - E_n^- = [t^{n-1}] \text{SgnAltrun}_{n,d,a}(t)$. The proof of the first part of the lemma is now complete by using Corollary 4.4.

We also see that any unimodal permutation has no valley and has at most 1 peak. Thus, we have $U_n^+ - U_n^- = [1] \text{SgnAltrun}_n(p, q) + [p] \text{SgnAltrun}_n(p, q)$. Therefore, by Theorem 1.1, we have $U_n^+ - U_n^- = 0$. Moreover, it is well-known that $U_n = 2^{n-1}$ (see [17]). Thus, $U_n^+ = U_n^- = 2^{n-2}$. This completes the proof. $\square$

With this preparation, we can now prove Theorem 1.6.

Proof of Theorem 1.6. We have

$$\sum_{n=0}^{\infty} E_n^\pm \frac{x^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} \pm \frac{1}{2} \sum_{n=0}^{\infty} (E_n^+ - E_n^-) \frac{x^n}{n!} = \frac{1}{2} (\sec x + \tan x \pm \cos x \pm x).$$

This completes the proof. $\square$

4.4. Central limit theorems

The following result about mean and variance of alternating runs is known, see [19, Page 46] and [7, Page 37].

Theorem 4.7. *Let permutations be sampled uniformly from $\mathfrak{S}_n$. When $n \geq 2$, the random variable X_{altruns} has mean $\mu_n = (2n-1)/3$ and when $n \geq 4$, it has variance $\sigma_n^2 = (16n-29)/90$. Thus, the distribution of the coefficients of $R_n(t)$ over $\mathfrak{S}_n$ is asymptotically normal.*

For positive integers n , let $F_n(t) = \sum_{k=0}^n f_{n,k} t^k$ and $G_n(t) = \sum_{k=0}^n g_{n,k} t^k$ be sequences of polynomials with $f_{n,k}, g_{n,k} \geq 0$ for all k . Further, for all positive integers n , let $F_n(1) > 0$ and $G_n(1) > 0$. We will consider the random variable X_f^n and X_g^n which takes the value k with probability $f_{n,k}/F_n(1)$ and $g_{n,k}/G_n(1)$ respectively. The following lemma follows from the proof of [16, Proposition 3.2] of Fulman, Petersen, Kim and Lee. It provides a condition for the moments of X_f^n and X_g^n to be identical. It appears in this form in [13, Lemma 8].

Lemma 4.8 (Fulman, Petersen, Kim and Lee). *Let $F_n(t)$ and $G_n(t)$ be as described above. Suppose we have a sequence of polynomials $H_n(t)$, a non-zero real number λ and a sequence of positive integers ℓ_n such that*

$$F_n(t) = \lambda G_n(t) \pm (1-t)^{\ell_n} H_n(t).$$

Then, for $r < \ell_n$, the r -th moment of X_f^n is identical to the r -th moment of X_g^n .

We can now prove our asymptotic normality results for X_{altruns} when sampled uniformly over $\mathcal{A}_n$ and $\mathfrak{S}_n - \mathcal{A}_n$.

Proof of Theorem 1.7. From the proof of Theorem 1.3, for $n \geq 1$, we clearly have

$$R_n^\pm(t) = \begin{cases} \frac{R_n(t) \pm 2t(1-t)^{2k}(1+t)^{2k-2}}{2} & \text{when } n = 4k, 4k+1. \\ \frac{R_n(t)}{2} & \text{when } n = 4k+2, 4k+3. \end{cases}$$

Thus, by Lemma 4.8, the first $\lfloor (n-1)/2 \rfloor$ moments of $R_n^\pm(t)$ are identical to the first $\lfloor (n-1)/2 \rfloor$ moments of $R_n(t)$. Since $R_n(t)$ is asymptotically normal, by the method of moments, $R_n^\pm(t)$ is also asymptotically normal. Further, they have the same expected value as $R_n(t)$ when $n \geq 3$ and the same variance as $R_n(t)$ when $n \geq 6$. The proof is complete. $\square$

5. Type B Coxeter groups

For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$, define $\text{Negs}(\pi) = \{\pi_i : i > 0, \pi_i < 0\}$ as the set of elements which occur in π with a negative sign. As defined in Petersen's book, [25, Page 294], define

$$\text{inv}_B(\pi) = |\{1 \leq i < j \leq n : \pi_i > \pi_j\}| + |\{1 \leq i < j \leq n : -\pi_i > \pi_j\}| + |\text{Negs}(\pi)|. \quad (28)$$

We refer to $\text{inv}_B(\pi)$ alternatively as the *length* of $\pi \in \mathfrak{B}_n$. Let $\mathfrak{B}_n^+ \subseteq \mathfrak{B}_n$ denote the subset of elements of $\mathfrak{B}_n$ with even length and let $\mathfrak{B}_n^- = \mathfrak{B}_n - \mathfrak{B}_n^+$. For $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$, let $\pi_0 = 0$ and define its set of type B peaks and type B valleys to be $\text{Peak}_B(\pi) = \{i \in [n-1] : \pi_{i-1} < \pi_i > \pi_{i+1}\}$ and $\text{Valley}_B(\pi) = \{i \in [n-1] : \pi_{i-1} > \pi_i < \pi_{i+1}\}$ respectively. Let $\text{pk}_B(\pi) = |\text{Peak}_B(\pi)|$ and $\text{val}_B(\pi) = |\text{Valley}_B(\pi)|$ be the cardinality of these sets. For $\pi \in \mathfrak{B}_n$, we say that π changes direction at index i if $i \in \text{Valley}_B(\pi) \cup \text{Peak}_B(\pi)$. We say that $\pi \in \mathfrak{B}_n$ has k type B alternating runs, denoted as $\text{altruns}_B(\pi) = k$ if it changes direction a total of $k-1$ times. For example, the permutation $5, 1, 4, \overline{3}, \overline{6}, 2 \in \mathfrak{B}_6$ has $\text{pk}_B(\pi) = 2$, $\text{val}_B(\pi) = 2$ and hence has $\text{altruns}_B(\pi) = 5$. Define $R_n^B(t) = \sum_{\pi \in \mathfrak{B}_n} t^{\text{altruns}_B(\pi)} = \sum_{k=1}^n R_{n,k}^B t^k$. As done for $\mathfrak{S}_n$, we are interested in enumerating the following:

$$\text{SgnBALtrun}_n(p, q) = \sum_{\pi \in \mathfrak{B}_n} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)}. \quad (29)$$

Recall that we had partitioned $\mathfrak{S}_n$ into four sets. We similarly partition $\mathfrak{B}_n$ into the following two sets: $\mathfrak{B}_{n,-,a} = \{\pi \in \mathfrak{B}_n : \pi_{n-1} < \pi_n\}$ and $\mathfrak{B}_{n,-,d} = \{\pi \in \mathfrak{B}_n : \pi_{n-1} > \pi_n\}$. Define the following polynomials:

$$\text{SgnBALtrun}_{n,-,a}(p, q) = \sum_{\pi \in \mathfrak{B}_{n,-,a}} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)}, \quad (30)$$

$$\text{SgnBALtrun}_{n,-,d}(p, q) = \sum_{\pi \in \mathfrak{B}_{n,-,d}} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)}. \quad (31)$$

We start with the following type B counterparts of Lemma 2.3 and Lemma 2.5 and connect the following polynomials $\text{SgnBALtrun}_{n,-,a}(p, q)$ and $\text{SgnBALtrun}_{n,-,d}(p, q)$.

Lemma 5.1. For positive integers n , we have

$$\text{SgnBALtrun}_{n,-,a}(p, q) = -\text{SgnBALtrun}_{n,-,d}(q, p) \text{ when } n \text{ is odd.} \quad (32)$$

$$\text{SgnBALtrun}_{n,-,a}(p, q) = \text{SgnBALtrun}_{n,-,d}(q, p) \text{ when } n \text{ is even.} \quad (33)$$

Proof. Let $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$. Define the map $\text{FlipSgn} : \mathfrak{B}_{n,-,a} \rightarrow \mathfrak{B}_{n,-,d}$ by

$$\text{FlipSgn}(\pi) = \overline{\pi_1}, \dots, \overline{\pi_2}, \dots, \overline{\pi_n}.$$

Clearly, FlipSgn is a bijection from $\mathfrak{B}_{n,-,a}$ to $\mathfrak{B}_{n,-,d}$. It is easy to see that flipping the sign of a single π_i changes the parity of inv_B (for example, see [27, Lemma 3]). Therefore, the map Compl_B preserves the parity of inv_B if and only if n is even. Moreover, it is easy to see that $\text{pk}_B(\pi) = \text{val}_B(\text{Compl}_B(\pi))$ and $\text{val}_B(\pi) = \text{pk}_B(\text{Compl}_B(\pi))$. As the remaining portion of the proof is similar to the proof of Lemma 2.3 and Lemma 2.5, we omit the details. The proof is complete. $\square$

We now consider the identity permutation $\text{id} = 1, 2, \dots, n-1, n \in \mathfrak{B}_n$ and the set $\mathfrak{B}_n(\text{id}) = \{\pi \in \mathfrak{B}_n : |\pi| = \text{id}\}$. Moreover, let $\mathfrak{B}_{n,-,a}(\text{id}) = \mathfrak{B}_{n,-,a} \cap \mathfrak{B}_n(\text{id})$ and $\mathfrak{B}_{n,-,d}(\text{id}) = \mathfrak{B}_{n,-,d} \cap \mathfrak{B}_n(\text{id})$. It is clear that $|\mathfrak{B}_n(\text{id})| = 2^n$.

Lemma 5.2. For positive integers n , we have $\sum_{\pi \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)} = 0$. Moreover,

$$\sum_{\pi \in \mathfrak{B}_{n,-,a} \setminus \mathfrak{B}_{n,-,a}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)} = 0, \quad \text{and} \quad \sum_{\pi \in \mathfrak{B}_{n,-,d} \setminus \mathfrak{B}_{n,-,d}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)} = 0.$$

Proof. Let $\pi = \pi_1, \dots, \pi_n \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\text{id})$. As $\pi \notin \mathfrak{B}_n(\text{id})$, there exists $1 \leq j \leq n$ such that $|\pi_j| \neq j$. Let $1 \leq k \leq n$ be the smallest index such that $|\pi_k| \neq k$. Let $|\pi_r| = k$. Thus $1 \leq r$. Further, as $\pi \in \mathfrak{B}_n \setminus \mathfrak{B}_n(\text{id})$, we also have $r \leq n$. We define f by flipping the sign of π_r in π . Thus, define $f : (\mathfrak{B}_n \setminus \mathfrak{B}_n(\text{id})) \rightarrow (\mathfrak{B}_n \setminus \mathfrak{B}_n(\text{id}))$ by

$$f(\pi_1, \dots, \pi_{r-1}, \pi_r, \pi_{r+1}, \dots, \pi_n) = \pi_1, \dots, \pi_{r-1}, \overline{\pi_r}, \pi_{r+1}, \dots, \pi_n.$$

In other words, $f(\pi)_i = \pi_i$ for $i \neq r$. Moreover, if $\pi_r = k$, we have $f(\pi)_r = \bar{k}$ and if $\pi_r = \bar{k}$, we have $f(\pi)_r = k$.

For example, if $\pi = 1, \bar{2}, 3, 5, \bar{4}$, then 4 is the smallest k with $|\pi_4| \neq 4$. Here, $|\pi_5| = 4$. So, we flip the sign of π_5 in π to obtain $f(\pi)$. Thus, $f(\pi) = 1, \bar{2}, 3, 5, 4$. For $\pi \in \mathfrak{B}_n$, it is easy to check that $\text{inv}_B(\pi) \not\equiv \text{inv}_B(f(\pi)) \pmod{2}$. That is, the map f flips the parity of the type B length (see [27, Lemma 3]). Recall r is the index with $|\pi_r| = |k|$. Note that $r - 1$ is a peak (valley, respectively) of π if and only if $r - 1$ is a peak (valley, respectively) of $f(\pi)$. When $r \neq n$, as $|\pi_{r+1}| > k$, we have $\pi_{r+1} > k$ if and only if $\pi_{r+1} > \bar{k}$. Thus, the map f preserves the number of peaks and valleys and is a sign-reversing involution. This completes the proof of the first part of the theorem. For the second part, we observe that the map f is invariant on the sets $\mathfrak{B}_{n,-a}$ and $\mathfrak{B}_{n,-d}$. The proof is complete. $\square$

Theorem 5.3. For positive integers $n \geq 2$, the following recurrence relations hold:

$$\text{SgnBALtrun}_{n,-a}(p, q) = (1 - pq) \text{SgnBALtrun}_{n-2,-a}(p, q), \quad (34)$$

$$\text{SgnBALtrun}_{n,-d}(p, q) = (1 - pq) \text{SgnBALtrun}_{n-2,-d}(p, q). \quad (35)$$

Therefore, we have

$$\begin{aligned} \text{SgnBALtrun}_{n,-a}(p, q) &= \begin{cases} (1 - q)(1 - pq)^{k-1} & \text{when } n = 2k \text{ is even,} \\ (1 - pq)^k & \text{when } n = 2k + 1 \text{ is odd.} \end{cases} \\ \text{SgnBALtrun}_{n,-d}(p, q) &= \begin{cases} (1 - p)(1 - pq)^{k-1} & \text{when } n = 2k \text{ is even,} \\ -(1 - pq)^k & \text{when } n = 2k + 1 \text{ is odd.} \end{cases} \\ \text{SgnBALtrun}_n(p, q) &= \begin{cases} (2 - p - q)(1 - pq)^{k-1} & \text{when } n = 2k \text{ is even,} \\ 0 & \text{when } n \text{ is odd.} \end{cases} \end{aligned}$$

Proof. We first consider (34). By Lemma 5.2, we have

$$\text{SgnBALtrun}_{n,-a}(p, q) = \sum_{\pi \in \mathfrak{B}_{n,-a}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)}$$

Let $\pi \in \mathfrak{B}_{n,-a}(\text{id})$. From definition of the set $\mathfrak{B}_{n,-a}(\text{id})$, we have that $\pi_i \in \{i, \bar{i}\}$ for $i \in [n - 1]$ and $\pi_n = n$. Thus, one of the following two possibilities can occur: (1) $\pi_{n-1} = n - 1$ and $\pi_n = n$, (2) $\pi_{n-1} = \bar{n} - 1$ and $\pi_n = n$. Thus, to compute $\sum_{\pi \in \mathfrak{B}_{n,-a}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)}$, one can inductively get all permutations in $\mathfrak{B}_{n,-a}(\text{id})$ by inserting $n - 1$, n or $\bar{n} - 1$, n at the end of the permutations of $\mathfrak{B}_{n-2}(\text{id})$.

Let $\pi_1, \dots, \pi_{n-2} \in \mathfrak{B}_{n-2,-a}(\text{id})$. Then, $\pi_1, \dots, \pi_{n-2}, n - 1, n \in \mathfrak{B}_{n,-a}(\text{id})$ and $\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n \in \mathfrak{B}_{n,-a}(\text{id})$. It is easy to see that $\text{pk}_B(\pi_1, \dots, \pi_{n-2}, n - 1, n) = \text{pk}_B(\pi_1, \dots, \pi_{n-2})$, $\text{val}_B(\pi_1, \dots, \pi_{n-2}, n - 1, n) = \text{val}_B(\pi_1, \dots, \pi_{n-2})$ and $\text{inv}_B(\pi_1, \dots, \pi_{n-2}, n - 1, n) = \text{inv}_B(\pi_1, \dots, \pi_{n-2})$. Moreover, we also have $\text{pk}_B(\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n) = \text{pk}_B(\pi_1, \dots, \pi_{n-2}) + 1$, $\text{val}_B(\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n) = \text{val}_B(\pi_1, \dots, \pi_{n-2}) + 1$ and $\text{inv}_B(\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n) \not\equiv \text{inv}_B(\pi_1, \dots, \pi_{n-2}) \pmod{2}$ by [27, Lemma 3].

We now consider $\pi_1, \dots, \pi_{n-2} \in \mathfrak{B}_{n-2,-d}(\text{id})$. Again, we have $\pi_1, \dots, \pi_{n-2}, n - 1, n \in \mathfrak{B}_{n,-a}(\text{id})$ and $\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n \in \mathfrak{B}_{n,-a}(\text{id})$. In this case, we have $\text{pk}_B(\pi_1, \dots, \pi_{n-2}, n - 1, n) = \text{pk}_B(\pi_1, \dots, \pi_{n-2})$, $\text{val}_B(\pi_1, \dots, \pi_{n-2}, n - 1, n) = \text{val}_B(\pi_1, \dots, \pi_{n-2}) + 1$ and $\text{inv}_B(\pi_1, \dots, \pi_{n-2}, n - 1, n) = \text{inv}_B(\pi_1, \dots, \pi_{n-2})$; and $\text{pk}_B(\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n) = \text{pk}_B(\pi_1, \dots, \pi_{n-2})$, $\text{val}_B(\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n) = \text{val}_B(\pi_1, \dots, \pi_{n-2}) + 1$ and $\text{inv}_B(\pi_1, \dots, \pi_{n-2}, \bar{n} - 1, n) \not\equiv \text{inv}_B(\pi_1, \dots, \pi_{n-2}) \pmod{2}$. Thus, we have

$$\begin{aligned} \sum_{\pi \in \mathfrak{B}_{n,-a}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)} &= (1 - pq) \sum_{\pi \in \mathfrak{B}_{n-2,-a}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)} \\ &\quad + (q - q) \sum_{\pi \in \mathfrak{B}_{n-2,-d}(\text{id})} (-1)^{\text{inv}_B(\pi)} p^{\text{pk}_B(\pi)} q^{\text{val}_B(\pi)} \\ &= (1 - pq) \text{SgnBALtrun}_{n-2,-a}(p, q) \end{aligned}$$

This completes the proof of (34). The proof of (35) follows from (34) by using the FlipSgn map. The following base cases are also easy to see. When $n = 1$, we have $\text{SgnBALtrun}_{n,-a}(p, q) = 1$ and $\text{SgnBALtrun}_{n,-d}(p, q) = -1$. Similarly, when $n = 2$, we have $\text{SgnBALtrun}_{2,-a}(p, q) = (1 - q)$ and $\text{SgnBALtrun}_{2,-d}(p, q) = (1 - p)$. Using the recurrences (34), (35) and Lemma 5.1, the proof is complete. $\square$

5.1. Exponent of $(1+t)$ that divides $R_n^{B,\pm}(t)$ and moment type identities

Setting $p = q = t$ in Theorem 5.3 and multiplying the result by t , we get the following.

Corollary 5.4. For positive integers n , we have

$$\text{SgnBAltrun}_n(t) = \begin{cases} 2t(1-t)(1-t^2)^{k-1} & \text{when } n = 2k \text{ is even,} \\ 0 & \text{when } n \text{ is odd.} \end{cases}$$

Zhao in [33] considering the following polynomial $R_n^{B,>}(t) = \sum_{\pi \in \mathfrak{B}_n: \pi_1 > 0} t^{\text{altrun}_B(\pi)} = \sum_{i=1}^n R_{n,i}^{B,>} t^i$, and in [33, Theorem 4.3.2], Zhao proved that for positive integers n , the polynomials $R_n^{B,>}(t)$ and hence $R_n^B(t)$ are divisible by $(1+t)^m$ where $m = \lfloor (n-1)/2 \rfloor$. From Corollary 5.4, we get the following refinement. As its proof is very similar to the proof of Theorem 1.3, we omit it.

Theorem 5.5. For positive integers n , both polynomials $R_n^{B,\pm}(t)$ are divisible by $(1+t)^m$ where $m = \lfloor (n-1)/2 \rfloor$.

From Lemma 4.6 and Theorem 5.5, the following is straightforward.

Corollary 5.6. For positive integers n and k with $n \geq 2k+3$, $\sum_{p=0}^n (2p+1)^k R_{n,2p+1}^{B,\pm} = \sum_{p=0}^n (2p+2)^k R_{n,2p+2}^{B,\pm}$.

5.2. Alternating and unimodal permutations in $\mathfrak{B}_n^\pm$

A permutation $\pi = \pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$ is called alternating if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \dots$, that is, $\pi_i < \pi_{i+1}$ for i even, and $\pi_i > \pi_{i+1}$ for i odd. Let, Alt_n^B and be the set of alternating permutations in $\mathfrak{B}_n$ and let $E_n^B = |\text{Alt}_n^B|$. The following counterpart of André's result is due to Steingrímsson [31, Theorem 34].

Theorem 5.7 (Steingrímsson). For positive integers n , we have $\sum_{n=0}^{\infty} E_n^B \frac{x^n}{n!} = \sec 2x + \tan 2x$.

Define $E_n^{B,+} = |\text{Alt}_n^B \cap \mathfrak{B}_n^+|$ and $E_n^{B,-} = |\text{Alt}_n^B \cap \mathfrak{B}_n^-|$. A permutation $\pi \in \mathfrak{B}_n$ is called unimodal if there exists an index j for which $\pi_1 < \pi_2 < \dots < \pi_j > \pi_{j+1} > \dots > \pi_n$. Moreover, let U_n^B , $U_n^{B,+}$, and $U_n^{B,-}$ denote the unimodal permutations in $\mathfrak{B}_n$, $\mathfrak{B}_n^+$, and $\mathfrak{B}_n^-$ respectively. Chow in [9, Theorem 2.3] proved that $U_n^B = 2^{2n-1}$. We start with the following Lemma.

Lemma 5.8. For positive integers n , we have $E_n^{B,+} = E_n^{B,-}$ and $U_n^{B,+} = U_n^{B,-}$.

Proof. Let, $\pi = \pi_1, \pi_2, \dots, \pi_j = x, \dots, \pi_n \in \text{Alt}_n^B \cap \mathfrak{B}_n^+$ where $|x| = 1$. Define $f : \text{Alt}_n^B \cap \mathfrak{B}_n^+ \rightarrow \text{Alt}_n^B \cap \mathfrak{B}_n^-$ by

$$f(\pi_1, \pi_2, \dots, \pi_j = x, \dots, \pi_n) = \pi_1, \pi_2, \dots, \pi_j = \bar{x}, \dots, \pi_n.$$

That is, f flips the sign of the smallest element in absolute value. Clearly, f is a bijection. When π is an alternating permutation, it is easy to note that $f(\pi)$ is also an alternating permutation. Moreover, $f(\pi) \in \mathfrak{B}_n^+$ iff $\pi \in \mathfrak{B}_n^-$. Therefore, $E_n^{B,+} = E_n^{B,-} = E_n^B/2$. The map f also shows that the number of unimodal elements in $\mathfrak{B}_n^+$ and $\mathfrak{B}_n^-$ are equinumerous. With these maps, the proof is complete. $\square$

Using the convention $E_0^{B,+} = 1$ and $E_0^{B,-} = 0$, and combining Lemma 4.8, and Lemma 5.8, we immediately get the following. As its proof is easy, we omit it.

Corollary 5.9. We have the following exponential generating functions: $\sum_{n=0}^{\infty} E_n^{B,\pm} \frac{x^n}{n!} = (\sec 2x + \tan 2x \pm 1)/2$. We further have

$$U_n^{B,+} = U_n^{B,-} = 2^{2n-2}.$$

5.3. Springer permutations in $\mathfrak{B}_n^\pm$

A snake of type B is a signed permutation $\pi_1, \pi_2, \dots, \pi_n \in \mathfrak{B}_n$ such that $0 < \pi_1 > \pi_2 < \pi_3 > \dots$. Let, Snake_n^B be the set of snakes in $\mathfrak{B}_n$ and let $S_n^B = |\text{Snake}_n^B|$. The numbers S_n^B are also called Springer numbers. Springer in [28] derived the following generating function $S(x) = \sum_{n \geq 0} S_n^B \frac{x^n}{n!} = \frac{1}{\cos x - \sin x}$.

Define $\text{Snake}_n^{B,+} = \text{Snake}_n^B \cap \mathfrak{B}_n^+$ to be the set of positive type B snakes and let $S_n^{B,+} = |\text{Snake}_n^{B,+}|$. Similarly, let $\text{Snake}_n^{B,-} = \text{Snake}_n^B \cap \mathfrak{B}_n^-$ and let $S_n^{B,-} = |\text{Snake}_n^{B,-}|$. We start by proving the following lemma. Let $\mathfrak{B}_n^> = \{\pi \in \mathfrak{B}_n : \pi_1 > 0\}$.

Lemma 5.10. For positive integers n , we have $S_n^{B,+} - S_n^{B,-} = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod{4}, \\ -1 & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$

Proof. Let $\text{id} = 1, 2, 3, 4, \dots, n-1, n \in \mathfrak{B}_n^>$ be the identity permutation. Consider the set $\mathfrak{B}_n^>(\text{id}) = \{\pi \in \mathfrak{B}_n^> : \pi_1 = 1 \text{ and } |\pi_i| = i \text{ for } i > 1\}$. For positive integers n , clearly, $|\mathfrak{B}_n^>(\text{id})| = 2^{n-1}$. Let $\pi = \pi_1, \pi_2, \dots, \pi_n \in \text{Snake}_n^{B,+} \cap (\mathfrak{B}_n - \mathfrak{B}_n^>(\text{id}))$. As $\pi \notin \mathfrak{B}_n^>(\text{id})$, there exists $j < n$ such that $|\pi_j| \neq j$. Let k be the smallest index such that $|\pi_k| \neq k$. Let $r > k$ be the index with $|\pi_r| = k$. Note that $r \geq 2$ and $r < n$. As k is the smallest non-fixed point in absolute value, we have $|\pi_i| = i$ for $1 \leq i \leq k-1$. As $2 \leq r < n$, both π_{r-1} and π_{r+1} exist and we have $|\pi_{r-1}| > k$ and $|\pi_{r+1}| > k$. Define $f : \text{Snake}_n^{B,+} \cap (\mathfrak{B}_n - \mathfrak{B}_n^>(\text{id})) \rightarrow \text{Snake}_n^{B,-} \cap (\mathfrak{B}_n - \mathfrak{B}_n^>(\text{id}))$ by

$$f(\pi_1, \pi_2, \dots, \pi_r = k, \dots, \pi_n) = \pi_1, \pi_2, \dots, \pi_r = \bar{k}, \dots, \pi_n.$$

When π is snake, it is easy to observe that $f(\pi)$ is also a snake. Further, as $\pi \in \mathfrak{B}_n^+$, we have $f(\pi) \in \mathfrak{B}_n^-$. Thus f is a well defined map and is clearly a bijection. Therefore, we have $|\text{Snake}_n^{B,+} \cap (\mathfrak{B}_n - \mathfrak{B}_n^>(\text{id}))| = |\text{Snake}_n^{B,-} \cap (\mathfrak{B}_n - \mathfrak{B}_n^>(\text{id}))|$. Thus, $S_n^{B,+} - S_n^{B,-} = |\text{Snake}_n^{B,+} \cap \mathfrak{B}_n^>(\text{id})| - |\text{Snake}_n^{B,-} \cap \mathfrak{B}_n^>(\text{id})|$.

We now count the number of snakes in the set $\mathfrak{B}_n^>(\text{id})$. Let π be a snake in $\mathfrak{B}_n^>(\text{id})$. We have $\pi_1 = 1$. As π is a snake in $\mathfrak{B}_n^>(\text{id})$, we have $\pi_2 < \pi_1$. As $\pi_2 \in \{2, \bar{2}\}$, $\pi_2 = \bar{2}$. Similarly, $\pi_3 = 3$. Continuing in this way, π must have $\pi_r = r$ when r is odd and $\pi_r = \bar{r}$ when r is even. Thus, for any positive integer n , the set $\mathfrak{B}_n^>(\text{id})$ has only 1 snake, namely $\pi = 1, \bar{2}, 3, \bar{4}, 5, \bar{6}, \dots$. The permutation $\pi \in \mathfrak{B}_n^+$ if and only if π has even number of negative entries which happens if and only if n is 0, 1 (mod 4). The proof is complete. $\square$

With initial values $S_0^{B,+} = 1$, $S_0^{B,-} = 0$, we get the following consequence of Lemma 5.10. As the proof is easy, we omit it.

Theorem 5.11. We have the following exponential generating functions:

$$\sum_{n=0}^{\infty} S_n^{B,+} \frac{x^n}{n!} = \frac{\cos^2 x}{\cos x - \sin x}, \quad \sum_{n=0}^{\infty} S_n^{B,-} \frac{x^n}{n!} = \frac{\sin^2 x}{\cos x - \sin x}.$$

5.4. Central limit theorems

In this Subsection, we give type B analogues of Theorem 1.7. The following result is due to Zhao [33, Proposition 4.5.1].

Theorem 5.12 (Zhao). The distribution of the coefficients of $R_n^B(t)$ is asymptotically normal as $n \rightarrow \infty$. The random variable X_{altruns_B} over $\mathfrak{B}_n$ has mean $(8n+5)/12$ for $n \geq 2$ and variance $(64n-71)/360$ for $n \geq 4$.

Combining Lemma 4.8 with Theorem 5.12 gives us the following. As its proof is similar to the proof of Theorem 1.7, we omit it.

Theorem 5.13. The distribution of the coefficients of $R_n^{B,\pm}(t)$ is asymptotically normal as $n \rightarrow \infty$. The random variable X_{altruns_B} over $\mathfrak{B}_n^\pm$ has mean $(8n+5)/12$ for $n \geq 3$ and variance $(64n-71)/360$ for $n \geq 5$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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