

Research Article

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Four-point condition matrices of edge-weighted trees

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Abstract: Formulas for the determinant of distance matrix D_T of tree T are known in the unweighted case and in the case when the edges of T have commuting variable weights. Associated with the *four-point condition* (4PC) and a tree T are two matrices, the Max4PC_T and the Min4PC_T . These are not full rank matrices and their rank, a basis B , and formulas for the determinant when restricted to the rows and columns of B are known. In this work, we generalize both these matrices to the case when the edges of T have commuting variable weights and determine edge-weighted counterparts of known results.

Keywords: distance matrix, tree, four-point condition, edge weights

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1 Introduction

Let $T = (V, E)$ be a tree on n vertices with edge set $E = \{e_1, \dots, e_{n-1}\}$. For $1 \leq i \leq n-1$, assign a variable α_i on the edge e_i . We assume that the α_i 's commute with each other and think of α_i as the weight on edge e_i . For two vertices $i, j \in T$, define their weighted distance $w_{i,j}$ as the sum of the weights of the variables on the edges in the unique path between i and j . We will need the unweighted distance $d_{i,j}$ between vertices i and j as well in this work. Recall that $d_{i,j}$ is the number of edges on the path between i and j . The weighted distance matrix WD of T is an $n \times n$ matrix whose (i, j) th entry is the weighted distance $w_{i,j}$. Clearly, when we set $\alpha_i = 1$ for $1 \leq i \leq n-1$, the matrix WD reduces to the unweighted distance matrix $D = (d_{i,j})$ of T . The following is a remarkable result of Graham and Pollak on the distance matrix D of a tree T [15].

Theorem 1. (Graham and Pollak) *Let T be a tree on n vertices with unweighted distance matrix D . Then, $\det D = (-1)^{n-1} 2^{n-2} (n-1)$.*

Thus, the value of $\det D$ is independent of the structure of the tree T and only depends on n , the number of vertices of T . Bapat et al. [6] considered the weighted distance matrix WD of a tree T on n vertices with edge $e_i \in E(T)$ having weight α_i for $1 \leq i \leq n-1$. The following two quantities appear multiple times in this work, and so for a tree T on n vertices, we define $\text{Prod}_T(\alpha) = \prod_{i=1}^{n-1} \alpha_i$ and $\text{Sum}_T(\alpha) = \sum_{i=1}^{n-1} \alpha_i$. With this notation, their result is the following.

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Theorem 2. (Bapat, Kirkland, and Neumann) *Let T be a tree on n vertices. For $1 \leq i \leq n - 1$, let edge $e_i \in E(T)$ have weight α_i , and let WD denote T 's weighted distance matrix. Then,*

$$\det WD = (-1)^{n-1} 2^{n-2} \cdot \text{Prod}_T(\alpha) \cdot \text{Sum}_T(\alpha).$$

Thus, the value of $\det WD$ is only a function of n and the edge weights (the α_i 's), and does not depend on the structure of T . Clearly, setting $\alpha_i = 1$ when $1 \leq i \leq n - 1$ in Theorem 2 gives us Theorem 1, and hence, Theorem 2 generalizes Theorem 1. These two results have inspired further studies on distances in trees (see, for instance, [5, 7–12, 17]).

Given a distance matrix D , a natural question is the following. When will entries of D arise from shortest path distances on a tree? Answering this, Buneman [14] showed this happens if and only if the distances satisfy the *four-point condition* (4PC). The 4PC states that for any four vertices i, j, k , and ℓ in T , the maximum value of the three terms $d_{i,j} + d_{k,\ell}$, $d_{i,k} + d_{j,\ell}$, and $d_{i,\ell} + d_{j,k}$ equals the second maximum value. Buneman mentioned in his study that the 4PC also holds in the weighted case when the edges have weights.

Let $V = \{1, 2, \dots, n\}$ be the set of vertices of T , and let $\mathbb{V}_n$ be the set of size-2 subsets of V . Three unweighted matrices related to the 4PC having rows and columns indexed by $\mathbb{V}_n$ are known in the literature. These are the Max4PC_T matrix, the Min4PC_T matrix, and the 2-Steiner distance matrix $\mathcal{D}_2(T)$ of T .

We describe the three matrices and give known results about them. Let $X, Y \in \mathbb{V}_n$ with $X = \{i, j\}$ and $Y = \{k, \ell\}$. The entry in row X and column Y in the maximum-4PC (respectively, minimum-4PC) matrix is denoted as $\text{Max4PC}(X, Y)$ (respectively, $\text{Min4PC}(X, Y)$) and is defined to be the maximum (respectively, minimum) value among three terms $d_{i,j} + d_{k,\ell}$, $d_{i,k} + d_{j,\ell}$, and $d_{i,\ell} + d_{j,k}$. The Steiner distance between X and Y is denoted by $\text{ST}(X, Y)$ and is defined as the number of edges in a minimum size subtree of T that contains the vertices in $X \cup Y$. The entry of the 2-steiner distance matrix $\mathcal{D}_2(T)$ in the row indexed by X and the column indexed by Y is $\text{ST}(X, Y)$. Azimi and Sivasubramanian [3] showed that as matrices,

$$\mathcal{D}_2(T) = \frac{1}{2}(\text{Max4PC}_T + \text{Min4PC}_T).$$

Thus, for all $X, Y \in \mathbb{V}_n$, we have

$$\text{ST}(X, Y) = \frac{1}{2}(\text{Max4PC}(X, Y) + \text{Min4PC}(X, Y)). \quad (1)$$

Bapat and Sivasubramanian [13] studied the Min4PC_T matrix for a tree T . The authors determined the rank of Min4PC_T , implicitly gave a basis B for its row space, and also determined its Smith normal form (SNF). Azimi et al. [2] determined the inverse of the submatrix of Min4PC_T when restricted to rows and columns indexed by this basis B . The Min4PC_T matrix is a hollow matrix as studied in Bapat and Kurata [18], and hence, it would be interesting to see if some of our results hold in the hollow matrix setting.

Azimi et al. [4] studied the matrix $\mathcal{D}_2(CT)$ of a caterpillar tree CT and among other results, determined the rank of $\mathcal{D}_2(CT)$. They asked for generalizing their results to all trees. Azimi and Sivasubramanian [3] considered $\mathcal{D}_2(T)$ for all trees T . The authors determined the rank of $\mathcal{D}_2(T)$ and gave a basis for its row space, determined the inertia, the determinant, and the inverse of the submatrix of $\mathcal{D}_2(T)$ when restricted to rows and columns from their basis.

The matrix Max4PC_T for a tree T has been recently studied by Azimi et al. [1]. The authors gave a formula for the rank of Max4PC_T , gave an algorithm that outputs a class $\mathfrak{B}$ of bases for the row space of Max4PC_T , and gave a formula for $\det \text{Max4PC}_T[B, B]$ for any basis $B \in \mathfrak{B}$. The authors also determined the inertia of Max4PC_T .

In this article, we consider edge-weighted versions of two of these matrices. Let T be a tree on n vertices with edge set $E = \{e_1, e_2, \dots, e_{n-1}\}$, and for $1 \leq i \leq n - 1$, let the edge e_i have variable weight α_i . As remarked earlier, we assume that the α_i 's commute with each other. For $U, V \in \mathbb{V}_n$, since the matrix entry indexed by (U, V) will be a polynomial with several variables (depending on both the tree T and the matrix considered), it is easy to see that setting $\alpha_i = 1$ for $1 \leq i \leq n - 1$ would give us the numerical matrices described earlier. However, we take care **not to set negative values** to any α_i as distances could then become negative and cease being a metric.

We consider edge-weighted versions of the matrices Max4PC_T and Min4PC_T and denote them as WtMax4PC_T and WtMin4PC_T , respectively. In this multi-variable setting, results on rank follow without major

difficulties, but a formula for the determinant obtains interesting and proofs tend to become involved. Hence, our main results in this work are similar to how Theorem 2 generalizes Theorem 1. Among other results, we mention the following two results on the WtMin4PC_T and the WtMax4PC_T matrices.

Theorem 3. *Let T be a weighted tree on $n > 2$ vertices with edge set E . For $1 \leq k \leq n - 1$, let edge $e_k \in E$ have weight α_k . Take $e = \{i, j\} \notin E$ and let $B = E \cup \{e\}$. Then, B is a basis for the row space of WtMin4PC_T . Furthermore,*

$$\det \text{WtMin4PC}_T[B, B] = (-1)^{n-1} 2^{n-2} (d_{i,j} - 1)^2 \cdot \text{Prod}_T(\alpha) \cdot \text{Sum}_T(\alpha).$$

In Theorem 3, when $e = \{i, j\} \notin E$ is chosen with $d_{i,j} = 2$ and $B = E \cup \{e\}$, it is easy to see that $\det \text{WtMin4PC}_T[B, B]$ is identical to the expression given in Theorem 2. Hence, in this case, the two weighted matrices $\text{WtMin4PC}_T[B, B]$ and WD have the same determinant value. We give an unweighted version in Theorem 6. It is simple to check that setting $\alpha_i = 1$ for $1 \leq i \leq n - 1$ in Theorem 3 gives Theorem 6. Theorem 3 is proved in Section 2.

We move on to the WtMax4PC_T matrix. As mentioned earlier, in [1, Theorem 20], an algorithm for constructing a family $\mathfrak{B}$ of bases was given. By part (a) of Theorem 24, any output B of this algorithm contains a unique pendant edge $e \in B$. The weight of this edge e plays a role in our formula for the weighted case. In Section 4, we prove the following.

Theorem 4. *Let T be a tree on $n \geq 3$ vertices with p leaves. Let $E = \{e_1, e_2, \dots, e_{n-1}\}$ be its edge set, and for $1 \leq i \leq n - 1$, let α_i be the weight of edge e_i . Let B be a basis for the row space of WtMax4PC_T output by [1, Lemma 9]. Let E_B denote the set of non-pendant edges of T that are in B , and let e be the unique pendant edge in B . Let the weight of e be α . Then,*

$$\det \text{WtMax4PC}_T[B, B] = (-1)^{n-p} 2^{2(n-p-1)} \alpha^2 \prod_{e_r \in E_B} \alpha_r^2.$$

Thus, for any pendant edge $e_i \notin B$, setting α_i to any positive real value, setting $\alpha = 1$, and for any non-pendant edge $e_j \in B$, setting $\alpha_j = 1$, we recover our earlier result [1, Theorem 2] (this result is also quoted as Theorem 20 in this work).

2 Rank and basis of WtMin4PC_T

In this section, we consider the WtMin4PC_T matrix for a tree T on n vertices. Bapat and Sivasubramanian [13] showed for a tree T that the rank of its unweighted Min4PC_T matrix is n . Recall that the rank of its usual distance matrix D is also n . Furthermore, the authors showed that both the matrices D and Min4PC_T have the same non-zero SNF values.

We first present a class $\mathfrak{B}$ of bases for the row space of both Min4PC_T and WtMin4PC_T . Additionally, we give a formula for the determinant of the submatrix of both Min4PC_T and WtMin4PC_T when we restrict these matrices to the indices in rows and columns from a basis $B \in \mathfrak{B}$.

Bapat and Sivasubramanian [13] showed that the determinant of $\text{Min4PC}_T[B, B]$ obtained by restricting Min4PC_T to the rows and columns contained in $B = \{e_1, \dots, e_{n-1}, g\}$, where $g \notin E(T)$ divides $2^{n-2}(n-1)$. We first ask when such a set B will be a basis for the row space of Min4PC_T ? We answer this by providing a formula for $\det \text{Min4PC}_T[B, B]$.

Theorem 5. (Bapat and Sivasubramanian [13, Lemma 3, Corollary 7, Theorem 9]) *Let T be an unweighted tree on n vertices and $E = \{e_1, \dots, e_{n-1}\}$ be its edge set. Then, $\text{rank}(\text{Min4PC}_T) = n$ and $\text{Min4PC}_T[E, E] = 2(J - I)$. Let the rows and columns of $\text{Min4PC}_T[E, E]$ be indexed sequentially by $e_1, e_2, \dots, e_{n-1}$. For $f = \{i, j\} \notin E$, define a vector $\mathbf{v}_f \in \mathbb{R}^{n-1}$ with its coordinates indexed by E as $\mathbf{v}_f^t = [\text{Min4PC}_T(f, e_1), \dots, \text{Min4PC}_T(f, e_{n-1})]$. Then, we have*

$$\mathbf{v}_f^t \text{Min4PC}_T[E, E]^{-1} \mathbf{v}_f = \frac{(n-1)(d_{i,j}-1)^2}{2(n-2)}.$$

Our next result describes a class $\mathfrak{B}$ of bases for the row space of Min4PC_T for a tree T . We also give the determinant of the submatrix of Min4PC_T when its rows and columns are restricted to elements of $B \in \mathfrak{B}$.

Theorem 6. Let T be a tree on n vertices and $E = \{e_1, \dots, e_{n-1}\}$ be the edge set. Take $f = \{i, j\} \notin E$ and let $B = E \cup \{f\}$. Let $\mathfrak{B}$ denote the collection of such B over all choices of $\{i, j\}$. Then, any $B \in \mathfrak{B}$ is a basis for the row space of Min4PC_T . Furthermore,

$$\det \text{Min4PC}_T[B, B] = (-1)^{n-1} 2^{n-2} (n-1)(d_{i,j}-1)^2.$$

Proof. It is easy to see that $\text{Min4PC}_T[B, B]$ can be written in block form as follows:

$$\text{Min4PC}_T[B, B] = \begin{bmatrix} \text{Min4PC}_T[E, E] & \mathbf{v}_f \\ \mathbf{v}_f^t & 0 \end{bmatrix}. \quad (2)$$

Since $\text{Min4PC}_T[E, E]$ is invertible, by the Schur complement formula for determinants (see Horn and Johnson [16]), we obtain

$$\begin{aligned} \det \text{Min4PC}_T[B, B] &= -(\mathbf{v}_f^t \text{Min4PC}_T[E, E]^{-1} \mathbf{v}_f) \det(\text{Min4PC}_T[E, E]) \\ &= -\frac{(n-1)(d_{i,j}-1)^2}{2(n-2)} \cdot (-1)^{n-2} 2^{n-1} (n-2) \\ &= (-1)^{n-1} 2^{n-2} (n-1)(d_{i,j}-1)^2. \end{aligned}$$

This completes the proof. $\square$

Remark 7. If we choose $f = \{i, j\}$ in Theorem 6 with $d_{i,j} = 2$, we obtain $\det \text{Min4PC}_T[B, B] = \det D$, where D is the usual distance matrix of T .

We generalize Theorem 6 to edge-weighted trees. Towards this, for a tree T on n vertices with edge weights α_i , define a function $\Gamma(T)$ as follows:

$$\Gamma(T) = -(n-2)(n-5) \text{Prod}_T(\alpha) + \sum_{i=1}^{n-1} \alpha_i^2 \left(\sum_{j=1, j \neq i}^{n-1} \left(\prod_{k=1, k \neq i, j}^{n-1} \alpha_k \right) \right). \quad (3)$$

Lemma 8. Let T be a tree on n vertices. Let $E = \{e_1, \dots, e_{n-1}\}$ be its edge set, and for $1 \leq i \leq n-1$, let edge e_i have weight α_i . Let $\text{WK} = \text{WtMin4PC}_T[E, E]$. Then, for $e_i \in E$, we have $\text{WK}(e_i, e_i) = 0$ and for $e_j \in E$ with $i \neq j$, we have $\text{WK}(e_i, e_j) = \alpha_i + \alpha_j$. Furthermore,

$$\det \text{WK} = (-1)^{n-2} 2^{n-3} \Gamma(T).$$

Proof. Let T be a tree on n vertices with edge set $E = \{e_1, \dots, e_{n-1}\}$ and let $\text{WK} = \text{WtMin4PC}_T[E, E]$. For an $(n-1) \times (n-1)$ matrix M and for $1 \leq i \leq n-1$, denote the i th column of M as $C_i(M)$. Note that WK can be expressed as a sum of two $(n-1) \times (n-1)$ matrices, i.e., we can write $\text{WK} = A + B$, where

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\alpha_1 & \alpha_1 & \dots & \alpha_1 \\ \alpha_2 & -\alpha_2 & \dots & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_{n-1} & \dots & \alpha_{n-1} \end{bmatrix}.$$

For $S \subseteq [n-1]$, let X_S be an $(n-1) \times (n-1)$ matrix defined by $C_i(X_S) = C_i(A)$ if $i \in S$ and $C_i(X_S) = C_i(B)$ if $i \notin S$. Note, by the multilinearity of the determinant, that

$$\det WK = \sum_{S \subseteq [n-1]} \det X_S. \quad (4)$$

If $|S| > 1$, then $\det X_S = 0$, as $a_j C_i(X_S) = a_i C_j(X_S)$ for some $i \neq j$. Thus, in (4), we only need to consider those $S \subseteq [n-1]$ with $|S| \leq 1$. Let J denote the $(n-1) \times (n-1)$ all ones matrix, and let I denote the $(n-1) \times (n-1)$ identity matrix. It is easy to see that

$$\det X_\emptyset = \det B = \text{Prod}_T(\alpha) \det(J - 2I) = (-1)^{n-2} 2^{n-2} (n-3) \text{Prod}_T(\alpha). \quad (5)$$

If $S = \{i\}$, note that $C_i(X_S) = \alpha_i \mathbf{1}$. Let $\text{Cofactor}(B)$ denote the cofactor matrix of B . We pull out a factor α_i and evaluate $\det X_S$ along its i th column. This gives us

$$\det X_{\{i\}} = \alpha_i (\mathbf{1}^t C_i(\text{Cofactor}(B))). \quad (6)$$

It is easy to check that

$$j \text{ th entry of } C_i(\text{Cofactor}(B)) = \begin{cases} (-1)^{n-2} (-2)^{n-3} \prod_{k \neq j} \alpha_k, & \text{when } i \neq j, \\ (-2)^{n-3} (n-4) \prod_{k \neq i} \alpha_k, & \text{when } i = j. \end{cases} \quad (7)$$

By (6) and (7), we have

$$\det X_{\{i\}} = (-2)^{n-3} \left((n-4) \prod_{k=1}^{n-1} \alpha_k - \alpha_i^2 \sum_{\substack{k=1 \\ k \neq i}}^{n-1} \prod_{\substack{j=1 \\ j \neq \{i, k\}}}^{n-1} \alpha_j \right). \quad (8)$$

Substituting (8) and (5) into (4) and using (3), we obtain

$$\det WK = \det X_\emptyset + \sum_{i=1}^{n-1} \det X_{\{i\}} = (-1)^{n-2} 2^{n-3} \Gamma(T).$$

This completes the proof. $\square$

Remark 9. By setting $\alpha_i = 1$ for $1 \leq i < n$, we obtain an unweighted tree T . In this case, we obtain $WK = \text{WtMin4PC}_T[E, E] = 2(J - I)$, and $\det WK = (-1)^{n-2} 2^{n-1} (n-2)$. Thus, Lemma 8 generalizes [13, Lemma 3].

Lemma 10. Let T be an edge-weighted tree on n vertices with edge set $E = \{e_1, \dots, e_{n-1}\}$. Let $WK = \text{WtMin4PC}_T[E, E]$. Define a vector $\mathbf{z} \in \mathbb{R}^{n-1}$, with its coordinates indexed by E . Denote by $z(e_i)$, the coordinate of $\mathbf{z}$ indexed by e_i , and define it as

$$z(e_i) = \left(\prod_{j=1, j \neq i}^{n-1} \alpha_j \right) \left(\sum_{j=1, j \neq i}^{n-1} \alpha_j - (n-4) \alpha_i \right).$$

Then, $WK\mathbf{z} = \Gamma(T)\mathbf{w}$, where $\mathbf{w}^t = [\alpha_1, \dots, \alpha_{n-1}]^t$.

Proof. The result follows from the fact that

$$\sum_{j=1}^{n-1} \left(\prod_{k \neq j} \alpha_k \sum_{k \neq j} \alpha_k \right) = \sum_{\substack{j=1 \\ |S|=n-3}}^{n-1} \sum_{S \subseteq [n-1] \setminus \{j\}} \left(\alpha_j^2 \prod_{k \in S} \alpha_k \right). \quad \square$$

For a tree T on n vertices, $\Gamma(T)$ as described in (3) can also be expressed as the following.

$$\Gamma(T) = \sum_{i=1}^{n-1} \left(\text{Sum}_T(\alpha) - \alpha_i \right) \left(\frac{\text{Prod}_T(\alpha)}{\alpha_i} \right) - (n-2)(n-5) \text{Prod}_T(\alpha). \quad (9)$$

Lemma 11. Let T be a tree on $n \geq 2$ vertices with edge set $E = \{e_1, \dots, e_{n-1}\}$. Then, $\Gamma(T) \neq 0$, and thus, $\text{WtMin4PC}_T[E, E]$ is invertible.

Proof. By (9), it is easy to see that

$$\Gamma(T) = \text{Sum}_T(\alpha) \text{Prod}_T(\alpha) \left(\sum_{i=1}^{n-1} \frac{1}{\alpha_i} \right) - (n-3)^2 \text{Prod}_T(\alpha).$$

As α_i for $1 \leq i \leq n-1$ are positive, by the AM $\geq$ HM inequality, we obtain

$$\text{Sum}_T(\alpha) \left(\sum_{i=1}^{n-1} \frac{1}{\alpha_i} \right) \geq (n-1)^2.$$

Hence, $\Gamma(T) > 0$ when $n > 2$. By Lemma 8, we have $\det \text{WtMin4PC}_T[E, E] \neq 0$. This completes the proof. $\square$

Let T be a tree on n vertices. Let i and j be two nonadjacent vertices in T and $e = \{i, j\}$. In our next result, we analyse the result of Lemma 10 carefully and find the unique solution x to the linear system of equations $\text{WK}x = \Gamma(T) \text{WtMin4PC}_T[E, e]$.

Lemma 12. Let T be a tree on n vertices with edge set $E = \{e_1, \dots, e_{n-1}\}$. For $1 \leq i \leq n-1$, let e_i have weight α_i . Let $\text{WK} = \text{WtMin4PC}_T[E, E]$ and $e = \{i, j\} \notin E$. Suppose that $z \in \mathbb{R}^{n-1}$ is the vector defined in Lemma 10. We define a vector $x_e \in \mathbb{R}^{n-1}$, with its coordinates indexed by E . For $f \in E$, denote the entry indexed by f in x_e as $x_e(f)$, and define it by

$$x_e(f) = \begin{cases} -(d_{i,j} - 1)z(f) + \Gamma(T), & \text{if } f \text{ is on the } ij\text{-path,} \\ -(d_{i,j} - 1)z(f), & \text{if } f \text{ is not on the } ij\text{-path.} \end{cases}$$

Then, we have $\text{WK}x_e = \Gamma(T) \text{WtMin4PC}_T[E, e]$.

Proof. Note for $i = 1, \dots, n-1$ that

$$\text{WtMin4PC}_T(e_i, e) = \begin{cases} w_{i,j} - \alpha_i, & \text{if } e_i \text{ is on the } ij\text{-path,} \\ w_{i,j} + \alpha_i, & \text{if } e_i \text{ is not on the } ij\text{-path.} \end{cases} \quad (10)$$

Recall $e = \{i, j\} \notin E$. Define $y_e \in \mathbb{R}^{n-1}$ as the vector with its r th coordinate being 1 if the edge e_r is on the ij -path and 0 otherwise. For $e_r \in E$, it is easy to note that

$$(\text{WK}y_e)(e_r) = \begin{cases} w_{i,j} + (d_{i,j} - 2)\alpha_r, & \text{if } e_r \text{ is on the } ij\text{-path,} \\ w_{i,j} + d_{i,j}\alpha_r, & \text{if } e_r \text{ is not on the } ij\text{-path.} \end{cases} \quad (11)$$

By Lemma 10, for $e_r \in E$, we have

$$(\text{WK}x_e)(e_r) = -(d_{i,j} - 1)\Gamma(T)\alpha_r + \Gamma(T)(\text{WK}y_e)(e_r). \quad (12)$$

Therefore, the result follows by substituting (11) into (12) and using the identity (10). $\square$

We will need the following result to determine both the rank of WtMin4PC_T and the determinant of $\text{WtMin4PC}_T[B, B]$, where B is a basis of the row space of WtMin4PC_T .

Lemma 13. Let T be a tree with n vertices with edge set E . For $1 \leq i \leq n-1$, let edge e_i have weight α_i . Let $a = \{i, j\} \notin E$ and $b = \{k, \ell\} \notin E$. Let z and x_a be the vectors defined Lemmas 10 and 12, respectively. Then,

$$w^t z = 2 \text{Sum}_T(\alpha) \text{Prod}_T(\alpha), \quad (13)$$

$$\mathbf{1}^t z = \Gamma(T) - 2(n-3) \text{Prod}_T(\alpha), \quad \text{and} \quad (14)$$

$$\text{WtMin4PC}_T[a, E]\mathbf{x}_b = \Gamma(T)\text{WtMin4PC}_T(a, b) + 2(d_{i,j} - 1)(d_{k,\ell} - 1)\text{Sum}_T(a)\text{Prod}_T(a), \quad (15)$$

where $\text{WtMin4PC}_T[a, E]$ is a submatrix of WtMin4PC_T restricted to the row indexed by a and the columns indexed by E .

Proof. It is easy to see that (13) follows from

$$\sum_{e_i \in E} \left(\sum_{e_j \in E \setminus \{e\}} \alpha_j - (n-4)\alpha_i \right) = 2\text{Sum}_T(a).$$

Furthermore, by (9), we have

$$\mathbf{1}^t \mathbf{z} = \sum_{j=1}^{n-1} \left(\prod_{k \neq j} \alpha_k \sum_{r \neq j} \alpha_r \right) - (n-4)(n-1)\text{Prod}_T(a) = \Gamma(T) - 2(n-3)\text{Prod}_T(a).$$

Let P_a and P_b denote the i, j -path and k, ℓ -path, respectively, in T . Define the vector $\tau_a \in \mathbb{R}^{n-1}$ with entries indexed by E as follows: $\tau_a(e_r) = -\alpha_r$ if $e_r \in P_a$ and $\tau_a(e_r) = \alpha_r$ if $e_r \notin P_a$. Define τ_b in an identical manner using P_b instead of P_a . Note that

$$\begin{aligned} \tau_a^t \mathbf{z} &= - \sum_{e_r \in P_a} \alpha_r \left[\prod_{j=1, j \neq r}^{n-1} \alpha_j \sum_{k=1, k \neq r} \alpha_k \right] + \sum_{e_s \notin P_a} \alpha_s \left[\prod_{j=1, j \neq s}^{n-1} \alpha_j \sum_{k=1, k \neq s} \alpha_k \right] + (n-4)\text{Prod}_T(a) \left[\sum_{e_s \notin P_a} \alpha_s - \sum_{e_r \in P_a} \alpha_r \right] \\ &= \text{Prod}_T(a) \left[\sum_{e_s \notin P_a} \sum_{j=1, j \neq s}^{n-1} \alpha_j - \sum_{e_r \in P_a} \sum_{k=1, k \neq r}^{n-1} \alpha_k \right] + (n-4)\text{Prod}_T(a) \left[\sum_{e_s \notin P_a} \alpha_s - \sum_{e_r \in P_a} \alpha_r \right] \\ &= \text{Sum}_T(a)\text{Prod}_T(a)(n-1-2d_{i,j}) - (n-3)\text{Prod}_T(a) \left[\sum_{e_s \notin P_a} \alpha_s - \sum_{e_r \in P_a} \alpha_r \right] \\ &= \text{Sum}_T(a)\text{Prod}_T(a)(n-1-2d_{i,j}) - (n-3)\text{Sum}_T(a)\text{Prod}_T(a) + 2(n-3)\text{Prod}_T(a)w_{i,j} \\ &= -2(d_{i,j} - 1)\text{Sum}_T(a)\text{Prod}_T(a) + 2(n-3)\text{Prod}_T(a)w_{i,j}. \end{aligned}$$

By (10), we have $\text{WtMin4PC}_T[E, a] = w_{i,j}\mathbf{1} + \tau_a$. Note that $\mathbf{x}_b = -(d_{k,\ell} - 1)\mathbf{z} + \Gamma(T)\mathbf{y}_b$, where $\mathbf{y}_b$ is the vector defined in the proof of Lemma 12. Clearly, $\mathbf{1}^t \mathbf{y}_b = d_{k,\ell}$. Thus, it follows that

$$\begin{aligned} \text{WtMin4PC}_T[a, E]\mathbf{x}_b &= (w_{i,j}\mathbf{1} + \tau_a^t)(-(d_{k,\ell} - 1)\mathbf{z} + \Gamma(T)\mathbf{y}_b) \\ &= -w_{i,j}(d_{k,\ell} - 1)\Gamma(T) + 2(n-3)w_{i,j}(d_{k,\ell} - 1)\text{Prod}_T(a) + \Gamma(T)d_{k,\ell}d_{i,j} + \Gamma(T)\tau_a^t \mathbf{y}_b \\ &\quad + 2(w_{i,j} - 1)(d_{k,\ell} - 1)\text{Sum}_T(a)\text{Prod}_T(a) - 2(n-3)(d_{k,\ell} - 1)w_{i,j}\text{Prod}_T(a) \\ &= \Gamma(T)(w_{i,j} + \tau_a^t \mathbf{y}_b) + 2(d_{i,j} - 1)(d_{k,\ell} - 1)\text{Sum}_T(a)\text{Prod}_T(a). \end{aligned}$$

Let $E(P_a)$ and $E(P_b)$ be the set of edges in the paths P_a and P_b , respectively. Note that

$$\tau_a^t \mathbf{y}_b = \sum_{e_r \in E(P_b) \setminus E(P_a)} \alpha_r - \sum_{e_s \in E(P_a) \cap E(P_b)} \alpha_s.$$

It follows that

$$w_{i,j} + \tau_a^t \mathbf{y}_b = \sum_{e_r \in E(P_a) \setminus E(P_b)} \alpha_r + \sum_{e_s \in E(P_b) \setminus E(P_a)} \alpha_s. \quad (16)$$

Furthermore, note that

$$\text{WtMin4PC}_T(a, b) = \begin{cases} w_{i,j} + w_{k,\ell}, & \text{if } E(P_a) \cap E(P_b) = \emptyset, \\ w_{i,j} + w_{k,\ell} - 2 \sum_{e_r \in E(P_a) \cap E(P_b)} \alpha_r, & \text{if } E(P_a) \cap E(P_b) \neq \emptyset. \end{cases} \quad (17)$$

By (16) and (17), we have $\text{WtMin4PC}_T(a, b) = \tau_a^t \mathbf{y}_b$, completing the proof. $\square$

Our next result generalizes [13, Theorem 9], and it shows that the rank of WtMin4PC_T is n , showing that it is independent of the structure of T and its edge weights.

Theorem 14. *Let T be a tree on $n > 2$ vertices with edge set E . Then, the rank of WtMin4PC_T is n .*

Proof. Let E^c be the set of edges in T^c and define the three matrices $\text{WK} = \text{WtMin4PC}_T[E, E]$, $\text{WK}_1 = \text{WtMin4PC}_T[E, E^c]$ and $\text{WK}_2 = \text{WtMin4PC}_T[E^c, E^c]$. Note that WtMin4PC_T can be written in block form as

$$\text{WtMin4PC}_T = \begin{bmatrix} \text{WK} & \text{WK}_1 \\ \text{WK}_1^t & \text{WK}_2 \end{bmatrix}.$$

Lemma 11 shows that $\det \text{WK} \neq 0$. Applying the Guttman rank additivity formula¹ [19, Equation (0.9.2)], we obtain

$$\text{rank}(\text{WtMin4PC}_T) = \text{rank}(\text{WK}) + \text{rank}(\text{WK}_2 - \text{WK}_1^t \text{WK}^{-1} \text{WK}_1). \quad (18)$$

Let $a = \{i, j\} \notin E$ and $b = \{k, \ell\} \notin E$. Let $\text{WtMin4PC}_T[E, a]$ be the submatrix of WtMin4PC_T restricted to the rows indexed by E and the column indexed by a . Hence, $\text{WtMin4PC}_T[E, a]$ is an $(n-1)$ -dimensional column vector. In an identical manner, define $\text{WtMin4PC}_T[E, b]$. Denote by $\text{Col}_a(\text{WK}_1) = \text{WtMin4PC}_T[E, a]$ and $\text{Col}_b(\text{WK}_1) = \text{WtMin4PC}_T[E, b]$ the two columns of WK_1 indexed by a and b respectively. Let $\mathbf{x}_a$ and $\mathbf{x}_b$ be vectors defined in Lemma 12 that are indexed by a and b respectively. Using Lemma 12, it is simple to check that $\text{WK}^{-1} \text{Col}_a(\text{WK}_1) = \frac{1}{\Gamma(T)} \mathbf{x}_a$. Therefore, by Lemma 13,

$$\text{Col}_b(\text{WK}_1)^t \text{WK}^{-1} \text{Col}_a(\text{WK}_1) = \text{WtMin4PC}_T(b, a) + (d_{i,j} - 1)(d_{k,\ell} - 1)X, \quad (19)$$

where $X = \text{Sum}_T(a) \text{Prod}_T(a) / \Gamma(T)$.

Define a vector $\boldsymbol{\mu} \in \mathbb{R}^{\binom{n}{2} - (n-1)}$ whose coordinates are indexed by the element of E^c as follows: for $e = \{s, t\} \notin E(T)$ define $\mu(e) = (d_{s,t} - 1)\sqrt{X}$. It is easy to note that

$$\text{WK}_2 - \text{WK}_1^t \text{WK}^{-1} \text{WK}_1 = \boldsymbol{\mu} \boldsymbol{\mu}^t.$$

Using (18), our proof is complete. $\square$

Corollary 15. *We need (19) when $a = b$. We note that as follows:*

$$\text{Col}_a(\text{WK}_1)^t \text{WK}^{-1} \text{Col}_a(\text{WK}_1) = (d_{i,j} - 1)^2 X.$$

In our last main result of this section, we describe a class $\mathfrak{B}$ of bases for the row space of WtMin4PC_T and also give a formula for the determinant of the matrix obtained by restricting WtMin4PC_T to the elements with both row and column indices from $B \in \mathfrak{B}$.

Proof of Theorem 3. The result follows by first applying the Schur complement formula to the matrix $\text{WtMin4PC}_T[B, B]$ as done in (2), and then using Corollary 15 along with Lemmas 8 and 12. $\square$

3 Rank of WtMax4PC_T

In [1, Theorem 1], Azimi et al. showed for a tree T with n vertices and p leaves that $\text{rank}(\text{Max4PC}_T) = 2(n - p)$. In the course of their proof, they designed an algorithm that outputs a class $\mathfrak{B}$ of bases for the row space of Max4PC_T . In this section, we extend this result to weighted trees. This shows that the rank of WtMax4PC_T is independent of the edge weights. Thus, our following result generalizes [1, Lemma 3] to weighted trees.

¹ The Guttman rank additivity formula is also known as the Schur complement formula for rank.

Lemma 16. Let T be a tree on n vertices. Let $n-1, n \in V(T)$ be such that vertex n is a leaf in T with unique neighbour $n-1$. If u is a vertex of T other than n and $n-1$, then for each pair $\{i, j\}$ of vertices, we have

$$\text{WtMax4PC}_T(\{u, n\}, \{i, j\}) = \text{WtMax4PC}_T(\{u, n-1\}, \{i, j\}) + w_{n-1, n}.$$

Proof. Denote $w_{n-1, n} = \alpha$ for brevity, and recall $u \neq n-1, n$. Clearly, if $v \in V(T)$ with $v \neq n$, then the v - n path must include the vertex $n-1$. We thus have

$$w_{v, n} = w_{v, n-1} + \alpha \quad \text{and} \quad w_{u, n} = w_{u, n-1} + \alpha. \quad (20)$$

For each pair $\{i, j\}$ of vertices in T , by the definition of the matrix WtMax4PC_T , we obtain

$$\text{WtMax4PC}_T(\{u, n\}, \{i, j\}) = \max\{w_{u, n-1} + w_{i, j} + \alpha, \quad w_{u, i} + w_{n, j}, \quad w_{u, j} + w_{n, i}\}. \quad (21)$$

We split our proof into two cases: when both i and j are different from n and when either $i = n$ or $j = n$.

Case 1: This is the case of $i \neq n$ and $j \neq n$. Substituting (20) into (21), we obtain

$$\begin{aligned} \text{WtMax4PC}_T(\{u, n\}, \{i, j\}) &= \max\{w_{u, n-1} + w_{i, j} + \alpha, \quad w_{u, i} + w_{n-1, j} + \alpha, \quad w_{u, j} + w_{n-1, i} + \alpha\} \\ &= \text{WtMax4PC}_T(\{u, n-1\}, \{i, j\}) + \alpha. \end{aligned}$$

Our proof is complete in this case.

Case 2: This is the case of $i = n$ or $j = n$. Without loss of generality, assume $j = n$, and thus, $i \leq n-1$. By the triangle inequality, we have

$$w_{u, n-1} + w_{i, n-1} \geq w_{u, i} \quad \text{and} \quad w_{u, n} + w_{i, n} \geq w_{u, i} + 2\alpha. \quad (22)$$

By (22), the entry of WtMax4PC_T indexed by the row $\{u, n\}$ and the column $\{i, n\}$ is

$$\text{WtMax4PC}_T(\{u, n\}, \{i, n\}) = \max\{w_{u, n} + w_{i, n}, \quad w_{u, i}\} = w_{u, n} + w_{i, n}.$$

Furthermore, if $k = \text{WtMax4PC}_T(\{u, n-1\}, \{i, n\})$, then we have

$$\begin{aligned} k &= \max\{w_{u, n-1} + w_{i, n}, \quad w_{u, i} + \alpha, \quad w_{u, n} + w_{n-1, i}\} \\ &= \max\{w_{u, n} + w_{i, n} - \alpha, \quad w_{u, i} + \alpha, \quad w_{u, n} + w_{n-1, i} - \alpha\} \quad [\text{by (20)}] \\ &= w_{u, n} + w_{i, n} - \alpha \quad [\text{by (20) and (22)}] \\ &= \text{WtMax4PC}_T(\{u, n\}, \{i, n\}) - \alpha. \end{aligned}$$

Thus, we have proved the result for the second case. Our proof is complete. $\square$

Our next lemma extends [1, Lemma 4] to weighted trees.

Lemma 17. Let T be a tree on n vertices. Let $y, z \in V(T)$ be such that y is a leaf and z is the quasi-pendant adjacent to y . Let $u \in V(T)$ be a neighbour of z other than y and B_u be the connected component of $T - z$ that contains the vertex u (Figure 1). Then,

$$\text{WtMax4PC}_T(\{y, z\}, \{i, j\}) = \begin{cases} \text{WtMax4PC}_T(\{u, z\}, \{i, j\}) + w_{y, z} + w_{u, z}, & \text{if } i, j \in B_u, \\ \text{WtMax4PC}_T(\{u, z\}, \{i, j\}) + w_{y, z} - w_{u, z}, & \text{otherwise.} \end{cases}$$

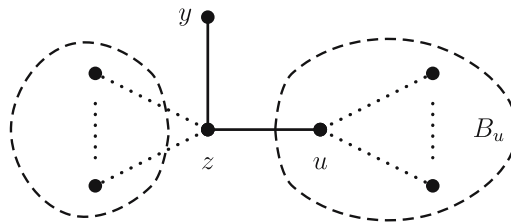


Figure 1: Illustrating the setup for Lemma 17.

Proof. Let i and j be two distinct vertices of T with $j \in B_u$. Using triangle inequality, it is simple to see that

$$w_{i,j} \leq w_{u,i} + w_{z,j} - w_{u,z}. \quad (23)$$

We first assume $i, j \in B_u \cap V(T)$. Clearly, $w_{y,v} = w_{u,v} + w_{y,z} + w_{u,z}$ for each $v \in B_u$. Therefore, we have

$$\begin{aligned} \text{WtMax4PC}_T(\{y, z\}, \{i, j\}) &= \max\{w_{y,z} + w_{i,j}, \quad w_{y,i} + w_{z,j}, \quad w_{y,j} + w_{z,i}\} \\ &= \max\{w_{y,z} + w_{i,j}, \quad w_{u,i} + w_{z,j} + w_{y,z} + w_{u,z}, \quad w_{u,j} + w_{z,i} + w_{y,z} + w_{u,z}\} \\ &= \max\{w_{i,j} - w_{u,z}, w_{u,i} + w_{z,j}, \quad w_{u,j} + w_{z,i}\} + w_{y,z} + w_{u,z} \\ &= \max\{w_{u,z} + w_{i,j}, \quad w_{u,i} + w_{z,j}, \quad w_{u,j} + w_{z,i}\} + w_{y,z} + w_{u,z} \quad [\text{by (23)}] \\ &= \text{WtMax4PC}_T(\{u, z\}, \{i, j\}) + w_{y,z} + w_{u,z}. \end{aligned}$$

Now, consider the case when $i \notin B_u$ and $j \in V(T)$. First, note that if $i = y$ and $j \in V(T) - \{y\}$, then $w_{y,j} + w_{u,z} = w_{z,j} + w_{y,u}$. Combining these with $w_{z,j} \leq w_{y,j}$ gives

$$\begin{aligned} \text{WtMax4PC}_T(\{y, z\}, \{y, j\}) &= \max\{w_{y,z} + w_{y,j}, \quad w_{z,j}, \quad w_{y,j} + w_{y,z}\} \\ &= \max\{w_{u,z} + w_{y,j} + w_{y,z} - w_{u,z}, \quad w_{y,j} + w_{z,u} + w_{y,z} - w_{u,z}\} \\ &= \max\{w_{u,z} + w_{y,j}, \quad w_{y,u} + w_{z,j}, \quad w_{y,j} + w_{z,u}\} + w_{y,z} - w_{u,z} \\ &= \text{WtMax4PC}_T(\{u, z\}, \{y, j\}) + w_{y,z} - w_{u,z}. \end{aligned}$$

We split the remaining part of our proof into two cases when $i \notin B_u \cup \{y\}$ with $j \in B_u$ and when $i \notin B_u \cup \{y\}$ with $j \notin B_u$.

Case 1: This is the case of $i \notin B_u \cup \{y\}$ and $j \in B_u$. In this case, it is easy to see that $w_{i,j} = w_{i,z} + w_{z,u} + w_{u,j}$, and so, we obtain

$$w_{u,i} + w_{z,j} = w_{i,j} + w_{u,z} = w_{u,j} + w_{z,i} + 2w_{u,z} \geq w_{i,j} - w_{u,z}. \quad (24)$$

Therefore,

$$\begin{aligned} \text{WtMax4PC}_T(\{y, z\}, \{i, j\}) &= \max\{w_{y,z} + w_{i,j}, \quad w_{y,i} + w_{z,j}, \quad w_{y,j} + w_{z,i}\} \quad [\text{by } w_{y,i} + w_{u,z} = w_{u,i} + w_{y,z}] \\ &= \max\{w_{y,z} + w_{i,j}, \quad w_{u,i} + w_{z,j} + w_{y,z} - w_{u,z}, \quad w_{u,j} + w_{z,i} + w_{y,z} + w_{u,z}\} \\ &= \max\{w_{u,z} + w_{i,j}, \quad w_{u,i} + w_{z,j}, \quad w_{u,j} + w_{z,i}\} + w_{y,z} - w_{u,z} \quad [\text{by (24)}] \\ &= \text{WtMax4PC}_T(\{u, z\}, \{i, j\}) + w_{y,z} - w_{u,z}. \end{aligned}$$

Case 2: This is the case of $i \notin B_u \cup \{y\}$ and $j \notin B_u$. In this case, note that if $j \neq y$, then $w_{y,i} + w_{u,z} = w_{u,i} + w_{y,z}$ and $w_{y,j} + w_{u,z} = w_{u,j} + w_{y,z}$. Using these, it is easy to see that

$$\begin{aligned} \text{WtMax4PC}_T(\{y, z\}, \{i, j\}) &= \max\{w_{y,z} + w_{i,j}, \quad w_{y,i} + w_{z,j}, \quad w_{y,j} + w_{z,i}\} \\ &= \max\{w_{u,z} + w_{i,j} + w_{y,z} - w_{u,z}, \quad w_{u,i} + w_{z,j} + w_{y,z} - w_{u,z}, \quad w_{u,j} + w_{z,i} + w_{y,z} - w_{u,z}\} \\ &= \text{WtMax4PC}_T(\{u, z\}, \{i, j\}) + w_{y,z} - w_{u,z}. \end{aligned}$$

Finally, if we have $j = y$, then $i \notin B_u \cup \{y\}$. In this case, one can check that $w_{y,i} = w_{u,i} + w_{y,z} - w_{u,z}$. Using this, we obtain

$$\begin{aligned} \text{WtMax4PC}_T(\{y, z\}, \{i, y\}) &= \max\{w_{y,z} + w_{i,y}, \quad w_{y,i} + w_{y,z}, \quad w_{z,i}\} \\ &= \max\{w_{u,z} + w_{i,y} + w_{y,z} - w_{u,z}, \quad w_{y,i} + w_{y,z} + w_{y,z} - w_{u,z}\} \quad [\text{as } w_{z,i} + w_{y,z} = w_{y,i}] \\ &= \max\{w_{u,z} + w_{i,y}, \quad w_{z,i} + w_{u,y}, \quad w_{u,i} + w_{y,z}\} + w_{y,z} - w_{u,z} \\ &= \text{WtMax4PC}_T(\{u, z\}, \{i, y\}) + w_{y,z} - w_{u,z}. \end{aligned}$$

This completes the proof. □

The next result is our main result for this section.

Theorem 18. Let T be a tree on $n \geq 3$ vertices with p leaves. Then, $\text{rank}(\text{WtMax4PC}_T) = 2(n - p)$.

Proof. We induct on n , the number of vertices in T . Our base case is when $n = 3$. The only possible tree is P_3 , the path tree on three vertices. Let the two edges of P_3 have weights a_1 and a_2 , respectively. Then, it can be verified that

$$\text{WtMax4PC}_{P_3} = \begin{bmatrix} 2a_1 & 2a_1 + a_2 & a_1 + a_2 \\ 2a_1 + a_2 & 2a_1 + 2a_2 & a_1 + 2a_2 \\ a_1 + a_2 & a_1 + 2a_2 & 2a_2 \end{bmatrix}.$$

It is easy to verify that $\text{rank}(\text{WtMax4PC}_{P_3}) = 2$. Thus, our base case is done.

Thus, assume that the result is true for all trees with $n - 1$ vertices. Let T be a tree on n vertices. Without loss of any generality, we assume that n is a leaf adjacent to $n - 1$. Let $\hat{T} = T - \{n\}$ be the tree obtained from T by deleting the vertex n from T . We divide the proof into two cases based on the degree of vertex $n - 1$ in T .

Case 1: This is the case when there exists a quasi pendant vertex $n - 1$ with degree two. Let $n - 2$ be the neighbour of $n - 1$ other than n . Let $w_{n-1,n} = a_{n-1}$ and $w_{n-2,n-1} = a_{n-2}$.

Let $\mathbb{V}_n$ be the collection of 2-sized subsets of $[n] = \{1, 2, \dots, n\}$ and $\mathbb{U}_{n-1} = \{\{i, n\} | i \in [n-1]\}$. For listing the rows and columns of WtMax4PC_T , we order the elements of $\mathbb{V}_n$ as $\mathbb{V}_n = (\mathbb{V}_{n-1}, \mathbb{U}_{n-1})$. For $R, C \subseteq \mathbb{V}_n$, let $\text{WtMax4PC}[R, C]$ denote the submatrix of WtMax4PC_T induced on the rows in R and the columns in C . With this notation, let $\text{WtMax4PC}_{12} = \text{WtMax4PC}_T[\mathbb{V}_{n-1}, \mathbb{U}_{n-1}]$ and $\text{WtMax4PC}_{22} = \text{WtMax4PC}_T[\mathbb{U}_{n-1}, \mathbb{U}_{n-1}]$. Then, WtMax4PC_T can clearly be written in the following partitioned form:

$$\text{WtMax4PC}_T = \begin{bmatrix} \text{WtMax4PC}_{\hat{T}} & \text{WtMax4PC}_{12} \\ \text{WtMax4PC}_{12}^t & \text{WtMax4PC}_{22} \end{bmatrix}.$$

In the matrix WtMax4PC_T , perform the elementary row operation $\text{Row}_{u,n} = \text{Row}_{u,n} - \text{Row}_{u,n-1}$ and the column operation $\text{Col}_{u,n} = \text{Col}_{u,n} - \text{Col}_{u,n-1}$ for $1 \leq u \leq n - 1$. Denoting equivalent matrices by $A \sim B$, Lemma 16 gives us

$$\begin{aligned} \text{WtMax4PC}_T &\sim \left[\begin{array}{c|c|c} \text{WtMax4PC}_{\hat{T}} & a_{n-1} \mathbf{J}_{\binom{n-1}{2} \times (n-2)} & \mathbf{u} \\ \hline a_{n-1} \mathbf{J}_{(n-2) \times \binom{n-1}{2}} & \mathbf{0}_{(n-2) \times (n-2)} & a_{n-1} \mathbf{1} \\ \hline \mathbf{u}^t & a_{n-1} \mathbf{1}^t & 2a_{n-1} \end{array} \right] \\ \text{WtMax4PC}_T &\sim \left[\begin{array}{c|c|c} \text{WtMax4PC}_{\hat{T}} & \begin{matrix} 0 & \cdots & 0 & a_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-1} \end{matrix} & \mathbf{u} \\ \hline \begin{matrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ a_{n-1} & \cdots & a_{n-1} \end{matrix} & \mathbf{0} & \begin{matrix} 0 \\ \vdots \\ 0 \\ a_{n-1} \end{matrix} \\ \hline \mathbf{u}^t & \begin{matrix} 0 & \cdots & 0 & a_{n-1} \end{matrix} & 2a_{n-1} \end{array} \right], \end{aligned}$$

where in the second step above, we perform the row operations $\text{Row}_{u,n} = \text{Row}_{u,n} - \text{Row}_{u,n-1}$ and identical column operations $\text{Col}_{u,n} = \text{Col}_{u,n} - \text{Col}_{u,n-1}$ for $1 \leq u \leq n - 2$. Recall that the vertex $n - 1$ is a leaf in the tree $\hat{T}$. Denote the row of a matrix M indexed by r as $M[r, :]$. Thus, by Lemma 16, for all $v \in \hat{T} - \{n - 1, n - 2\}$, we obtain

$$\text{WtMax4PC}_{\hat{T}}(\{n - 1, v\}, :) = \text{WtMax4PC}_{\hat{T}}(\{n - 2, v\}, :) + a_{n-2} \mathbf{1}^t. \quad (25)$$

Let $n - 3$ be a neighbour of $n - 2$ in T and let $w_{n-3,n-2} = a_{n-3}$. By the definition of WtMax4PC_T , it is simple to see that

$$\text{WtMax4PC}_T(\{n - 1, n\}, \{n - 1, n - 3\}) = a_{n-1} + a_{n-2} + a_{n-3} \quad \text{and} \quad (26)$$

$$\text{WtMax4PC}_T(\{n - 1, n\}, \{n - 2, n - 3\}) = a_{n-1} + 2a_{n-2} + a_{n-3}. \quad (27)$$

We perform the row operation $\text{Row}_{n-2,n} = \alpha_{n-2}\text{Row}_{n-2,n} - \alpha_{n-1}(\text{Row}_{n-1,n-3} - \text{Row}_{n-2,n-3})$ and an identical column operation $\text{Col}_{n-2,n} = \alpha_{n-2}\text{Col}_{n-2,n} - \alpha_{n-1}(\text{Col}_{n-1,n-3} - \text{Col}_{n-2,n-3})$. Using (25)–(27) and the above row and column operation, we obtain the following matrix that is row and column equivalent to WtMax4PC_T .

$$\begin{aligned} \text{WtMax4PC}_T &\sim \left[\begin{array}{c|ccc|c} \text{WtMax4PC}_{\hat{T}} & 0 & \cdots & 0 & 0 & \mathbf{u} \\ \hline 0 & \cdots & 0 & & & 0 \\ \vdots & \ddots & \vdots & & & \vdots \\ 0 & \cdots & 0 & & & 0 \\ 0 & \cdots & 0 & & & 2\alpha_{n-1}\alpha_{n-2} \\ \hline \mathbf{u}^t & 0 & \cdots & 0 & 2\alpha_{n-1}\alpha_{n-2} & 2\alpha_{n-1} \end{array} \right] \\ &\sim \left[\begin{array}{c|cc|c} \text{WtMax4PC}_{\hat{T}} & \mathbf{0} & & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & & 2\alpha_{n-1}\alpha_{n-2} \\ \hline \mathbf{0} & \mathbf{0} & 2\alpha_{n-1}\alpha_{n-2} & 2\alpha_{n-2} \end{array} \right] \\ &\sim \left[\begin{array}{c|cc|c} \text{WtMax4PC}_{\hat{T}} & \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -2\alpha_{n-1}^2\alpha_{n-2} & 0 \\ \hline \mathbf{0} & \mathbf{0} & 0 & 2\alpha_{n-2} \end{array} \right]. \end{aligned}$$

This completes the proof of Case 1.

Case 2: This is the case when any quasi-pendant vertex in T has degree at least three. Let $[v_1, v_2, \dots, v_k]$ be a maximum length path in T . Hence, its length equals the diameter of T . Clearly, v_1 is a pendant vertex and v_2 is a quasi-pendant vertex in T . Since the degree of v_2 in T is at least three, there is another leaf y with y being adjacent to v_2 and $y \neq v_1$. By relabelling, we assume that n and $n-2$ are two leaves in T that are both adjacent to $n-1$. Let $w_{n-1,n} = \alpha_{n-1}$ and $w_{n-2,n-1} = \alpha_{n-2}$. When $1 \leq i < j \leq n$ and $u \notin \{n, n-1\}$, Lemma 16 gives

$$\text{WtMax4PC}_T(\{u, n\}, \{i, j\}) = \text{WtMax4PC}_T(\{u, n-2\}, \{i, j\}) + \alpha_{n-1} \quad \text{and} \quad (28)$$

$$\text{WtMax4PC}_T(\{u, n-1\}, \{i, j\}) = \text{WtMax4PC}_T(\{u, n-2\}, \{i, j\}) + \alpha_{n-2}. \quad (29)$$

Let vertex $n-3$ be adjacent to $n-2$ in T , and let $w_{n-3,n-2} = \alpha_{n-3}$. Let B_{n-3} be the connected component of $T - \{n-2\}$ that contains the vertex $n-3$. By Lemma 17, we have

$$\text{WtMax4PC}_T(\{n-2, n\}, \{i, j\}) = \begin{cases} \text{WtMax4PC}_T(\{n-3, n-2\}, \{i, j\}) + \alpha_{n-1} + \alpha_{n-3}, & \text{if } i, j \in B_{n-3} \\ \text{WtMax4PC}_T(\{n-3, n-2\}, \{i, j\}) + \alpha_{n-1} - \alpha_{n-3}, & \text{otherwise.} \end{cases} \quad (30)$$

$$\text{WtMax4PC}_T(\{n-2, n-1\}, \{i, j\}) = \begin{cases} \text{WtMax4PC}_T(\{n-3, n-2\}, \{i, j\}) + \alpha_{n-2} + \alpha_{n-3}, & \text{if } i, j \in B_{n-3} \\ \text{WtMax4PC}_T(\{n-3, n-2\}, \{i, j\}) + \alpha_{n-2} - \alpha_{n-3}, & \text{otherwise.} \end{cases} \quad (31)$$

Using (28) and (29), we obtain $\text{Row}_{n-3,n-1} - \text{Row}_{n-3,n-2} = \alpha_{n-2}\mathbf{1}^t$ and combining this with (30) and (31), we obtain $(\text{Row}_{n-2,n} - \text{Row}_{n-3,n-2}) - (\text{Row}_{n-2,n-1} - \text{Row}_{n-3,n-2}) = (\alpha_{n-1} - \alpha_{n-2})\mathbf{1}^t$. Furthermore, for all $i \in V(T)$ with $i \notin \{n, n-2\}$, by (28)–(31), it is simple to check the following:

$$\begin{aligned} \alpha_{n-2}\text{Row}_{i,n} &= \alpha_{n-2}(\text{Row}_{i,n} - \text{Row}_{i,n-2}) - \alpha_{n-1}(\text{Row}_{n-3,n-1} + \text{Row}_{n-3,n-2}) \quad \text{and} \\ \alpha_{n-2}\text{Col}_{i,n} &= \alpha_{n-2}(\text{Col}_{i,n} - \text{Col}_{i,n-2}) - \alpha_{n-1}(\text{Col}_{n-3,n-1} + \text{Col}_{n-3,n-2}). \end{aligned}$$

Furthermore, note that

$$\begin{aligned} \alpha_{n-2}\text{Row}_{n-2,n} &= \alpha_{n-2}(\text{Row}_{n-2,n} - \text{Row}_{n-3,n-2}) - \alpha_{n-2}(\text{Row}_{n-2,n-1} - \text{Row}_{n-3,n-2}) \\ &\quad - (\alpha_{n-1} - \alpha_{n-2})(\text{Row}_{n-3,n-1} - \text{Row}_{n-3,n-2}) \quad \text{and} \\ \alpha_{n-2}\text{Col}_{n-2,n} &= \alpha_{n-2}(\text{Col}_{n-2,n} - \text{Col}_{n-3,n-2}) - \alpha_{n-2}(\text{Col}_{n-2,n-1} - \text{Col}_{n-3,n-2}) \\ &\quad - (\alpha_{n-1} - \alpha_{n-2})(\text{Col}_{n-3,n-1} - \text{Col}_{n-3,n-2}). \end{aligned}$$

Using the aforementioned row operations and the identical column operations, it is easy to see that the matrix WtMax4PC_T is similar to the following block matrix:

$$\begin{bmatrix} \text{WtMax4PC}_{\hat{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Thus, Case 2 is proved. This completes our proof. $\square$

The following result is an easy consequence of Theorem 18.

Corollary 19. *Let T be a tree on n vertices with $n > 3$. Suppose there exist two leaves u and v adjacent to the same vertex. Then, we have*

$$\text{rank}(\text{WtMax4PC}_T) = \text{rank}(\text{WtMax4PC}_{T-u}) = \text{rank}(\text{WtMax4PC}_{T-v}).$$

4 Basis for the row space of WtMax4PC_T and determinant

Let T be a weighted tree on n vertices with α_i being the weight on edge e_i for $1 \leq i \leq n-1$. Let WtMax4PC_T be its weighted maximum-4PC matrix. When $\alpha_i = 1$ for $1 \leq i \leq n-1$, in [1, Lemma 9], Ali et al. constructed a class $\mathfrak{B}$ of bases for the row space of Max4PC_T . They also gave a formula for the determinant of $\text{Max4PC}_T[B, B]$ for all $B \in \mathfrak{B}$. As we will compare our weighted version with the unweighted version, we quote their unweighted result below.

Theorem 20. [1, Lemma 9 and Theorem 11] *Let T be a tree on n vertices, and let Max4PC_T be its maximum-4PC matrix. Let $\mathfrak{B}$ be the class of bases given in [1, Lemma 9] for the row space of Max4PC_T . If p is the number of leaves in T , then for all $B \in \mathfrak{B}$, $|B| = 2(n-p)$ and $\det \text{Max4PC}_T[B, B] = (-1)^{n-p} 2^{2(n-p-1)}$.*

For $1 \leq i \leq n-1$, if $\alpha_i > 0$, we first show that each $B \in \mathfrak{B}$ from Theorem 20, is a basis for the row space of WtMax4PC_T . For proving Theorem 20, an algorithm inspired by depth first search was given. The output of this algorithm was shown to be a basis for all trees T other than the star trees. Star trees were handled separately. As there are some choices in listing adjacent blocks, the output of our algorithm is not unique, and so we obtain a class $\mathfrak{B}$ with all $B \in \mathfrak{B}$ being a basis. Thus, $\mathfrak{B}$ is a class of bases that our algorithm outputs.

We move to our edge-weighted version and again separate star trees from the rest. Let S_n denote the star tree on n vertices. We start by giving an identical class of bases for the row space of WtMax4PC_{S_n} . We denote the degree of the vertex u in T as $\deg(u)$.

Theorem 21. *Let S_n be the weighted star tree on vertex set $[n]$ and let $\deg(1) = n-1$. For $1 < i \leq n$, let α_i be the weight on the edge $\{1, i\}$. Then, $\text{rank}(\text{WtMax4PC}_{S_n}) = 2$ and for i, j with $1 < i < j \leq n$, the set of rows indexed by $\{1, i\}$ and $\{i, j\}$ forms a basis B for the row space of WtMax4PC_{S_n} . Denote by $\mathfrak{B}$, the class of such B over the choices of i, j . Furthermore, $\det \text{WtMax4PC}_{S_n}[B, B] = -\alpha_i^2$ for all $B \in \mathfrak{B}$.*

Proof. From Corollary 19, it follows when $n \geq 3$ that $\text{rank}(\text{WtMax4PC}_{S_n}) = \text{rank}(\text{WtMax4PC}_{S_3}) = 2$. To see this, note that

$$\text{WtMax4PC}_{S_3} = \begin{matrix} & \{1, 2\} & \{1, 3\} & \{2, 3\} \\ \begin{matrix} \{1, 2\} \\ \{1, 3\} \\ \{2, 3\} \end{matrix} & \begin{bmatrix} 2\alpha_2 & \alpha_2 + \alpha_3 & 2\alpha_2 + \alpha_3 \\ \alpha_2 + \alpha_3 & 2\alpha_3 & \alpha_2 + 2\alpha_3 \\ 2\alpha_2 + \alpha_3 & \alpha_2 + 2\alpha_3 & 2\alpha_2 + 2\alpha_3 \end{bmatrix} \end{matrix},$$

and $\alpha_2 \times (\text{Row}_{\{2,3\}} - \text{Row}_{\{1,2\}}) = \alpha_3 \times (\text{Row}_{\{2,3\}} - \text{Row}_{\{1,3\}})$, where $\text{Row}_{\{i,j\}}$ is the row of WtMax4PC_{S_3} indexed by the pair $\{i, j\}$. When the tree is S_n , one can verify that $\alpha_i \times (\text{Row}_{\{i,j\}} - \text{Row}_{\{1,i\}}) = \alpha_j (\text{Row}_{\{i,j\}} - \text{Row}_{\{1,j\}})$ for i, j with

$1 < i < j \leq n$. Furthermore, if $B = \{\{1, i\}, \{i, j\}\}$ for some $i, j \in S_n$, then $\det \text{WtMax4PC}_{S_n}[B, B] = -\alpha_j^2 \neq 0$ whenever $\alpha_j \neq 0$. Thus, the rows indexed by $\{1, i\}$ and $\{i, j\}$ form a basis for the row space of WtMax4PC_{S_n} . This completes the proof. $\square$

One can ask when the rows indexed by $\{1, i\}$ and $\{j, k\}$ would be linearly independent.

Remark 22. For $1 < i \leq j < k \leq n$, the rows indexed by $\{1, i\}$ and $\{j, k\}$ are linearly independent if $\alpha_i \neq \alpha_j + \alpha_k$, since each 2×2 minor with rows and columns indexed by $\{1, i\}$ and $\{j, k\}$ has a factor $(\alpha_j + \alpha_k - \alpha_i)$.

Setting $\alpha_i = 1$ for $1 \leq i \leq n - 1$, it is easy to see that [1, Lemma 7] is a direct consequence of Theorem 21.

Corollary 23. [1, Lemma 7] Let S_n be the star tree on n vertices with $\deg(1) = n - 1$. Then, $\text{rank}(\text{Max4PC}_{S_n}) = 2$ and for $1 < i < j \leq n$, the rows indexed by $\{1, i\}$ and $\{i, j\}$ are linearly independent. Let $\mathfrak{B}$ the class of bases defined in Theorem 21. Then, for each $B \in \mathfrak{B}$, we have $\det \text{Max4PC}_{S_n}[B, B] = -1$.

We introduce a term “pendant edge” for our next result. In a tree T , an edge is known as a *pendant edge* if one of its end points is a leaf (i.e., one of its end points is a pendant vertex). If both end points of an edge e have degree strictly greater than one, then such an e will be called as a *non-pendant edge*. In our next result, we show that the set B output by an algorithm described in [1, Theorem 20] is a basis for the row space of WtMax4PC_T as well.

Theorem 24. Let T be a tree on $n > 3$ vertices with p leaves. We assume that $T \neq S_n$ and for $1 \leq i \leq n - 1$, edge e_i has weight α_i . Let $\mathfrak{B}$ be the collection of bases as described in [1, Theorem 20]. For each $B \in \mathfrak{B}$, the following statements are true.

- (a) There exist a unique pendant edge $e = \{u, v\} \in B \cap E(T)$ such that $\deg(u) = 1$ with $\{u, w\} \in B$, where w is a neighbour of v and $\deg(w) > 1$. Moreover, if $\ell \neq u$ is a leaf in T and $\{x, \ell\} \in B \cap E(T)$ for some $x \in T$, then $\{x, y\} \in B \cap E(T)$ for some neighbour y of x with $\deg(y) \neq 1$.
- (b) The number of elements in B is $2(n - p)$.
- (c) The set B forms a basis for the row space of WtMax4PC_T .

Proof. The proofs of items (a) and (b) do not involve weights on the edges and so directly follow from [1, Theorem 11].

Proof of Item (c). We induct on n . When $n = 4$, if we exclude the star tree, we are left with P_4 , the path on 4 vertices. Thus, our base case is P_4 . Suppose P_4 has edges $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$ with respective weights α_1, α_2 and α_3 . If we have $B = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\} \in \mathfrak{B}$, then it can easily be verified that $\det \text{WtMax4PC}_{P_4}[B, B] = 4\alpha_1^2\alpha_2^2 \neq 0$. Thus, from Theorem 18, the result is true for our base case.

We now assume that the result holds for all trees on $n - 1$ vertices. Let tree $T \neq S_n$ be a tree on n vertices with edge set $E(T)$. Let $\mathfrak{B}$ be a collection of sets output by the algorithm described in Theorem 20, and let $B \in \mathfrak{B}$. From part (a), there exist unique triplet of vertices u, v , and w in T such that $\{u, v\}, \{u, w\} \in B$ with

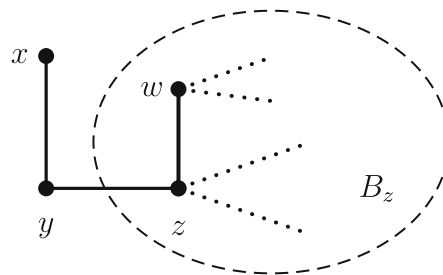


Figure 2: Illustrative figure for the argument above.

$\deg(u) = 1$ and $\{u, v\}, \{v, w\} \in E(T)$. Without loss of generality, we assume that $u = 1, v = 2$, and $w = 3$. Let the weight on the edge $\{1, 2\}$ be α_1 . Therefore from Lemma 16, one can see that

$$\text{WtMax4PC}_T[B, \{1, 3\}] = \text{WtMax4PC}_T[B, \{2, 3\}] + \alpha_1 \mathbf{1}. \quad (32)$$

From Item (a), recall that if $\ell \neq 1$ is a leaf in T and $\{v_1, \ell\} \in B$ for some $v_1 \in T$, then it follows that $\{v_1, \ell\} \in E(T)$ and $\{v_1, v_2\} \in B$ for some neighbour v_2 of v_1 other than ℓ . Without loss of generality, assume that x is a leaf located on a path whose length is the diameter of T with $\{x, y\} \in E(T)$ for some $y \in T$. Note that if there are multiple leaves attached to y , then by the induction hypothesis and applying Corollary 19, it follows that B is indeed a basis for the row space of WtMax4PC_T . We next assume that $\deg(y) = 2$ and $\{y, z\} \in E(T)$ with $\{x, y\}, \{y, z\} \in B$ and with $z \neq x$. Furthermore, let w be the neighbour of z (other than y) such that $\{z, w\} \in B$ (Figure 2).

Consider the set $\widehat{B}$ obtained by applying our algorithm as described in Theorem 20 on $T - x$ that makes the same choices, which resulted in the set B . By the induction hypothesis, $\widehat{B}$ is a basis for the row space of WtMax4PC_{T-x} . Suppose the weights on $\{x, y\}, \{y, z\}$, and $\{z, w\}$ are α, β and γ , respectively. Now, we divide the remaining part of the proof into two cases depending on whether $\{y, z\} \in \widehat{B}$ or $\{y, z\} \notin \widehat{B}$.

Case 1: This is the case of $\{y, z\} \in \widehat{B}$. From our algorithm given in Theorem 20, it is simple to check that $\{y, w\} \in B$. Thus, $\widehat{B} = B \setminus \{\{x, y\}, \{y, w\}\}$. If $\mathbf{u} = \text{WtMax4PC}_T[\widehat{B}, \{x, y\}]$ and $\mathbf{v} = \text{WtMax4PC}_{T-x}[\widehat{B}, \{y, w\}]$ are the column vectors of $\text{WtMax4PC}_T[\widehat{B}, \widehat{B}]$ indexed by the columns $\{x, y\}$ and $\{y, w\}$, respectively, then, it is easy to see that the matrix $\text{WtMax4PC}_T[B, B]$ has the following partitioned form:

$$\text{WtMax4PC}_T[B, B] = \begin{matrix} & \{x, y\} & \{y, w\} \\ \begin{matrix} \{x, y\} \\ \{y, w\} \end{matrix} & \begin{bmatrix} \text{WtMax4PC}_{T-x}[\widehat{B}, \widehat{B}] & \mathbf{u} & \mathbf{u} \\ \mathbf{u}^t & 2\alpha & \alpha + \beta + \gamma \\ \mathbf{v}^t & \alpha + \beta + \gamma & 2(\beta + \gamma) \end{bmatrix} \end{matrix}.$$

From Lemma 16, one can see that

$$\text{WtMax4PC}_{T-x}[\widehat{B}, \{y, w\}] = \text{WtMax4PC}_{T-x}[\widehat{B}, \{z, w\}] + \beta \mathbf{1}. \quad (33)$$

Furthermore, it is easy to see that

$$\text{WtMax4PC}_T(\{z, w\}, \{x, y\}) = \alpha + 2\beta + \gamma \quad \text{and} \quad \text{WtMax4PC}_T(\{z, w\}, \{y, w\}) = 2\beta + \gamma.$$

Perform the row operation $\text{Row}_{\{y, w\}} = \text{Row}_{\{y, w\}} - \text{Row}_{\{z, w\}} - \frac{\beta}{\alpha_1}(\text{Row}_{\{1, 3\}} - \text{Row}_{\{2, 3\}})$ and an identical column operation in $\text{WtMax4PC}_T[B, B]$. Using (32) and (33) in addition, we obtain the following:

$$\begin{matrix} & \{x, y\} & \{y, w\} \\ \begin{matrix} \{x, y\} \\ \{y, w\} \end{matrix} & \begin{bmatrix} \text{WtMax4PC}_{T-x}[\widehat{B}, \widehat{B}] & \mathbf{u} & \mathbf{0} \\ \mathbf{u}^t & 2\alpha & -2\beta \\ \mathbf{0}^t & -2\beta & 0 \end{bmatrix} \end{matrix}.$$

It follows that $\det \text{WtMax4PC}_T[B, B] = -4\beta^2 \det \text{WtMax4PC}_{T-x}[\widehat{B}, \widehat{B}]$. Hence, using the induction hypothesis, B is a basis for the row space of WtMax4PC_T .

Case 2: This is the case of $\{y, z\} \notin \widehat{B}$. Using our algorithm from Theorem 20, as done in Case 1, the basis $\widehat{B}$ for the row space of WtMax4PC_{T-x} can be determined from B . We have $\widehat{B} = B \setminus \{\{y, z\}, \{x, y\}\}$, and the matrix $\text{WtMax4PC}_T[B, B]$ can be partitioned as follows:

$$\text{WtMax4PC}_T[B, B] = \begin{matrix} & \{y, z\} & \{x, y\} \\ \begin{matrix} \{y, z\} \\ \{x, y\} \end{matrix} & \begin{bmatrix} \text{WtMax4PC}_{T-x}[\widehat{B}, \widehat{B}] & \mathbf{w} & \mathbf{u} \\ \mathbf{w}^t & 2\beta & \alpha + \beta \\ \mathbf{u}^t & \alpha + \beta & 2\alpha \end{bmatrix} \end{matrix},$$

where $\mathbf{u} = \text{WtMax4PC}_T[\widehat{B}, \{x, y\}]$ and $\mathbf{w} = \text{WtMax4PC}_{T-x}[\widehat{B}, \{y, z\}]$.

Since $\{y, z\} \notin \widehat{B}$, it follows that if $i, j \in B_z$ for each $\{i, j\} \in \widehat{B}$. Thus, by applying Lemma 17, we have $\mathbf{u} = \mathbf{w} + (\alpha + \beta)\mathbf{1}$. Performing the row operation $\text{Row}_{\{x, y\}} = \text{Row}_{\{x, y\}} - \text{Row}_{\{y, z\}} - \frac{(\alpha + \beta)}{a_1}(\text{Row}_{\{1, 3\}} - \text{Row}_{\{2, 3\}})$ and an identical column operation in $\text{WtMax4PC}_T[B, B]$ give us the following equivalent form:

$$\begin{array}{c} \{y, z\} \\ \{x, y\} \end{array} \begin{array}{c} \{y, z\} \quad \{x, y\} \\ \left[\begin{array}{cc|cc} \text{WtMax4PC}_{T-x}[\widehat{B}, \widehat{B}] & \mathbf{w} & \mathbf{0} & \\ \mathbf{w}^t & 2\beta & -2\beta & \\ \mathbf{0}^t & -2\beta & 0 & \end{array} \right] \end{array}.$$

Thus, $\det \text{WtMax4PC}_T[B, B] = -4\beta^2 \det \text{WtMax4PC}_{T-x}[\widehat{B}, \widehat{B}]$. By induction, we obtain that B is a basis for the row space of WtMax4PC_T , completing the proof. $\square$

We are finally ready to prove Theorem 4.

Proof of Theorem 4. We use induction on n , where again, our base case is when $n = 4$ and $T = P_4$. Using the base case of our proof of Item (c) of Theorem 24, we have $\det \text{WtMax4PC}_{P_4}[B, B] = 4\alpha_1^2\alpha_2^2$, where α_1 and α_2 are weights of the unique pendant edge $e \in B$ mentioned in Item (a) of Theorem 24 and of the non-pendant edge of P_4 respectively. Assume that the result is true for all trees with $n - 1$ vertices. Let $T \neq S_n$ be a tree with n vertices. Then, the result follows from the proof of Item (c) of Theorem 24 and the induction hypothesis. The proof is complete. $\square$

Note that Theorem 4 is also true for the star tree S_n . Thus, combining Theorems 21 and 24, we obtain Theorem 20 by setting each weight $\alpha_i = 1$ for all $i \geq 1$.

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